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ON GENERALIZATIONS OF THE HILBERT NULLSTELLENSATZ FOR INFINITY DIMENSIONS (A SURWAY)

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Abstract. The paper contains a proof of Hilbert Nullstellensatz for the polynomials on infinite-dimensional complex spaces and for a symmetric and a block-symmetric polynomials.

Keywords: polynomials, symmetric polynomials, block-symmetric polynomials, algebra of polynomials, Hilbert Nullstellensatz, algebraic basis.

1. INTRODUCTION

The Hilbert Nullstellensatz is a classical princip in Algebraic Geometry and actually its starting point. It provides a bijective correspondence between affine varieties, which are geometric objects and radical ideals in a polynomials ring, wich are algebraic objects. For the proof and applications of the Hilbert Nullstellensatz we refer the reader to [6].

The question whether a bounded polynomial functionalon a complex Banach space X is determined by its kernel the set of zeros under te assumption that all the factors of its decomposition into irreducible factors are simple was posed by Mazur and Orlich (see also Problem 27 in [10]). A positive answer to this question it follows from Theorem 2 of the present paper. Moreover, this result remains valid even when the ring of bounded polynomial functionals is replaced by any ring of polynomials for which there exists a decomposition into ireducible factors satisfying the following condition along with each polynomial $P(x)$ that it contains the ring also contains the polynomial $P_{\lambda;x_0}(x) = P(x_0 + \lambda x)$, where $x \in X$ and $\lambda \in \mathbb{C}$.

Let X and Y be vector spaces over the field $\mathbb{C}$ of complex numbers. A mapping $\overline{P}_k(x_1, \ldots, x_k)$ from the Cartesian product X^k into Y is $k - linear$ if it is linear in each component. The restriction P_k of the k-linear operator $\overline{P}_k$ to the diagonal $\triangle = \{(x_1, \ldots, x_k) \in X^k : x_1 = \ldots = x_k\}$, which can be naturally identified with X, is a homogeneous polynomial of degree k (briefly, a k-monomial). A finite sum of k-monomials, $0 \leq k \leq n$, $P(x) = P_0(x) + P_1(x) + \ldots + P_n(x)$, $P_n \neq 0$ is a polynomial of degree n. For general properties of polynomials on abstract linear spaces we refer the reader to [4].

This paper is devoted to generalizations of the Hilbert Nullstellensatz of infinite dimensional spaces. In Section 2 we consider the case of abstract infinite dimension complex linear spaces. Section 3 is devoted to continuous polynomials on complex Banach spaces. In Section 4 we examin symmetric polynomials on ℓ_p and Section 5 contains some new results about Nullstellensatz for block-symmetric polynomials.

2. The Nullstellensatz on Infinite-Dimensional Complex Spaces

All results of this section are proved in [15].

Let us denote by X a complex vector space, by $\mathcal{P}(X)$ the algebra of all complex-valued polynomials on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$ satisfying the following conditions:

(1) If $P(x) \in \mathcal{P}_0(X)$, then $P_{x_0;\lambda}(x) = P(\lambda x + x_0) \in \mathcal{P}_0(X)$ for any $x_0 \in X$ and $\lambda \in \mathbb{C}$.
(2) If $P \in \mathcal{P}_0(X)$, $P = P_1 P_2; P_1 \neq 0, P_2 \neq 0$, then $P_1 \in \mathcal{P}_0(X)$ and $P_2 \in \mathcal{P}_0(X)$.

That is, the algebra $\mathcal{P}_0(X)$ is factorial and closed under translation. We shall agree to call such algebras of polynomials *FT-algebra*.

It is obvious that $\mathcal{P}(X)$ is an FT-algebra. A typical example of an FT-algebra is algebra of bounded polynomials (on bounded subset) on a locally convex space X. We shall denote this algebra by $\mathcal{P}_b(X)$. Anothe example of an FT-algebra is provided by the polynomials formed by finite sums of finite products of continuous linear functionals on X (polynomials of finite type). If Y is subspace of X, we take $\mathcal{P}_0(Y)$ to mean the restrictions of the polynomials of $\mathcal{P}_0(X)$ to Y. It easy to see that $\mathcal{P}_b(Y)$ coincides with the algebra of bounded polynomials on Y.

Let $P_\gamma(x) \in \mathcal{P}_0(X)$ be a family of polynomials, where γ belongs to an index set Γ. We recall that an ideal (P_γ) in $\mathcal{P}_0(X)$ is a set $J = \left\{ P \in \mathcal{P}_0(X) : P = \sum_{\gamma\in\Gamma} Q_\gamma(x) P_\gamma(x), Q_\gamma \in \mathcal{P}_0(X) \right\}$, where the sum $\sum_{\gamma\in\Gamma} Q_\gamma(x) P_\gamma(x)$ contains only a finite number of terms that are not identically zero. A linearly independent subset $\{P_{\gamma_\beta}\}$ of the set $\{P_\gamma\}$ such that $(P_\gamma) = (P_{\gamma_\beta})$ is a linear basis of the ideal J. For an ideal $J \in \mathcal{P}_0(X)$, $V(J)$ denotes the *zero* of J, that is, the common set of zeros of all polynomials in J. Let G be a subset of X. Then $I(G)$ denotes the *hull* of G, that is, a set of all polynomials in $\mathcal{P}_0(X)$ which vanish on G. The set $radJ$ is called the *radical* of J if $P^k \in J$ for some positive integer k implies $P \in radJ$. P is called a *radical* polynomial if it can be represented by a product of mutually different irreducible polynomials. In the case $(P) = rad(P)$.

It is easy to see that $I(G)$ is an ideal in $\mathcal{P}_0(X)$. The main problem that we shall solve consists of establising conditions under wich the equality

$$I(V(J)) = J$$

holds for the ideal $J \in \mathcal{P}_0(X)$ that is, an ideal in $\mathcal{P}_0(X)$ is uniquely determined by its set of zeros.

In the finite-dimensional case the answer to this question is provided by the Hilbert Nullstellensatz, which asserts that a necessary and sufficient condition for this to happen is that the ideal J be equal to its radical (which we shall define below). We remark that for the infinite-dimensional case this condition is not sufficient. (A counterexample will be given).

Lemma 2.1. *Let $P_1, \dots, P_n$ be polynomials on X and* $\deg P_1 \geq \deg P_2 \geq \dots \geq \deg P_n > 0$. *Then there exists an element $h \in X$ such that for any $x \in X$ the degree of the scalar-valued polynomial $P_1(x+th)$ in t is* $\deg P_1$, *and the polynomials $P_2, \dots, P_n$ depend on h, that is, for each $P_i, i = 2, \dots, n$, there exists x_i such that the scalar-valued polynomial $P_i(x_i + th)$ in t is of positive degree.*

Proof. For $n = 1$ the assertion of the lemma is obvious. Assume it is true for $n-1$. Let h_1 be the required element for $P_1, \dots, P_{n-1}$. Assume that P_n is independent of h_1, that is, $P_n(x+th_1) = P_n(x)\ \forall x \in X$. Let h_2 be an elementof X such that P_n depends on h_2. We make the definition $h(\lambda) := h_1 + \lambda h_2, \lambda \in \mathbb{C}$. Consider the family of scalar-valued polynomials $P_1(x + th(\lambda))$ in t with parameters λ, x. For any x there is only a finite set of λ, at which the polynomial $P_1(x + th(\lambda))$ is of degree less than $\deg P_1$ in t.

Indeed, let $\deg P_1 = m$, and let $P_1 = \sum\limits_{i=0}^{m} f_i$ be an expansion in monomials. Then $P_1(x + th(\lambda))$ can be given in the following form:

$$\begin{aligned} P_1(x + th(\lambda)) &= \sum_{i=1}^{m} f_i(x + th(\lambda)) = \sum_{i=1}^{m} t^j \bar{f}_i(x, \dots, x, \overbrace{h(\lambda), \dots, h(\lambda)}^{j}) \\ &= t^m f_m(h(\lambda)) + \sum_{k<m} \sum_{j \leq k} t^j q_j(x + h(\lambda)), \end{aligned}$$

where $\bar{f}_i$ are $i-$ linear forms corresponding to the monomials f_i;

$$q_j = \sum_i \bar{f}_i(x, \dots, x, \overbrace{h(\lambda), \dots, h(\lambda)}^{j}).$$

If $\deg P_1(x + th(\lambda')) < m$ for some value λ' of the parameter λ, then $f_m(h(\lambda')) = f_m(h_1 + \lambda' h_2) = 0$. But, since $f_m(h_1 + \lambda h_2)$ is polynomial in the variable λ (for fixed h_1 and h_2), it can have only a finite number of zeros without being identically zero. Assume that $f_m(h_1 + \lambda h_2) \equiv 0$. Then this relation also holds for $\lambda = 0$. Hence $\deg P_1(x + th(0)) = \deg P_1(x + th_1) < m$, which contradicts the choice of h_1.

Similarly, for each $i = 2, \dots, n-1$ there exists a finite set of values of the parameter λ at which the polynomials $P_i(x + th(\lambda))$ have smaller degree in t than $\deg P_i$, in particular, degree 0. Hence there exists a number $\lambda_0 \neq 0$ such that $\deg P_1(x + th(\lambda_0)) = m$ with respect to t, and the polynomials P_i depend on $h(\lambda_0)$ for $1 < i < n$. Moreover, P_n also depends on $h(\lambda_0)$, since $P_n(x + th(\lambda_0)) = P_n(x + t\lambda_0 h_2)$. Therefore, $h := h(\lambda_0)$ is the required element for n. The lemma is now proved. □

Theorem 2.2. *Let X be a complex vector space of arbitrary (possibly infinite) dimension, and let $P_1(x), \dots, P_n(x) \in \mathcal{P}_0(X)$, where $\mathcal{P}_0(X)$ is an FT-algebra. Then there exists an element $h \in X$, a subspace Z complementary to $\mathbb{C}h$ in X, and polynomial functionals $G_1, \dots, G_{n-1} \in \mathcal{P}_0(X)$ such that:*

(1) $g_k(z + th) = g_k(z)\ \forall z \in Z, t \in \mathbb{C}, k = 1, \dots, n-1.$
(2) *All G_k belong to the ideal $(P_1, \dots, P_n)$ in the algebra $\mathcal{P}_0(X)$.*
(3) *The set of zeros of the ideal $(g_1, \dots, g_{n-1})$ in the algebra $\mathcal{P}_0(Z)$ is the projection of the zeros of the ideal $(P_1, \dots, P_n)$ in $\mathcal{P}_0(X)$ onto the subspace Z along h.*
(4) *If $g_k \equiv 0, k = 1, \dots, n-1$, then $P_1, \dots, P_n$ have a common divisor.*

Proof. Let $\deg P_1 = \max_i \deg P_i$ and let $h \in X$ be an element such that the degree of the polynomial $P_1(x+th)$ in the variable $t \in \mathbb{C}$ equals $\deg P_1$ for all $x \in X$ and the polynomials $P_1, \ldots, P_n$ depend on h. Such an element exists in accordance with Lemma 2.1. Concider the polynomials $P_1, \ldots, P_n$ as elements of the algebra $(\mathcal{P}_0(Z))[t]$, where Z is a closed subspace complementary to $\mathbb{C}h$ in X. That is, the elements of the algebra $(\mathcal{P}_0(Z))[t]$ are polynomials of t with coefficients in the fieldof quotients of elements of $\mathcal{P}_0(Z)$. We shall denote them by $\tilde{P}_1(t), \ldots, \tilde{P}_n(t)$ respectively. We may assume that $\deg \tilde{P}_1(t) \geq \tilde{P}_2(t) \geq \ldots \geq \tilde{P}_n(t)$. Division with remainder holds in the algebra $(\mathcal{P}_0(Z))[t]$. Therefore for $\tilde{P}_1(t)$ and $\tilde{P}_2(t)$ there exist $P_2^1(t)$ and $P_2^1(t)$ in $(\mathcal{P}_0(Z))[t]$ such that

$$\tilde{P}_1 - Q_2^1 \tilde{P}_2 = P_2^1. \tag{2.1}$$

If $\deg P_2^1 \geq \deg \tilde{P}_3$, there exist Q_3^1 and P_3^1 in $(\mathcal{P}_0(Z))[t]$ such that

$$P_2^1 - Q_3^1 \tilde{P}_3 = P_3^1. \tag{2.2}$$

When $\deg P_2^1 < \deg \tilde{P}_3$, we set $Q_3^1 = 0, P_3^1 = P_2^1$. Continuing this process, we obtain the following relations:

$$\begin{array}{c} P_3^1 - Q_4^1 \tilde{P}_4 = P_4^1, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \tag{2.3}$$

$$P_{n-1}^1 - Q_n^1 \tilde{P}_n = \tilde{P}_{n+1}, \tag{2.4}$$

where $\deg \tilde{P}_{n+1} < \deg \tilde{P}_n$. From relations (2.1)-(2.4), we have:

$$\tilde{P}_1 - \sum_{i=2}^{n} Q_i^1 \tilde{P}_i = \tilde{P}_{n+1}.$$

For the elements $\tilde{P}_2, \ldots, \tilde{P}_{n+1} \in (\mathcal{P}_0(Z))[t]$ we obtain similarly the relations

$$\tilde{P}_2 - \sum_{i=3}^{n+1} Q_i^2 \tilde{P}_i = \tilde{P}_{n+2},$$

$\deg \tilde{P}_{n+2} < \deg \tilde{P}_{n+1}$; for $\tilde{P}_3, \ldots, \tilde{P}_{n+2}$:

$$\tilde{P}_3 - \sum_{i=4}^{n+2} Q_i^3 \tilde{P}_i = \tilde{P}_{n+3},$$

$\deg \tilde{P}_{n+3} < \deg \tilde{P}_{n+2}$, and so on.

Since the sequence $\deg \tilde{P}_{n+1}, \deg \tilde{P}_{n+2}, \ldots$ is strictly decreasing, by continuing this process we obtain for a coefficient k_1 :

$$\tilde{P}_{k_1-1} - \sum_{i=k_1}^{n+k_1-2} Q_i^{k_1} \tilde{P}_i = \tilde{P}_{n+k_1-1}.$$

Moreover, $\deg \tilde{P}_{n+k_1-1} = 0$, that is, $\tilde{P}_{n+k_1-1} \in \mathcal{P}_0(Z)$. We introduce the notation $G_1 = \tilde{P}_{n+k_1-1}$. Consider the elements $\tilde{P}_{k_1}(t), \ldots, \tilde{P}_{n+k_1-2}(t) \in (\mathcal{P}_0(Z))[t]$. There are $n-1$ of them, all depending on t. Applying the preceding reasoning to them, we obtain for some $k_2 > k_1$:

$$\tilde{P}_{k_2-1} - \sum_{i=k_2}^{n+k_2-3} Q_i^{k_2} \tilde{P}_i = \tilde{P}_{n+k_2-2},$$

where $\tilde{P}_{n+k_2-2} \in \mathcal{P}_0(Z)$, $\deg \tilde{P}_{n+k_2-2} < \deg \tilde{P}_{n+k_2-3} < \dots$. We introduce the notation $G_2 = \tilde{P}_{n+k_2-2}$. Consider the polynomials $\tilde{P}_{k_2}(t), \dots, \tilde{P}_{n+k_2-3}(t) \in (\mathcal{P}_0(Z))[t]$. There are $n-2$ of them, all dependent on t, and the preceding reasoning is applicable to them.

Thus at step r we obtain, for some $k_r > k_{r-1>\dots>k_1}$:

$$\tilde{P}_{k_r-1} - \sum_{i=k_r}^{n+k_r-r-2} Q_i^{k_r} \tilde{P}_i = \tilde{P}_{n+k_r-r-1},$$

where $\tilde{P}_{n+k_r-r-1} \in \mathcal{P}_0(Z)$. We introduce the notation $G_r = \tilde{P}_{n+k_r-r-1}$. At step $r = n-1$ our algorithm coincides with the Euclidean algorithm for the polynomials $\tilde{P}_{k_{n-1}}(t), \tilde{P}_{k_{n-1}+1}(t)$. That is, for some $k_{n-1} > \dots > k_1$ we find:

$$\tilde{P}_{k_{n-1}} - Q_{k_{n-1}+1}^{k_{n-1}} \tilde{P}_{k_{n-1}+1} = \tilde{P}_{k_{n-1}+2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\tilde{P}_{k_n-4} - Q_{k_n-3}^{k_n-4} \tilde{P}_{k_n-3} = \tilde{P}_{k_n-2}, \tag{2.5}$$

$$\tilde{P}_{k_n-3} - Q_{k_n-2}^{k_n-3} \tilde{P}_{k_n-2} = \tilde{P}_{k_n-1}, \tag{2.6}$$

where $\tilde{P}_{k_n-1} \in \mathcal{P}_0(Z)$. We introduce the notation $G_{n-1} = \tilde{P}_{k_n-1}$.

It is clear from the algorithm that all the polynomials $\tilde{P}_i \in (\mathcal{P}_0(Z))[t]$ belong to the ideal $(\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_n)$ in the algebra $(\mathcal{P}_0(Z))[t]$. In particular, this is true also for $G_r = \tilde{P}_{n+k_r-1}$. That is, there exist polynomials V_i^k, $k = 1, \dots, n-1$, $i = 1, \dots, n$, in the algebra $(\mathcal{P}_0(Z))[t]$ such that

$$\sum_{i=1}^{n} \tilde{P}_i V_i^k = G_k$$

for $k = 1, \dots, n-1$. Multiplying each of these equalities by the common denominator $a_k \in \mathcal{P}_0(Z)$ of the coefficients of the terms of degree t in $\mathcal{P}_0(Z)$ and passing to the algebra $\mathcal{P}_0(X)$, we find that there exist polynomials $v_i^k \in \mathcal{P}_0(X)$, such that

$$\sum_{i=1}^{n} P_i v_i^k = g_k, \tag{2.7}$$

where $g_k = G_k a_k$.

Thus we have found a sequence of polynomials $g_1, \dots, g_{n-1}$, that actually belong to $\mathcal{P}_0(Z)$, more precisely: $g_k(z+th) = g_k(z) \forall z \in Z$. In addition, all g_k belong to the ideal $(P_1, \dots, P_n)$. Let $z_0 \in Z$ be a common zero of the polynomials g_k. Then $z_0 + th$ is a common zero of g_k, $k = 1, \dots, n-1$, for any $t \in \mathbb{C}$. We multiply Eq. (2.6) by the common denominator $b_1 \in \mathcal{P}_0(Z)$ of the coefficients of the powers of t and pass to the algebra $\mathcal{P}_0(Z)$. Then,

$$P_{k_n-3} - q_{k_n-2}^{k_n-3} P_{k_n-2} = g_{n-1},$$

where $P_i = \tilde{P}_i b_1$, $q_i = Q_i b_1$. Therefore $P_{k_n-3}(z_0+th)$ is divisible by $P_{k_n-2}(z_0+th)$ (since $g_{n-1}(z_o+th) = 0$). Let us multiply Eq. (2.5) by b_2, the common denominator of the powers of t in (2.5), and substitute the value of P_{k_n-3} in place of P_{k_n-3} itself:

$$P_{k_n-4} - q_{k_n-3}^{k_n-4}(g_{n-1} + q_{k_n-2}^{k_n-3} P_{k_n-2}) - P_{k_n-2} = 0.$$

Taking account of the relation $g_{n-1}(z_0+th) = 0$, we find that $P_{k_n-4}(z_0+th)$ is divisible by $P_{k_n-2}(z_0+th)$. Working from bottom to top, we find that the polynomials $b(z_0+th)P_1(z_0+th), \dots, b(z_0+th)P_n(z_0+th)$ are divisible by $P_{k_n-2}(z_0+th)$, where b is polynomial in $\mathcal{P}_0(Z)$.

Assume that $P_{k_n-2}(z_0+th) \equiv const$ (with respect t). This means that the degree of the polynomial $P_{k_n-2}(z_0+th)$ is less than the degree of the polynomial $\tilde{P}_{k_n-2}(t) \in (\mathcal{P}_0(Z))[t]$, since by construction $\deg \tilde{P}_{k_n-2} > 0$. Then we also have $\deg P_{k_n-3}(z_0+th) < \deg \tilde{P}_{k_n-3}(t)$. Working from the bottom upward, we find that $\deg P_1(z_0+th)$, as a polynomial in t, is less than $\deg \tilde{P}_1 = \deg P_1$. But the equality $\deg \tilde{P}_1 = \deg P_1$ (which holds by the choice of h) means that the monomial of highest degree in t in the polynomial $P_1(z_0+th)$ is independent of $z \in Z$, so that this is impossible. Hence $P_{k_n-2}(z_0+th) \neq const$, and therefore, first of all, the fact that $b(z_0+th)P_i(z_0+th)$ is divisible by $P_{k_n-2}(z_0+th)$ for $1 \leq i \leq n$ implies that $P_i(z_0+th)$ is divisible by $P_{k_n-2}(z_0+th)$, $1 \leq i \leq n$, since b is independent of h and $P_{k_n-2}(z_0+th)$ depends on h; second there exists $t_0 \in \mathbb{C}$ such that $P_{k_n-2}(z_0+th) = 0$. Thus $x_0 = z_0 + t_0 h$ is a common zero of the polynomials $P_1, \dots, P_n$.

As a result we have the following: if z_0 is a zero of the ideal $(g_1, \dots, g_{n-1})$, then for some t_0 we find that $x_0 = z_0 + t_0$ is a zero of the ideal $(P_1, \dots, P_n)$. It follows from Eqs. (2.7) that the converse is also true: every zero of the ideal $(P_1, \dots, P_n)$ is a zero of the ideal $(g_1, \dots, g_{n-1})$, and hence its projection of the zeros of the ideal $(P_1, \dots, P_n)$ on the subspace Z along h. In the case when $g_k \equiv 0$ for all k we find that all P_i, $i = 1, \dots, n$, are divisible by $P_{k_n-2}(z+th)$ for every $z \in Z$, that is, $(P_1(x), \dots, P_n(x))$ have the common divisor $P_{k_n-2}(x)$. The theorem is now proved. $\square$

Remark 1. In the case $\dim X = 1$ the proposed algorithm becomes the general Euclidean algorithm for finding a common divisor for n polynomials in one variable

Corollary 2.3. *Let $J = (P_1, \dots, P_n)$ be an ideal of polynomials in $\mathcal{P}_0(X)$ and $\dim X \geq n$. Then there exist elements $h_1, \dots, h_m \in X$, a subspace $W \subset X$ of codimension $m \leq n-1$, and a polynomial $f \in \mathcal{P}_0(X)$ such that:*

(1) *$f \in J$.*

(2) *f is independent of $h_1, \dots, h_m$, that is, for any $w \in W$ $f(w + t_1 h_1 + \ldots + t_m h_m) = f(w)$, where $t_1, \dots, t_n$ are arbitrary elements of $\mathbb{C}$.*

(3) *The kernel of f is the projection of the set $V(J)$ on W along the subspace $H_m = lin(h_1, \dots, h_m)$.*

Proof. We apply Theorem 2.2 to the ideal $J = (P_1, \dots, P_n)$. Let $g_1, \dots, g_{n-1}$ be polynomials, h an element of X, Z is the subspace of X whose existence is guaranteed by the theorem. We revise the notation for $g_i^1 := g_i$, $i = 1, \dots, n-1$, $h_1 := h$, $Z_1 = Z$. Applying Theorem 2.2 to the polynomials $g_1^1, \dots, g_{n-1}^1$, we obtain polynomials $g_1^2, \dots, g_{n-2}^2$, element $h_2 \in X$, and a subspace $Z_2 \subset X$. Here h_2 can be chosen from the subspace Z_1 and $Z_2 \subset Z_1$. Applying Theorem 2.2 several times at step $m \leq n-1$, we obtain a polynomial $g_1^m =: f \in \mathcal{P}_0(X)$ such that $f \in J$. Indeed

$$J = (P_1, \dots, P_n) \supset (g_1^1, \dots, g_{n-1}^1) \supset \ldots \supset (g_1^m) = (f), \quad f \in (f). \tag{2.8}$$

Let $w_0 \in \ker f$. Then by Theorem 2.2 we have $w_0 + t_m^0 \in V((g_1^{m-1}, g_2^{m-1}))$ for some t_m^0. Then $w_0 + t_m^0 + t_{m-1}^0 \in V((g_1^{m-1}, g_2^{m-1}, g_3^{m-1}))$ for some t_{m-1}^0. Continuing, we find that $w_0 + t_1^0 + \ldots + t_m^0 \in V(J)$ for some $t_1^0, \dots, t_m^0$ in the other hand, if $x_0 \in V(J)$, then $x_0 \in \ker f$. Moreover, it follows from the inclusions (2.8) and Theorem 2.2 the independent of $h_1, \dots, h_m$, so that the projection of x_0 on $W := Z_m$ belongs to the kernel of f. The corollary proved. $\square$

We now recall some definitions from ideal theory.

Definition 2.4. The ideal $radJ$ is the radical of the ideal J, if for any positive integer k the relation $P^k \in J$ implies $P \in radJ$. If $J = radJ$, then J is a radical ideal.

Definition 2.5. An ideal J is prime if $\mathcal{P}_0(X)/J$ is integral domain, that is the algebra $\mathcal{P}_0(X)/J$ has no zero divisor ideal is maximal if $\mathcal{P}_0(X)/J$ is a field.

Theorem 2.6 (The Hilbert Nullstellensatz.). *Let J be an ideal the FT-algebra $\mathcal{P}_0(X)$, $J = (P_1, \ldots, P_n)$. Then:*

(1) *If $V(J) = \varnothing$, then $J =$ (2.1).*
(2) *$I(V(J)) = radJ$.*

Proof. Since this theorem is well known for the case $\dim X < \infty$, we can assume that $\dim X = \infty$ (hence $> n$). I follows immediately from Corollary 2.3. Therefore only Point 2 requires proof.

We apply reasoning that is well known for the finite-dimensional case [12]. Let f be an arbitrary polynomial algebra $\mathcal{P}_0(X)$. Assume that $f(x) = 0 \forall x \in V(J)$. Let $y \in \mathbb{C}$ be an additional independent variable. Consider $\mathcal{P}_0(X + y)$ of polynomials on the space $X \oplus \mathbb{C}y$, that are polynomials in $\mathcal{P}_0(X)$ for each fixed $y \in \mathbb{C}$ and polynomials in $\mathbb{C}[y]$, the algebra of all polynomials in y, for each fixed $x \in X$. The algebra $\mathcal{P}_0(X + y)$ is obviously an FT-algebra. Theorem 2.2 holds in it. The polynomials $P_1, \ldots, P_n$ and $fy - 1$ have no common zeros. By Point 1 of the there exist polynomials $g_1, \ldots, g_{n+1} \in \mathcal{P}_0(X + y)$, such that

$$\sum_{i=1}^{n} P_i q_i + (fy - 1)q_{n+1} \equiv 1,$$

and $g_1, \ldots, g_{n+1}$ depend on $x \in X$ and y. Since this is an identity, it remains valid also for rational functionals the substitute $y = \frac{1}{f}$. Hence,

$$\sum P_i q_i(x, \frac{1}{f}) = 1.$$

Reducing these to a common denominator, we find that for some N

$$\sum P_i q_i'(x) f^{-N} = 1$$

or

$$\sum P_i q_i'(x) = f^N,$$

where $q_i'(x) = q_i(x, f^{-1}) f^N \in \mathcal{P}_0(X)$. But this means that f^N belongs to the ideal J. Hence $f \in radJ$ theorem is now proved. □

We now give an example of an ideal generated by an infinite number of polynomials for which the Nullstellensatz does not hold.

Example 2.7. Let H be a separable Hilbert space. Consider the ideal J generated by finite sums of polynomials $f_i(x) = (x, e_i) + a_i$, where $(\ ,\ )$ is the inner product, (e_i) is an orthonormal basis in H, and $a_i \in \mathbb{C}$. The only zero that this ideal can have is an element $\sum_i a_i e_i$. But if (a_i) are chosen so that this sum diverges in H, the ideal J has no zeros. But it is obvious that the ideal J contains no units.

In the case $n = 2$ the next corollary gives a positive answer to Problem 27 of [10] (see also [13]).

Corollary 2.8. *Let $P_1, \ldots, P_n$ be continuous polynomials on the Banach space X. Assume that there exists a sequence of elements $(x_i)_{i=1}^{\infty}$, $||x_i|| = 1$, such that $P_k(x_i) \to 0$ as $i \to \infty$, $1 \leq k \leq n$. Then the polynomials $P_1, \ldots, P_n$ have a common zero.*

Proof. Suppose such is not the case. Since the algebra $\mathcal{P}_b(X)$ is an FT-algebra, according to Theorem 2.2 there exist continuous polynomials $q_1, \ldots, q_n$ such that

$$P_1 q_1 + \ldots + P_n q_n \equiv 1,$$

and this contradicts the fact that $P_k(x_i) \to \infty$, $1 \le k \le n$. The corollary is now proved. □

Now consider the topology σ on X whose closed sets are the kernels of polynomials in $\mathcal{P}_0(X)$, along with finite unions and arbitrary intersections of them. It is easy to see that this is indeed a topology. By analogy with the finite-dimensional case we call this topology the Zariski topology. We remark that for different FT-algebras we obtain different Zariski topologies. In the case of the algebra of continuous polynomials on X the Zariski topology is strictly weaker than the topology on X. In this connection the following question arises.

3. The Nullstellensatz for Algebras of Polynomials on Banach Spaces

All results of this section are proved in [14].

Let X be a Banach space, and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$.

Theorem 3.1. [2] *Let Y be a complex vector space. Let A be an algebra of functions on Y such that the restriction of each $f \in A$ to any finite dimensional subspace of Y is an analytic polynomial. Let I be a proper ideal in A. Then there is a net (y_α) in Y such that $f(y_\alpha) \to 0$ for all $f \in I$.*

Here we need a technical lemma.

Lemma 3.2. [2] *Let Y be a complex vector space. Let $F = (f_1, \ldots, f_n)$ be a map from Y to $\mathbb{C}^n$ such that the restriction of each f_i to any finite dimensional space of Y is a polynomial. Then the closure of the range of F, $F(X)^-$ is an algebraic variety. Moreover there exists a finite dimensional subspace $Y_0 \subset X$ such that $F(Y_0)^- = F(X)^-$.*

Theorem 3.3. *Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$ with unity which contains all finite type polynomials. Let J be an ideal in $\mathcal{P}_0(X)$ which is generated by a finite number of polynomials $P_1, \ldots, P_n \in \mathcal{P}_0(X)$. If the polynomials $P_1, \ldots, P_n$ have no common zeros, then J is not proper.*

Proof. According to Lemma 3.2 there exists a finite dimensional subspace $Y_0 = \mathbb{C}^m \subset X$ such that $F(Y_0)^- = F(X)^-$ where $F(x) = (P_1(x), \ldots, P_n(x))$. Let $e_1, \ldots, e_m$ be a basis in Y_0 and $e_1^*, \ldots, e_m^*$ be the coordinate functionals. Denote by $P_k|_{Y_0}$ the restriction of P_k to Y_0. Since $\dim Y_0 = m < \infty$, there exists a continuous projection $T : X \to Y_0$. So any polynomial $Q \in \mathcal{P}(Y_0)$ can be exended to a polynomial $\hat{Q} \in \mathcal{P}_0(X)$ by formula $\hat{Q} = Q(T(x))$. $\hat{Q}$ belongs to $\mathcal{P}_0(X)$ becouse it is a finite type polynomial. Let us consider the map

$$G(x) = (P_1(x), \ldots, P_n(x), \hat{e}_1^*(x), \ldots, \hat{e}_m^*(x)) : X \to \mathbb{C}^{m+n}.$$

By definition of G, $G(X)^- = G(Y_0)^-$.

Suppose that J is a proper ideal in $\mathcal{P}_0(X)$ and so J is contained in a maximal ideal J_M. Let ϕ be a complex homomorphism such that $J_M = \ker \phi$. By Theorem 3.1 there exists a $\mathcal{P}_0$- convergent net (x_α) such that $\phi(P) = \lim_\alpha P(x_\alpha)$ for every $\mathcal{P}_0(X)$. Since $G(X)^- = G(Y_0)^-$, there is a net $(z_\beta \subset Y_0)$ such that $\lim_\alpha G(x_\alpha) = \lim_\beta G(z_\beta)$. Note that each polynomial $Q \in \mathcal{P}(Y_0)$ is generated by the coordinate functionals. Thus $\lim_\beta Q(z_\beta) = \lim_\alpha \hat{Q}(x_\alpha) = \phi(Q)$. Also $\lim_\beta P_k|_{Y_0}(z_\beta) = \lim_\alpha P_k(x_\alpha) = \phi(P_k)$, $k = 1, \ldots, n$. On the other hand, every $\mathcal{P}_0$-convergent net on a finite dimensional subspace is weakly convergent and so it converges to a point $x_0 \in Y_0 \subset X$. Thus $P_k(x_0) = 0$ for $1 \leq k \leq n$ that contradicts the assumption that $P_1, \dots, P_n$ have no common zeros. □

A subalgebra A_0 of an algebra A is called *factorial* if for every $f \in A_0$ the equality $f = f_1 f_2$ implies that $f_1 \in A_0$ and $f_2 \in A_0$.

Theorem 3.4 (Hilbert Nullstellensatz Theorem). *Let $\mathcal{P}_0(X)$ be a factorial subalgebra in $\mathcal{P}(X)$ which contains all polynomials of finite type and let J be an ideal of $\mathcal{P}_0(X)$ which is generated by a finite sequence of polynomials $P_1, \dots, P_n$. Then $\mathrm{rad} J \subset \mathcal{P}_0(X)$ and*

$$I[V(J)] = \mathrm{rad} J$$

in $\mathcal{P}_0(X)$.

Proof. Since $\mathcal{P}_0(X)$ is factorial, $\mathrm{rad} J \subset \mathcal{P}_0(X)$ for every ideal $J \in \mathcal{P}_0(X)$. Evidently, $I[V(J)] \supset \mathrm{rad} J$. Let $P \in \mathcal{P}_0(X)$ and $P(x) = 0$ for every $x \in V(J)$. Let $y \in \mathbb{C}$ be an additional "independent variable" which is associated with a basis vector e of an extra dimension. Consider a Banach space $X \oplus \mathbb{C}e = \{x + ye : x \in X, y \in \mathbb{C}\}$. We denote by $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ the algebra of polynomials on $X \oplus \mathbb{C}e$ such that every polynomial in $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ belongs to $\mathcal{P}_0(X)$ for arbitrary $y \in \mathbb{C}$. The polynomials $P_1, \dots, P_n, Py - 1$ have no common zeros. By Theorem 3.3 there are polynomials $Q_1, \dots, Q_{n+1} \in \mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ such that

$$\sum_{i=1}^{n} P_i Q_i + (Py - 1) Q_{n+1} \equiv 1.$$

Since it is an identity it will be still true for all vectors x such that $P(x) \neq 0$ if we substitute $y = 1/P(x)$. Thus

$$\sum_{i=1}^{n} P_i(x) Q_i(x, 1/P(x)) = 1.$$

Taking a common denominator, we find that for some positive integer N,

$$\sum_{i=1}^{n} P_i(x) Q_i'(x) P^{-N}(x) = 1$$

or

$$\sum_{i=1}^{n} P_i(x) Q_i'(x) = P^N(x), \tag{3.1}$$

where $Q'(x) = Q(x, P^{-1}) P^N(x) \in \mathcal{P}_0(X)$. The equality (3.1) holds on an open subset $X \ \ker P$, so it holds for every $x \in X$. But it means that P^N belongs to J. So $P \in \mathrm{rad} J$. □

4. The Nullstellensatz for Algebras of Symmetric Polynomials on ℓ_p

Let X be a Banach space, and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. A sequence $(G_i)_i$ of polynomials is called an *algebraic basis* of $\mathcal{P}_0(X)$ if for every $P \in \mathcal{P}_0(X)$ there is $q \in \mathcal{P}(\mathbb{C})$ for some n such that $P(x) = q(G_1(x), \dots, G_n(x))$; in other words, if G is the mapping $x \in X \rightsquigarrow G(x) := (G_1(x), \dots, G_n(x)) \in \mathbb{C}^n$, then $P = q \circ G$.

Let $\mathcal{P}_s(X)$ be the algebra of all symmetric polynomials. Let $\langle p \rangle$ be the smallest integer that is greater than or equal to p. In [5], it is proved that the polynomials $F_k(\sum a_i e_i) = \sum a_i^k$ for $k = \langle p \rangle, \langle p \rangle + 1, \ldots$ form an algebraic basis in $\mathcal{P}_s(\ell_p)$. So there are no symmetric polynomials of degree less than $\langle p \rangle$ in $\mathcal{P}_s(\ell_p)$ and if $\langle p_1 \rangle = \langle p_2 \rangle$, then $\mathcal{P}_s(\ell_{p_1}) = \mathcal{P}_s(\ell_{p_2})$. Thus, without loss of generality we can consider $\mathcal{P}_s(\ell_p)$ only for integer values of p. Throughout, we shall assume that p is an integer, $1 \leq p < \infty$.

It is well known [8] that for $n < \infty$ any polynomial in $\mathcal{P}_s(\mathbb{C}^n)$ is uniquely representable as a polynomial in the *elementary symmetric polynomials* $(R_i)_{i=1}^n$, $R_i(x) = \sum_{k_1 < \ldots < k_i} x_{k_1} \ldots x_{k_i}$.

In paper [1] was proof next results.

Lemma 4.1. *Let $\{G_1, \ldots, G_n\}$ be an algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, there is $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $G_i(x) = \xi_i$, $i = 1, \ldots, n$. If for some $y = (y_1, \ldots, y_n)$, $G_i(y) = \xi_i$ $i = 1, \ldots, n$, then $x = y$ up to a permutation.*

Proof. First, we suppose that $G_i = R_i$. Then, according to the Vieta formulae [8], the solutions of the equation

$$x^n - \xi_1 x^{n-1} + \ldots + (-1)^n \xi_n = 0$$

satisfy the conditions $R_i(x) = \xi_i$, and so $x = (x_1, \ldots, x_n)$ as required. Now let G_i be an arbitrary algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. Then $R_i(x) = v_i(G_1(x), \ldots, G_n(x))$ for some polynomials v_i on $\mathbb{C}^n$. Setting v as the polynomial mapping $x \in \mathbb{C}^n \rightsquigarrow v(x) := (v_1(x), \ldots, v_n(x)) \in \mathbb{C}^n$, we have $R = v \circ G$.

As the elementary symmetric polynomials also form a basis, there is a polynomial mapping $w : \mathbb{C}^n \to \mathbb{C}^n$ such that $G = w \circ R$; hence $R = (v \circ w) \circ R$ so $v \circ w = \mathrm{id}$. Then v and w are inverse to each other, since $w \circ v$ coincides with the identity on the open set, $\mathrm{Im}(w)$. In particular, v is one-to-one.

Now, the solutions $x_1, \ldots, x_n$ of the equation

$$x^n - v_1(\xi_1, \ldots, \xi_n) x^{n-1} + \ldots + (-1)^n v_n(\xi_1, \ldots, \xi_n) = 0$$

satisfy the conditions $R_i(x) = v_i(\xi)$, $i = 1, \ldots, n$. That is, $v(\xi) = R(x) = v(G(x))$, and hence $\xi = G(x)$. $\square$

Corollary 4.2. *Given $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, there is $x \in \ell_p^{n+p-1}$ such that*

$$F_p(x) = \xi_1, \ldots F_{p+n-1}(x) = \xi_n.$$

Proposition 4.3 (Nullstellensatz)**.** *Let $P_1, \ldots, P_m \in \mathcal{P}_s(\ell_p)$ be such that* $\ker P_1 \cap \ldots \cap \ker P_m = \varnothing$. *Then there are $Q_1, \ldots, Q_m \in \mathcal{P}_s(\ell_p)$ such that $\sum_{i=1}^{m} P_i Q_i \equiv 1$.*

Proof. Let $n = \max_i(\deg P_i)$. We may assume that $P_i(x) = q_i(F_p(x), \ldots, F_n(x))$ for some $g_i \in \mathcal{P}(\mathbb{C}^{n-p+1})$. Let us suppose that at some point $\xi \in \mathbb{C}^{n-p+1}$, $\xi = (\xi_1, \ldots, \xi_{n-p+1})$ and $g_i(\xi) = 0$. Then by Corollary 4.2 there is $x_0 \in \ell_p$ such that $F_i(x_0) = \xi_i$. So the common set of zeros of all q_i is empty. Thus by the Hilbert Nullstellensatz there are polynomials $q_1, \ldots, q_m$ such that $\sum_i g_i q_i \equiv 1$. Put $Q_i(x) = q_i(F_p(x), \ldots, F_n(x))$. $\square$

5. The Nullstellensatz for Algebras of Block-Symmetric Polynomials

Let

$$\mathcal{X}^2 = \oplus_{\ell_1}\mathbb{C}^2$$

be an infinite ℓ_1-sum of copies of Banach space $\mathbb{C}^2$. So any element $\overline{x} \in \mathcal{X}^2$ can be represented as a sequence $\overline{x} = (x_1, \ldots, x_n, \ldots)$, where $x_n \in \mathbb{C}^2$, with the norm $\|\overline{x}\| = \sum\limits_{k=1}^{\infty} \|x_k\|$.

A polynomial P on the space $\mathcal{X}^2$ is called *block-symmetric* (or *vector-symmetric*) if:

$$P\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots\right) = P\left(x_1, \ldots, x_n, \ldots\right),$$

where $x_i \in \mathbb{C}^2$ for every permutation σ on the set $\mathbb{N}$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X}^2)$ the algebra of block-symmetric polynomials on $\mathcal{X}^2$.

In paper [7] it was shown that the algebraic basis of algebra $\mathcal{P}_{vs}(\mathcal{X}^2)$ is form by polynomials

$$H^{p,n-p}(x,y) = \sum_{i=1}^{\infty} x_i^p y_i^{n-p},$$

where $0 \le p \le n$, $(x_i, y_i) \in \mathbb{C}^2$.

Let us denote by $\mathcal{P}_{vs}^m(\mathcal{X}^2)$ the subalgebra of $\mathcal{P}_{vs}(\mathcal{X}^2)$ which is generated by polynomials

$$H^{1,0,}(x,y), \ldots, H^{p,n-p}(x,y).$$

The number of these elements is equal to m and we denote by τ_{vs}^m the system of generators of algebra $\mathcal{P}_{vs}^m(\mathcal{X}^2)$.

Let $(x,y), (z,t) \in \mathcal{X}^2$,

$$(x,y) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots\right)$$

and

$$(z,t) = \left(\begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots\right)$$

where $(x_i, y_i), (z_i, t_i) \in \mathbb{C}^2$. We put

$$(x,y) \bullet (z,t) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots\right)$$

and define

$$\mathcal{T}_{(z,t)}(f)(x,y) := f((x,y) \bullet (z,t)). \tag{5.1}$$

We will say that $(x,y) \to (x,y) \bullet (z,t)$ is the block symmetric translation and the operator $\mathcal{T}_{(z,t)}$ is the symmetric translation operator. Evidently, we have that

$$H^{k_1,k_2}((x,y) \bullet (z,t)) = H^{k_1,k_2}(x,y) + H^{k_1,k_2}(z,t)$$

for all k_1, k_2.

For some positive number k denote by $\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{k-1,k}$ complex k^{th} roots of the unity, namely $\alpha_{m,k} = e^{2mi\pi/k}$. The following lemma is well known.

Lemma 5.1. *For some positive integer number n*

$$\sum_{m=0}^{k-1} \alpha_{m,k}^n = h \begin{cases} k & \text{if} \quad n = 0 \mod k \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.2. *For any $H^{p,n-p} \in \tau_{vs}^m$ on $\mathcal{X}^2$ and for any $\xi_{p,n-p}$ there exist a vector*

$$(x,y)_{p,n-p} = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \dots, \begin{pmatrix} x_{N_{p,n-p}} \\ y_{N_{p,n-p}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots \right)$$

in $\mathcal{X}^2$ such that $H^{p,n-p} = \xi_{p,n-p}$, $H^{l_1,l_2} = 0$ for all $l_1 \neq p, l_2 \neq n-p$.

Proof. Let us consider two cases:

(1) $p = 0$ or $n = p$;

(2) $p \neq 0, n \neq p$.

1. If $p = 0$ or $n = p$, then the polynomials $H^{0,n}(x,y) = F_p(y)$ and $H^{p,0}(x,y) = F_p(x)$ are symmetric relatively vectors $y = (y_1, \dots, y_n, \dots)$, $x = (x_1, \dots, x_n, \dots)$ respectively. In the paper [1, p. 57] is proof that for symmetric polynomial $F_k(x) = \sum_{i=1}^{\infty} x_i^k$ exist the vector $x_0 = (x_0^1, x_0^2, \dots, x_0^n, \dots) \in \ell_1$ such that $F_k(x_0) = \xi_{k0}$, $F_j(x_0) = 0$. Then for the polynomial $H^{p,0}(x,y)$ there exists vector $(x_0, 0)_{p,0}$ such that $H^{p,0}((x_0,0)_{p,0}) = \xi_{p,0}$ and $H^{l_1,l_2}((x_0,0)_{p,0}) = 0$ for all $l_1 \neq p, l_2 \neq 0$. If we have $p = 0$ then there exists vector $(0, y_0)_{0,n}$ such that $H^{0,n}((0,y_0)_{0,n}) = \xi_{0,n}$ and $H^{l_1,l_2}((0,y_0)_{0,n}) = 0$ for all $l_1 \neq 0, l_2 \neq n$.

2. For the second case we consider polynomials

$$H^{p,k-p}(x,y) = \sum_{i=1}^{\infty} x_i^p y_i^{k-p} \in \tau_{vs}^m$$

of degree k, where $1 \leq p < k$. First we assume that $p \geq k-p$, $p \geq \frac{k}{2}$ and consider the vector

$$(\overline{x}, \overline{y}) = \left(\begin{pmatrix} a\big(\alpha_{0,p(n+1)}\big)^{n+1-(k-p)} \\ b\big(\alpha_{0,p(n+1)}\big)^{p} \end{pmatrix}, \begin{pmatrix} a\big(\alpha_{1,p(n+1)}\big)^{n+1-(k-p)} \\ b\big(\alpha_{1,p(n+1)}\big)^{p} \end{pmatrix}, \dots, \right.$$

$$\left. \begin{pmatrix} a\big(\alpha_{p(n+1)-1,p(n+1)}\big)^{n+1-(k-p)} \\ b\big(\alpha_{p(n+1)-1,p(n+1)}\big)^{p} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots \right),$$

where $\alpha_{i,p(n+1)}$ is the i^{th} roots of complex $p(n+1)$ roots of the unity.

According to Lemma 5.1 we have $H^{p,k-p}(\overline{x}, \overline{y}) = p(n+1)a^p b^{k-p}$. On the system of generating τ_{vs}^m there exists a polynomial which is not equal to zero at $(\overline{x}, \overline{y})$. Let us denote by $H^{p_1,k_1-p_1}, \dots, H^{p_l,k_l-p_l}$, $k_l \leq n$ the polynomials such that

$$p_i(n+1-(k-p)) + p(k_i - p_i) = 0 \mod p(n+1), \qquad i = 1, \dots, l.$$

For this polynomials we have $H^{p_i,k_i-p_i}(\overline{x}, \overline{y}) = p_i(n+1)a^{p_i}b^{k_i-p_i}$, $i = 1, \dots, l$. All other polynomials of the system τ_{vs}^m are equal of zero at $(\overline{x}, \overline{y})$.

We note that for all $i = 1, \dots, l$ $k_i \neq k$. Indeed let $k_1 = k$. In the case $p_1 < p$ we obtain

$$p_1(n+1-(k-p)) + p(k_1 - p_1) = (n+1)p_1 + k(p - p_1) = (n+1-k)p_1 + kp < p(n+1).$$

From this inequality it follows

$$p_1(n+1-(k-p)) + p(k_1-p_1) \neq 0 \mod p(n+1),$$

that contradicts above hypothesis.

In the case $p_1 > p$ we obtain

$$p_1(n+1-(k-p)) + p(k_1-p_1) = (n+1)p_1 + k(p-p_1) = (n+1-k)p_1 + kp < p_1(n+1).$$

From this inequality it follows that for the condition $p_1(n+1) = 0 \mod p(n+1)$ necessary $p_1 = sp, s > 1, s \in \mathbb{N}$.

Since $p > \frac{k}{2}$, then $p_1 > s\frac{k}{2}$. Since $s > 1$ and $s \in \mathbb{N}$, then if $s_{\min} = 2$ we obtain that $p_1 > k$, wich is impossible. Therefore, $k_i \neq k$.

Now we show that $k < k_i$ for all $i = 1, \ldots, l$. Indeed let $i = 1$ $k_1 < k$. For the polynomial H^{p_1,k_1-p_1} we have $p_1(n+1-(k-p)) + p(k_1-p_1) = 0 \mod p(n+1)$. From inequality $k_1 < k$ it follows that

$$p_1(n+1-(k-p)) + p(k_1-p_1) = p_1(n+1-k) + pk. \tag{5.2}$$

If $p_1 < p$ we obtain:

$$p_1(n+1-k) + pk < p(n+1).$$

Therefore $p_1(n+1-(k-p)) + p(k_1-p_1) \neq 0 \mod p(n+1)$.

If $p_1 \geq p$, then

$$p_1(n+1-k) + pk \leq p_1(n+1).$$

In order to last expression of inequality will evenly divided on $p(n+1)$ necessary that $p_1 = sp$. Since $p > \frac{k}{2}$, then $p > \frac{k_1}{2}$, then $p_1 > s\frac{k_1}{2}$. If $s = 1$ we obtain that $p_1(n+1-(k-p)) + p(k_1 - p_1) = p(n+1-(k-p)) + p(k_1-p) = p(n+1) - p(k-k_1) < p(n+1)$. Therefore on this case $p_1(n+1-(k-p)) + p(k_1-p_1) \neq 0 \mod p(n+1)$. If $s \geq 2$ we obtain $p_1 > k_1$, wich is impossible. Therefore $k < k_i$ for all $i = 1, \ldots, l$.

We will show that $p_i = sp$, $k_i = sk$ for all $i = 1, \ldots, l$. Indeed from

$$p(n+1-(k-p)) + p(k-p) = 0 \mod p(n+1)$$

it follows that

$$mp(n+1-(k-p)) + pm(k-p) = 0 \mod p(n+1),$$

where $m > 1$ (the case $m < 1$ is impossible because $mk < k$). Therefore we obtain the polynomials $H^{mp,m(k-p)}$, which will be among the polynomials $H^{p_1,k_1-p_1}, \ldots, H^{p_l,k_l-p_l}$. We suppose that there exist polynomials $H^{p+s_1,k-p+s_2}$, where $s_1 < p$, $s_2 < k-p$.

Then

$$\begin{aligned}(p+s_1)(n+1-(k-p)) + p(k-p+s_2) &= p(n+1-(k-p)) + p(k-p) \\ &+ s_1(n+1-(k-p))ps_2.\end{aligned}$$

Since $p(n+1-(k-p)) + p(k-p) = 0 \mod p(n+1)$, then should performed the codition

$$s_1(n+1-(k-p)) + ps_2 = 0 \mod p(n+1),$$

wich is impossible because $s_1(n+1-(k-p)) + ps_2 < p(n+1-(k-p)) + p(k-p) = p(n+1)$. Therefore all polynomials $H^{p_1,k_1-p_1}, \ldots, H^{p_l,k_l-p_l}$ are of the form $H^{mp,m(k-p)}$, $m = 2, \ldots, w$ where $wk < n+1$. Therefore the polynomials $H^{p_1,k_1-p_1}, \ldots, H^{p_l,k_l-p_l}$ we can mark as

$$H^{p_1,k_1-q_1} = H^{2p,2(k-p)}, H^{p_2,k_2-p_2} = H^{3p,3(k-p)}, \ldots, H^{p_l,k_l-p_l} = H^{(l+1)p,(l+1)(k-p)},$$

where $(l+1)k < n+1$.

Next we concider the vector

$$(\overline{\overline{x}},\overline{\overline{y}})=\left(\begin{pmatrix} a\sqrt[2k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix},\ldots,\begin{pmatrix} a\sqrt[2k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix},\ldots,\right.$$

$$\begin{pmatrix} a\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix},\ldots,\begin{pmatrix} a\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix},$$

$$\begin{pmatrix} a\sqrt[2k]{-1}\sqrt[3k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\sqrt[3k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix},\ldots,\begin{pmatrix} a\sqrt[2k]{-1}\sqrt[3k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\sqrt[3k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix},\ldots,$$

$$\begin{pmatrix} a\sqrt[(i+1)k]{-1}\sqrt[(j+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[(i+1)k]{-1}\sqrt[(j+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix},\ldots,$$

$$\begin{pmatrix} a\sqrt[(i+1)k]{-1}\sqrt[(j+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[(i+1)k]{-1}\sqrt[k(j+1)]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix},\ldots,$$

$$\begin{pmatrix} a\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix},\ldots,$$

$$\begin{pmatrix} a\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix},\ldots,$$

$$\begin{pmatrix} a\sqrt[i_2k]{-1}\ldots\sqrt[i_lk]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[i_1k]{-1}\ldots\sqrt[i_{l-1}k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix},\ldots,$$

$$\begin{pmatrix} a\sqrt[i_1k]{-1}\ldots\sqrt[i_{l-1}k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[i_1k]{-1}\ldots\sqrt[i_{l-1}k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix},\ldots,$$

$$\begin{pmatrix} a\sqrt[2k]{-1}\ldots\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\ldots\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix},\ldots,$$

$$\left.\begin{pmatrix} a\sqrt[2k]{-1}\ldots\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\ldots\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix},\ldots\right).$$

Then we obtain that

$$\begin{aligned} H^{ip,i(k-p)}((\overline{x},\overline{y})\bullet(\overline{\overline{x}},\overline{\overline{y}})) &= a^{ip}b^{i(k-p)}\Bigg(p(n+1)-p(n+1)+p(n+1)\sum_{\substack{j=1\\ j\neq i}}^{l}\left(\sqrt[jk]{-1}\right)^{ik} \\ &- p(n+1)\sum_{\substack{j=1\\ j\neq i}}^{l}\left(\sqrt[jk]{-1}\right)^{ik}+\dots \\ &+ p(n+1)\sum_{\substack{j_1<\dots<j_{l-1}\\ j_m\neq i}}^{l}\left(\sqrt[j_1k]{-1}\dots\sqrt[j_{l-1}k]{-1}\right)^{ik} \\ &- p(n+1)\sum_{\substack{j_1<\dots<j_{l-1}\\ j_m\neq i}}^{l}\left(\sqrt[j_1k]{-1}\dots\sqrt[j_{l-1}k]{-1}\right)^{ik}\Bigg)=0. \end{aligned}$$

For $H^{p,k-p}$ we obtain

$$\begin{aligned} H^{p,k-p}((\overline{x},\overline{y})\bullet(\overline{\overline{x}},\overline{\overline{y}})) &= p(n+1)a^{p}b^{k-p}\Bigg(1+\sum_{j=1}^{l}\sqrt[j+1]{-1}+\dots \\ &+ \sum_{j_1<\dots<j_{l-1}}^{l}\sqrt[j_1]{-1}\dots\sqrt[j_{l-1}]{-1}+\sqrt[2]{-1}\dots\sqrt[l]{-1}\sqrt[l+1]{-1}\Bigg). \end{aligned} \tag{5.3}$$

We denote by M the next condition

$$M=1+\sum_{j=1}^{l}\sqrt[j+1]{-1}+\dots+\sum_{j_1<\dots<j_{l-1}}^{l}\sqrt[j_1]{-1}\dots\sqrt[j_{l-1}]{-1}+\sqrt[2]{-1}\dots\sqrt[l]{-1}\sqrt[l+1]{-1}.$$

If we choice $(j+1)^{th}$ a complex root of -1, $j=1,\dots ł$ such that $M\neq 0$ to zero, we obtain $H^{p,k-p}((\overline{x},\overline{y})\bullet(\overline{\overline{x}},\overline{\overline{y}}))\neq 0$.

If we substitute to (5.3)

$$a=\frac{1}{\sqrt[p]{(k-p)(n+1)M}}\sqrt[p]{\xi_{p,k-p}}, \qquad b=1$$

we obtain

$$H^{p,k-p}((\overline{x},\overline{y})\bullet(\overline{\overline{x}},\overline{\overline{y}}))=H^{p,k-p}((x,y)_{p,k-p})=\xi_{p,k-p}.$$

In the case $p<k-p$ we consider the vector

$$\begin{aligned} (\overline{x},\overline{y}) = \Bigg(&\begin{pmatrix} a\left(\alpha_{0,(k-p)(n+1)}\right)^{k-p} \\ b\left(\alpha_{0,(k-p)(n+1)}\right)^{n+1-p}\end{pmatrix},\begin{pmatrix} a\left(\alpha_{1,(k-p)(n+1)}\right)^{k-p} \\ b\left(\alpha_{1,(k-p)(n+1)}\right)^{n+1-p}\end{pmatrix},\dots, \\ &\begin{pmatrix} a\left(\alpha_{(k-p)(n+1)-1,(k-p)(n+1)}\right)^{k-p} \\ b\left(\alpha_{(k-p)(n+1)-1,(k-p)(n+1)}\right)^{n+1-p}\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\dots\Bigg), \end{aligned}$$

where $\alpha_{i,(k-p)(n+1)}$ is i^{th} root of $(k-p)(n+1)$ complex root of the unity. For this case the proof is the same like in the case $p\geq k-p$. $\square$

Corollary 5.3. *Let* $\tau_{vs}^m = \{H^{\widetilde{p},j-\widetilde{p}}(x,y), 0 \leq \widetilde{p} \leq j, j = 1, \ldots, n\}$, $j \leq m$. *Then for each* $\xi = (\xi_{1,0}, \ldots, \xi_{p,k-p}, \ldots, \xi_{p',k'-p'}) \in \mathbb{C}^m$ *there is* $(x,y)_{p,k-p} \in \mathcal{X}^2$ *such that* $H^{p,k-p}((x,y)_{pq}) = \xi_{p,k-p}$.

Proposition 5.4. *Let* $P_1, \ldots, P_m \in \mathcal{P}_{vs}(\mathcal{X}^2)$ *such that* $\ker P_1 \cap \ldots \cap \ker P_m = \varnothing$. *Then there are* $Q_1, \ldots, Q_m \in \mathcal{P}_{vs}(\mathcal{X}^2)$ *such that*

$$\sum_{i=1}^m P_i Q_i = 1.$$

Proof. For the proof we use the same method as in [1, p. 58]. Let $n = \max_i(\deg P_i)$. We may assume that $P_i(x,y) = q_i(H^{1,0}, \ldots, H^{l_1,k-l_1})$ for some $q_i \in \mathcal{P}(\mathbb{C}^n)$, where $0 \leq l_1 \leq k$, n is number of polynomials $H^{l_1,k-l_1}$. Let us suppose that at some point $\xi \in \mathbb{C}^n$, $\xi = (\xi_{1,0}, \ldots, \xi_{p,k-p}, \ldots, \xi_{p',k'-p'})$ and $g_i(\xi) = 0$. Then by Corollary 5.3 there is $(x,y)_{p,k-p} \in \mathcal{X}^2$ such that $H^{p,k-p}((x,y)_{p,k-p}) = \xi_{p,k-p}$. So the common set of zeros of all q_i is empty. Thus by the Hilbert Nullstellensatz there are polynomials $g_1, \ldots, g_m$ such that $\sum_i q_i g_i \equiv 1$. Put $Q_i(x,y) = g_i(H^{1,0}, \ldots, H^{l_1,k-l_1})$. □

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У роботі доведено теореми Гільберта про нулі для поліномів на нескінченно вимірному комплексному просторі, для симетричних та блочно-симетричних поліномів.

Ключові слова: поліноми, симетричні поліноми, блочно-симетричні поліноми, алгебра поліномів, теорема Гільберта про нулі, алгебраїчний базис.