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SYMMETRIC POLYNOMIALS AND HOLOMORPHIC FUNCTIONS ON INFINITE DIMENSIONAL SPACES

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Abstract. A survey of general results about spectra of uniform algebras of symmetric holomorphic functions and algebras of symmetric analytic functions of bounded type on Banach spaces is given.

Keywords: polynomials and analytic functions on Banach spaces, symmetric polynomials, spectra of algebras.

1. SYMMETRIC POLYNOMIALS ON REARRANGEMENT-INVARIANT FUNCTION SPACES

Let X, Y be Banach spaces over the field $\mathbb{K}$ of real or complex numbers. A mapping $P : X \to Y$ is called an *n-homogeneous polynomial* if there exists a symmetric n-linear mapping $A : X^n \to Y$ such that for all $x \in X$ $P(x) = A(x, \ldots, x)$.

A *polynomial of degree n* on X is a finite sum of k-homogeneous polynomials, $k = 0, \ldots, n$. Let us denote by $\mathcal{P}(^n X, Y)$ the space of all n-homogeneous continuous polynomials $P : X \to Y$ and by $\mathcal{P}(X, Y)$ the space of all continuous polynomials.

It is well known ([13], XI §52) that for $n < \infty$ any symmetric polynomial on $\mathbb{C}^n$ is uniquely representable as a polynomial in the elementary symmetric polynomials $(G_i)_{i=1}^n$, $G_i(x) = \sum_{k_1 < \cdots < k_i} x_{k_1} \ldots x_{k_i}$.

Symmetric polynomials on ℓ_p and $L_p[0,1]$ for $1 \le p < \infty$ were first studied by Nemirovski and Semenov in [16]. In [11] González, Gonzalo and Jaramillo investigated algebraic bases of various algebras of symmetric polynomials on so called rearrangement-invariant function spaces, that is spaces with some symmetric structure. Up to some inessential normalisation, the study of rearrangement-invariant function spaces is reduced to the study of the following three cases:

1. $I = \mathbb{N}$ and the mass of every point is one;
2. $I = [0, 1]$ with the usual Lebesgue measure;

3. $I = [0, \infty)$ with the usual Lebesgue measure.

We shall say that σ is an *automorphism* of I, if it ia a bijection of I, so that both σ and σ^{-1} are measurable and both preserve measure. We denote by $\mathcal{G}(I)$ the group of all automorphisms of I. If $X(I)$ is a rearrangement-invariant function space on I and $f \in X(I)$, then f is a real-valued measurable function on I and $f \circ \sigma \in X(I)$ for all $\sigma \in X(I)$. Also, there is an equivalent norm on $X(I)$ verifying that

$$\|f \circ \sigma\| = \|f\|$$

for all $\sigma \in \mathcal{G}(I)$ and all $f \in X(I)$. We always consider $X(I)$ endowed with this norm.

Following [16], we say that a polynomial P on $X(I)$ is *symmetric* if

$$P(f \circ \sigma) = P(f)$$

for all $\sigma \in \mathcal{G}(I)$ and all $f \in X(I)$.

In the same way, if $\mathcal{G}_0$ is a subgroup of $\mathcal{G}(I)$, a polynomial is said to be $\mathcal{G}_0$-*invariant* if $P(f) = P(f \circ \sigma)$ for all $\sigma \in \mathcal{G}_0$ and all $f \in X(I)$.

Let $X(I)$ be a rearrangement-invariant function space on I and consider the set

$$\mathcal{J}(X) = \{r \in \mathbb{N} : X(I) \subset L_r(I)\}.$$

Note that if $\mathcal{J}(X) \neq$ we can consider, for each $r \in \mathcal{J}(X)$, the polynomials

$$P_r(f) = \int_I f^r.$$

These are well-defined symmetric polynomials on $X(I)$ and we will call them the *elementary symmetric polynomials* on $X(I)$.

1.1. Symmetric Polynomials on Spaces with a Symmetric Basis

Let $X = X(\mathbb{N})$ be a Banach space with a symmetric basis $\{e_n\}$. A polynomial P on X is symmetric if for every permutation $\sigma \in \mathcal{G}(\mathbb{N})$

$$P\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = P\Big(\sum_{i=1}^{\infty} a_i e_{\sigma(i)}\Big).$$

We consider the finite group $\mathcal{G}_n(\mathbb{N})$ of permutations of $\{1, \dots, n\}$ and the σ-finite group $\mathcal{G}_0(\mathbb{N}) = \cup_n \mathcal{G}_n(\mathbb{N})$ as subgroups of $\mathcal{G}(\mathbb{N})$. By continuity, a polynomial is symmetric if and only if it is $\mathcal{G}_0(\mathbb{N})$-invariant. Indeed, if P is $\mathcal{G}_0(\mathbb{N})$-invariant and $\sigma \in \mathcal{G}(\mathbb{N})$,

$$P\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = \lim_{n\to\infty} P\Big(\sum_{i=1}^{n} a_i e_i\Big) = \lim_{n\to\infty} P\Big(\sum_{i=1}^{n} a_i e_{\sigma(i)}\Big) = P\Big(\sum_{i=1}^{\infty} a_i e_{\sigma(i)}\Big).$$

Recall that a sequence $\{x_n\}$ is said to have a *lower p-estimate* for some $p \geq 1$, if there is a constant $C > 0$ such that

$$C\Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p} \leq \Big\|\sum_{i=1}^{n} a_i x_i\Big\|$$

for all $a_1, \dots, a_n \in \mathbb{R}$.

Note that $X \subset \ell_r$ if and only if the basis has a lower r-estimate, and therefore we have in this case

$$\mathcal{J}(X) = \{r \in \mathbb{N} : \{e_n\} \quad \text{has a lower r-estimate}\}.$$

Now we define

$$n_0(X) = \inf \mathcal{J}(X),$$

where we understand that the infimum of the empty set is ∞. The elementary symmetric polynomials are then

$$P_r\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = \sum_{i=1}^{\infty} a_i^r,$$

where $r \geq n_0(X)$.

Theorem 1.1. [11] *Let X be a Banach space with a symmetric basis e_n, let P be a symmetric polynomial on X and consider $k = \deg P$ and $N = n_0(X)$.*
1. If $k < N$, then $P = 0$.
2. If $k \geq N$, then there exists a real polynomial q of several real variables such that

$$P\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = q\Big(\sum_{i=1}^{\infty} a_i^N, \ldots, \sum_{i=1}^{\infty} a_i^k\Big)$$

for every $\sum_{i=1}^{\infty} a_i e_i \in X$.

1.2. Symmetric Polynomials on $X[0,1]$ and $X[0,\infty)$

Let $X[0,1]$ be a separable rearrangement-invariant function space on $[0,1]$. Note that the set $\mathcal{J}(X)$ is never empty since we always have $X[0,1] \subset L_1[0,1]$.
We define

$$n_\infty(X) = \sup\{r \in \mathbb{N} : X[0,1] \subset L_r[0,1]\}.$$

Therefore the elementary symmetric polynomials on $X[0,1]$ are

$$P_r(f) = \int_0^1 f^r$$

for each integer $r \leq n_\infty(X)$.

Theorem 1.2. [11] *Let $X[0,1]$ be a separable rearrangement-invariant function space on $[0,1]$ and consider the index $n_\infty(X)$ as above. Let P be a $\mathcal{G}_0[0,1]$-invariant polynomial on $X[0,1]$ and let $k = \deg P$. Then there exists a real polynomial q in several real variables such that*

$$P(f) = q\Big(\int_0^1 f, \ldots, \int_0^1 f^m\Big)$$

for all $f \in X$, where $m = \min\{n_\infty(X), k\}$.

Theorem 1.3. [11] *Let $X[0,\infty)$ be a separable rearrangement-invariant function space, let P be a $\mathcal{G}_0$-invariant polynomial on $X[0,\infty)$ and consider $k = \deg P$. Let n_0 and n_∞ be defined as above.*
1. If either $n_0 > n_\infty$, or $k < n_0 \leq \infty$, then $P = 0$.
2. If $n_0 \leq n_\infty$ and $n_0 \leq k$, then there is a real polynomial q in several real variables such that

$$P(f) = q\Big(\int_0^\infty f^{n_0}, \ldots, \int_0^\infty f^m\Big),$$

where $m = \min\{n_\infty, k\}$.

2. Uniform Algebras of Symmetric Holomorphic Functions

Let X be a Banach sequence space with a symmetric norm, that is, for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$, and $x = (x_n) \in B$ also $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots) \in B$, where B is an open unit ball.

A holomorphic function $f : B \to \mathbb{C}$ is called symmetric if for all $x \in B$ and all permutations $\sigma : \mathbb{N} \to \mathbb{N}$, the following holds:

$$f(x_1, \ldots, x_n, \ldots) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots).$$

Our interest throughout this section will be in the set

$$\mathcal{A}_{us}(B) = \{f : B \to \mathbb{C} | f \text{ is holomorphic, uniformly continuous, and symmetric on } B\}.$$

The following result is straightforward.

Proposition 2.1. [4] *$\mathcal{A}_{us}(B)$ is a unital commutative Banach algebra under the supremum norm. Each function $f \in \mathcal{A}_{us}(B)$ admits a unique (automatically symmetric) extension to $\overline{B}$.*

Let us give some examples of $\mathcal{A}_{us}(B)$ when B is the open unit ball of some classical Banach spaces X.

Example 2.2. $X = c_0$.

Theorem 2.3. [5] *Let $P : c_0 \to \mathbb{C}$ be an n-homogeneous polynomial and $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ and an n-homogeneous polynomial $Q : \mathbb{C}^N \to \mathbb{C}$ such that for all $x = (x_1, \ldots, x_N, x_{N+1}, \ldots) \in B$, $|P(x) - Q(x_1, \ldots, x_N)| < \varepsilon$.*

Corollary 2.4. [4] *For all $n \in \mathbb{N}$, $n \geq 1$, the only n-homogeneous symmetric polynomial $P : c_0 \to \mathbb{C}$ is $P = 0$.*

Since any function $f \in \mathcal{A}_{us}(B)$ can be uniformly approximated on B by finite sums of symmetric homogeneous polynomials, it follows that $\mathcal{A}_{us}(B)$ consists of just the constant functions when B is the open unit ball of c_0.

Example 2.5. [4] $X = \ell_p$ for some p, $1 \leq p < \infty$.

The linear ($n = 1$) case. Let $\varphi \in \ell_p^*$ be a symmetric 1-homogeneous polynomial on ℓ_p; that is, φ is a symmetric continuous linear form. Since φ can be regarded as a point $(y_1, \ldots, y_m, \ldots) \in \ell_p^*$ and since $y_j = \varphi(e_1)$ for all j, we see that $y_1 = \ldots = y_m = \ldots$. Therefore, the set of symmetric linear forms φ on ℓ_1 consists of the 1-dimensional space $\{b(1, \ldots, 1, \ldots) | b \in \mathbb{C}\}$. For $p > 1$, the above shows that there are no non-trivial symmetric linear forms on ℓ_p.

The quadratic ($n = 2$) case. Let $P : \ell_p \to \mathbb{C}$ be a symmetric 2-homogeneous polynomial, and let $A : \ell_p \times \ell_p \to \mathbb{C}$ be the unique symmetric bilinear form associated to P, using the polarization formula and $P(x) = A(x, x)$ for all $x \in \ell_p$. Now, $P(e_1) = P(e_j)$ for all $j \in \mathbb{N}$. Moreover,

$$\begin{aligned} P(e_1 + e_2) &= A(e_1 + e_2, e_1 + e_2) = A(e_1, e_1) + 2A(e_1, e_2) + A(e_2, e_2) \\ &= P(e_1) + 2A(e_1, e_2) + P(e_2) \end{aligned}$$

and likewise

$$P(e_j + e_k) = P(e_j) + 2A(e_j, e_k) + P(e_k),$$

for all j and $k \in \mathbb{N}$. Therefore $A(e_j, e_k) = A(e_1, e_2)$.

So, for all N,

$$P(x_1, \ldots, x_N, 0, 0, \ldots) = a \sum_{j=1}^{N} x_j^2 + b \sum_{j \neq k} x_j x_k,$$

where $a = P(e_1)$ and $b = A(e_j, e_k)$.

From this, we can conclude, that for $X = \ell_1$, the space of symmetric 2-homogeneous polynomials on ℓ_1, $\mathcal{P}_s(^2\ell_1)$, is 2-dimensional with basis $\{\sum_j x_j^2, \sum_{j\neq k} x_j x_k\}$. On the other hand, the corresponding space $\mathcal{P}_s(^2\ell_2)$ of symmetric 2-homogeneous polynomials on ℓ_2, is 1-dimensional with basis $\{\sum_j x_j^2\}$. For $1 < p < 2$, $\mathcal{P}_s(^2\ell_p)$ is also the one-dimensional space generated by $\sum_j x_j^2$, while $\mathcal{P}_s(^2\ell_p) = \{0\}$ for $p > 2$.

This argument can be extended to all n and all p, and we can conclude that for all n, p, the space of symmetric n-homogeneous polynomials on ℓ_p, $\mathcal{P}_s(^n\ell_p)$, is finite dimensional. Consequently, since for all $f \in \mathcal{A}_{us}(B)$, f is a uniform limit of symmetric n-homogeneous polynomials, we have reasonably good knowledge about the functions in $\mathcal{A}_{us}(B)$. So we can say that $\mathcal{A}_{us}(B)$, for B the open unit ball of an ℓ_p space, is a "small" algebra.

2.1. THE SPECTRUM OF $\mathcal{A}_{us}(B)$

Recall that the *spectrum* (or *maximal ideal space*) of a Banach algebra $\mathcal{A}$ with identity e is the set $\mathcal{M}(\mathcal{A}) = \{\varphi : \mathcal{A} \to \mathbb{C} \quad | \varphi \quad \text{is a homomorphism and} \quad \varphi(e) = 1\}$. We recall that if $\varphi \in \mathcal{M}(\mathcal{A})$, then φ is automatically continuous with $\|\varphi\| = 1$. Moreover, when we consider it as a subset of $\mathcal{A}^*$ with the weak-star topology, $\mathcal{M}(\mathcal{A})$ is compact.

We will examine $\mathcal{M}(\mathcal{A}_{us}(B))$ when $B = B_{\ell_p}$. The most obvious element in $\mathcal{M}(\mathcal{A}_{us}(B))$ is the evaluation homomorphism δ_x at a point x of $\overline{B}$ (recalling that since the functions in $\mathcal{A}_{us}(B)$ are uniformly continuous, they have unique continuous extensions to $\overline{B}$). Of course, if $x, y \in B$ are such that y can be obtained from x by a permutation of its coordinates, then $\delta_x = \delta_y$. It is natural to wonder whether $\mathcal{M}(\mathcal{A}_{us}(B))$ consists of only the set of equivalence classes $\{\delta_{\tilde{x}} | x \in \overline{B}\}$, where $x \sim y$ means that x and y differ by a permutation.

Example 2.6. [1, 4]

For every $n \in \mathbb{N}$ define $F_n : B \to \mathbb{C}$ by $F_n(x) = \sum_{j=1}^{\infty} x_j^n$. To simplify, we take $B = B_{\ell_2}$ (so that F_n will be defined only for $n \geq 2$). It is known that the algebra generated by $\{F_n | n \geq 2\}$ is dense in $\mathcal{A}_{us}(B)$. For each $k \in \mathbb{N}$, let

$$v_k = \frac{1}{\sqrt{k}}(e_1 + ... + e_k).$$

It is routine that each v_k has norm 1, that $\delta_{v_k}(F_2) = 1$ for all $k \in \mathbb{N}$, and that for all $n \geq 3$,

$$\delta_{v_k}(F_n) = F_n(v_k) = \frac{1}{(\sqrt{k})^n} k \to 0 \text{ as } k \to \infty.$$

Since $\mathcal{M}(\mathcal{A}_{us}(B))$ is compact, the set $\{\delta_{v_k} | k \in \mathbb{N}\}$ has an accumulation point $\varphi \in \mathcal{M}(\mathcal{A}_{us}(B))$. It is clear that $\varphi(F_2) = 1$ and $\varphi(F_n) = 0$ for all $n \geq 3$. It is not difficult to verify that $\varphi \neq \delta_x$ for every $x \in \overline{B}$. This construction could be altered slightly, by letting $v_k = \frac{1}{\sqrt{k}}(\alpha_1 e_1 + ... + \alpha_k e_k)$, where each $|\alpha_j| \leq 1$. Thus, with this method we give a small number of additional homomorphisms in $\mathcal{M}(\mathcal{A}_{us}(B))$ that do not correspond to point evaluations.

It should be mentioned that it is not known whether $\mathcal{M}(\mathcal{A}_{us}(B_{\ell_p}))$ contains other points. However, in [1] was given a different characterization of $\mathcal{M}(\mathcal{A}_{us}(B_{\ell_p}))$. In order to do this, we first simplify our notation by considering only B_{ℓ_1}. For each $n \in \mathbb{N}$, define $\mathcal{F}^n : B_{\ell_1} \to \mathbb{C}^n$ as follows:

$$\mathcal{F}^n(x) = (F_1(x), \dots, F_n(x)) = \Big(\sum_j x_j, \dots, \sum_j x_j^n\Big).$$

Let $D_n = \mathcal{F}^n(B_{\ell_1})$, and let $[D_n]$ be the polynomially convex hull of D_n (see, e.g., [12]). Let

$$\Sigma_1 = \{(b_i)_{i=1}^\infty \in \ell_\infty : (b_i)_{i=1}^n \in [D_n], \text{ for all } n \in \mathbb{N}.\}$$

In other words, Σ_1 is the inverse limit of the sets $[D_n]$, endowed with the natural inverse limit topology.

Theorem 2.7. [1, 4] *Σ_1 is homeomorphic to $\mathcal{M}(\mathcal{A}_{us}(B_{\ell_1}))$.*

The analogous results, and the analogous definitions, are valid for Σ_p and $\mathcal{M}(\mathcal{A}_{us}(B_{\ell_p}))$.

The basic steps in the proof of Theorem 2.7 are as follows: First, since the algebra generated by $\{F_n | n \geq 1\}$ is dense in $\mathcal{A}_{us}(B_{\ell_1})$, each homomorphism $\varphi \in \mathcal{M}(\mathcal{A}_{us}(B_{\ell_1}))$ is determined by its behavior on $\{F_n\}$. Next, every symmetric polynomial P on ℓ_1 can be written as $P = Q \circ \mathcal{F}^n$ for some $n \in \mathbb{N}$ and some polynomial $Q : \mathbb{C}^n \to \mathbb{C}$. Finally, to each $(b_i) \in \Sigma_1$, one associates $\varphi = \varphi_{(b_i)} : \mathcal{A}_{us}(B_{\ell_1}) \to \mathbb{C}$ by $\varphi(P) = Q(b_1, \dots, b_n)$. This turns out to be a well-defined homomorphism, and the mapping $(b_i) \in \Sigma_1 \rightsquigarrow \varphi_{(b_i)} \in \mathcal{M}(\mathcal{A}_{us}(B_{\ell_1}))$ is a homeomorphism.

2.2. The Spectrum of $\mathcal{A}_{us}(B)$ in the Finite Dimensional Case

Let us turn to $\mathcal{A}_{us}(B)$, where B is the open unit ball of $\mathbb{C}^n$, endowed with a symmetric norm. Because of finite dimensionality, $\mathcal{A}_{us}(B) = \mathcal{A}_s(B)$, where $\mathcal{A}_s(B)$ is the Banach algebra of symmetric holomorphic functions on B that are continuous on $\overline{B}$.

Unlike the infinite dimensional case, the following result holds.

Theorem 2.8. [1, 4] *Every homomorphism $\varphi : \mathcal{A}_s(B) \to \mathbb{C}$ is an evaluation at some point of $\overline{B}$.*

We describe below the main ideas in the proof of this result.

Proposition 2.9. [1, 4] *Let $C \subset \mathbb{C}^n$ be a compact set. Then C is symmetric and polynomially convex if and only if C is polynomially convex with respect to only the symmetric polynomials.*

In other words, C is symmetric and polynomially convex if and only if

$$C = \{z_0 \in \mathbb{C}^n : |P(z_0)| \leq \sup_{z \in C} |P(z)|, \text{ for all symmetric polynomials } P\}.$$

For $i \in \mathbb{N}$, let

$$R_i(x) = \sum_{1 \leq k_1 \leq \ldots \leq k_i \leq n} x_{k_1} \cdots x_{k_j}.$$

Proposition 2.10. [1, 4] *Let B be the open unit ball of a symmetric norm on $\mathbb{C}^n$. Then the algebra generated by the symmetric polynomials $R_1, \dots, R_n$ is dense in $\mathcal{A}_s(B)$.*

Lemma 2.11. *(Nullstellensatz for symmetric polynomials)*[1, 4] *Let $P_1, \dots, P_m$ be symmetric polynomials on $\mathbb{C}^n$ such that*

$$\ker P_1 \cap \cdots \cap \ker P_m = \emptyset.$$

Then there are symmetric polynomials $Q_1, \dots, Q_m$ on $\mathbb{C}^n$ such that

$$\sum_{j=1}^m P_j Q_j \equiv 1.$$

To prove Theorem 2.8, let us consider the symmetric polynomials $P_1 = R_1 - \varphi(R_1), \dots, P_m = R_m - \varphi(R_m)$. If $\ker P_1 \cap \cdots \cap \ker P_m = \emptyset$, then Lemma 2.11 implies that there are symmetric polynomials $Q_1, \dots, Q_m$ on $\mathbb{C}^n$ such that $\sum_{j=1}^m P_j Q_j \equiv 1$. This is impossible, since $\varphi(P_j Q_j) = 0$. Therefore, there exists some $x \in \mathbb{C}^n$ such that $P_j(x) = 0$ for all j, which means $\varphi(R_j) = R_j(x)$ for all j. By Proposition 2.10, $\varphi(P) = P(x)$, for all symmetric polynomials $P : \mathbb{C}^n \to \mathbb{C}$.

So, for all such P, $|\varphi(P)| = |P(x)| \leq \|P\|$. This means that x belongs to the symmetrical polynomial convex hull of $\overline{B}$. Since $\overline{B}$ is symmetric and convex, it is symmetrically polynomially convex (by Proposition 2.9). Thus $x \in \overline{B}$. □

3. The Algebra of Symmetric Analytic Functions on ℓ_p

Let us denote by $\mathcal{H}_{bs}(\ell_p)$ the algebra of all symmetric analytic functions on ℓ_p that are bounded on bounded sets endowed with the topology of the uniform convergence on bounded sets and by $\mathcal{M}_{bs}(\ell_p)$ the spectrum of $\mathcal{H}_{bs}(\ell_p)$, that is, the set of all non-zero continuous complex-valued homomorphisms.

3.1. The Radius Function on $\mathcal{M}_{bs}(\ell_p)$

Following [3] we define the *radius function* R on $\mathcal{M}_{bs}(\ell_p)$ by assigning to any complex homomorphism $\phi \in \mathcal{M}_{bs}(\ell_p)$ the infimum $R(\phi)$ of all r such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB_{ℓ_p}, that is $|\phi(f)| \leq C_r \|f\|_r$. Further, we have $|\phi(f)| \leq \|f\|_{R(\phi)}$.

As in the non symmetric case, we obtain the following formula for the radius function

Proposition 3.1. [6] *Let $\phi \in \mathcal{M}_{bs}(\ell_p)$ then*

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n}, \tag{3.1}$$

where ϕ_n is the restriction of ϕ to $\mathcal{P}_s(^n\ell_p)$ and $\|\phi_n\|$ is its corresponding norm.

Proof. To prove (3.1) we use arguments from [3, 2.3. Theorem]. Recall that

$$\|\phi_n\| = \sup\{|\phi_n(P)| : P \in \mathcal{P}_s(^n\ell_p) \text{ with } \|P\| \leq 1\}.$$

Suppose that

$$0 < t < \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Then there is a sequence of homogeneous symmetric polynomials P_j of degree $n_j \to \infty$ such that $\|P_j\| = 1$ and $|\phi(P_j)| > t^{n_j}$. If $0 < r < t$, then by homogeneity,

$$\|P_j\|_r = \sup_{x \in rB_{\ell_p}} |P_j(x)| = r^{n_j},$$

so that

$$|\phi(P_j)| > (t/r)^{n_j} \|P_j\|_r,$$

and ϕ is not continuous for the $\|\ \|_r$ norm. It follows that $R(\phi) \geq r$, and on account of the arbitrary choice of r we obtain

$$R(\phi) \geq \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Let now be $s > \limsup_{n\to\infty} \|\phi_n\|^{1/n}$ so that $s^m \geq \|\phi_m\|$ for m large. Then there is $c \geq 1$ such that $\|\phi_m\| \leq cs^m$ for every m. If $r > s$ is arbitrary and $f \in \mathcal{H}_{bs}(\ell_p)$ has Taylor series expansion $f = \sum_{n=1}^{\infty} f_n$, then

$$r^m\|f_m\| = \|f_m\|_r \leq \|f\|_r, \quad m \geq 0.$$

Hence

$$|\phi(f_m)| \leq \|\phi_m\|\|f_m\| \leq \frac{cs^m}{r^m}\|f\|_r$$

and so

$$\|\phi(f)\| \leq c\Big(\sum \frac{s^m}{r^m}\Big)\|f\|_r.$$

Thus ϕ is continuous with respect to the uniform norm on rB, and $R(\phi) \leq r$. Since r and s are arbitrary,

$$R(\phi) \leq \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

□

3.2. An Algebra of Symmetric Functions on the Polydisk of ℓ_1

Let us denote

$$\mathbb{D} = \Big\{x = \sum_{i=1}^{\infty} x_i e_i \in \ell_1 \colon \ \sup_i |x_i| < 1\Big\}.$$

It is easy to see that $\mathbb{D}$ is an open unbounded set. We shall call $\mathbb{D}$ the polydisk in ℓ_1.

Lemma 3.2. [6] *For every $x \in \mathbb{D}$ the sequence $\mathcal{F}(x) = (F_k(x))_{k=1}^{\infty}$ belongs to ℓ_1.*

Proof. Let us firstly consider $x \in \ell_1$, such that $\|x\| = \sum_{i=1}^{\infty} |x_i| < 1$ and let us calculate $\mathcal{F}(x) = (F_k(x))_{k=1}^{\infty}$. We have

$$\begin{aligned} \|\mathcal{F}(x)\| &= \sum_{k=1}^{\infty} |F_k(x)| = \sum_{k=1}^{\infty} \Big|\sum_{i=1}^{\infty} x_i^k\Big| \leq \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} |x_i|^k \\ &\leq \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{\infty} |x_i|\Big)^k = \sum_{k=1}^{\infty} \|x\|^k = \frac{\|x\|}{1-\|x\|} < \infty. \end{aligned}$$

In particular, $\|\mathcal{F}(\lambda e_k)\| = \frac{|\lambda|}{1-|\lambda|}$ for $|\lambda| < 1$.

If x is an arbitrary element in $\mathbb{D}$, pick $m \in \mathbb{N}$ so that $\sum_{i=m+1}^{\infty} |x_i| < 1$. Put $u = x - (x_1, \ldots, x_m, 0 \ldots)$ and and notice that $F_k(x) = F_k(x_1 e_1) + \cdots + F_k(x_m e_m) + F_k(u)$ with $\|x_k e_k\| < 1$, $k = 1, \ldots, m$ as well as $\|u\| < 1$. Also, $\|\mathcal{F}(x_k e_k)\| \leq \frac{\|x\|_\infty}{1-\|x\|_\infty}$. Hence,

$$\|\mathcal{F}(x)\| = \Big\|\sum_{k=1}^{m} \mathcal{F}(x_k e_k) + \mathcal{F}(u)\Big\| \leq \sum_{k=1}^{m} \|\mathcal{F}(x_k e_k)\| + \|\mathcal{F}(u)\| < \infty.$$

□

Note that $\mathcal{F}$ is an analytic mapping from $\mathbb{D}$ into ℓ_1 since $\mathcal{F}(x)$ can be represented as a convergent series $\mathcal{F}(x) = \sum_{k=1}^{\infty} F_k(x)e_k$ for every $x \in \mathbb{D}$ and $\mathcal{F}$ is bounded in a neighborhood of zero (see [9], p. 58).

Proposition 3.3. [6] *Let $g_1, g_2 \in \mathcal{H}_b(\ell_1)$. If $g_1 \neq g_2$, then there is $x \in \mathbb{D}$ such that $g_1(\mathcal{F}(x)) \neq g_2(\mathcal{F}(x))$.*

Proof. It is enough to show that if for some $g \in \mathcal{H}_b(\ell_1)$, we have $g(\mathcal{F}(x)) = 0 \ \forall x \in \mathbb{D}$, then $g(x) \equiv 0$.

Let $g(x) = \sum_{n=1}^{\infty} Q_n(x)$ where $Q_n \in \mathcal{P}(^n\ell_1)$ and

$$Q_n\Big(\sum_{n=1}^{\infty} x_i e_i\Big) = \sum_{k_1+\ldots+k_n=n} \sum_{i_1<\ldots<i_n} q_{n,i_1\ldots i_n} x_{i_1}^{k_1} \ldots x_{i_n}^{k_n}.$$

For any fixed $x \in \mathbb{D}$ and $t \in \mathbb{C}$ such that $tx \in \mathbb{D}$, let $g(\mathcal{F}(tx)) = \sum_{j=1}^{\infty} t^j r_j(x)$ be the Taylor series at the origin. Then

$$\sum_{n=1}^{\infty} Q_n(\mathcal{F}(tx)) = g(\mathcal{F}(tx)) = \sum_{j=1}^{\infty} t^j r_j(x).$$

Let us compute $r_m(x)$. We have

$$r_m(x) = \sum_{\substack{k<m \\ k_1 i_1+\ldots k_n i_n=m}} q_{k,i_1\ldots i_n} F_{i_1}^{k_1}(x) \ldots F_{i_n}^{k_n}(x). \tag{3.2}$$

It is easy to see that the sum on the right hand side of (3.2) is finite.

Since $g(\mathcal{F}(x)) = 0$ for every $x \in \mathbb{D}$, then $r_m(x) = 0$ for every m. Further being $F_1, \ldots, F_n$ algebraically independent $q_{k,i_1\ldots i_n} = 0$ in (3.2) for an arbitrary $k < m$, $k_1 i_1 + \ldots + k_n i_n = m$. As this is true for every m then $Q_n \equiv 0$ for $n \in \mathbb{N}$. So $g(x) \equiv 0$ on ℓ_1. ☐

Let us denote by $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ the algebra of all symmetric analytic functions which can be represented by $f(x) = g(\mathcal{F}(x))$, where $g \in \mathcal{H}_b(\ell_1)$, $x \in \mathbb{D}$. In other words, $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ is the range of the one-to-one composition operator $C_{\mathcal{F}}(g) = g \circ \mathcal{F}$ acting on $\mathcal{H}_b(\ell_1)$. According to Proposition 3.3 the correspondence $\Psi : f \mapsto g$ is a bijection from $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ onto $\mathcal{H}_b(\ell_1)$. Thus we endow $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ with the topology that turns the bijection Ψ an homeomorphism. This topology is the weakest topology on $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ in which the following seminorms are continuous:

$$q_r(f) := \|(\Psi(f))\|_r = \|g\|_r = \sup_{\|x\|_{\ell_1} \le r} |g(x)|, \quad r \in \mathbb{Q}.$$

Note that Ψ is a homomorphism of algebras. So we have proved the following proposition:

Proposition 3.4. [6] *There is an onto isometric homomorphism between the algebras $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ and $\mathcal{H}_b(\ell_1)$.*

Corollary 3.5. [6] *The spectrum $\mathcal{M}(\mathcal{H}_s^{\ell_1}(\mathbb{D}))$ of $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ can be identified with $\mathcal{M}_b(\ell_1)$. In particular, $\ell_1 \subset \mathcal{M}(\mathcal{H}_s^{\ell_1}(\mathbb{D}))$, that is, for arbitrary $z \in \ell_1$ there is a homomorphism $\psi_z \in \mathcal{M}(\mathcal{H}_s^{\ell_1}(\mathbb{D}))$, such that $\psi_z(f) = \Psi(f)(z)$.*

The following example shows that there exists a character on $\mathcal{H}_s^{\ell_1}(\mathbb{D})$, which is not an evaluation at any point of $\mathbb{D}$.

Example 3.6. [6] Let us consider a sequence of real numbers (a_n), $0 \le |a_n| < 1$ such that $(a_n) \in \ell_2 \setminus \ell_1$ and that the series $\sum_{n=1}^{\infty} a_n$ conditionally converges to some number C. Despite $(a_n) \notin \ell_1$, evaluations on (a_n) are determined for every symmetric polynomial on ℓ_1. In particular, $F_1((a_n)) = C$, $F_k((a_n)) = \sum a_n^k < \infty$ and $\{F_k((a_n))\}_{k=1}^{\infty} \in \ell_1$. So (a_n) "generates" a character on $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ by the formula $\varphi(f) = \Psi(f)(\mathcal{F}((a_n)))$.

Since $(a_n) \in \ell_2$, then $F_k((a_{\pi(n)})) = F_k((a_n))$, $k > 1$. Notice that there exists a permutation on the set of positive integers, π, such that $\sum_{n=1}^{\infty} a_{\pi(n)} = C' \neq C$. For such a permutation π we may do the same construction as above and obtain a homomorphism φ_π "generated by evaluation at $(a_{\pi(n)})$", $\varphi(f) = \Psi(f)(\mathcal{F}((a_{\pi(n)})))$.

Let us suppose that there exist $x, y \in \mathbb{D}$ such that $\varphi(f) = f(x)$ and $\varphi_\pi(f) = f(y)$ for every function $f \in \mathcal{H}_s^{\ell_1}(\mathbb{D})$. Since $\varphi(F_k) = \varphi_\pi(F_k)$, $k \ge 2$, then by [1] Corollary 1.4, it follows that there is a permutation of the indices that transforms the sequence x into the sequence y. But this cannot be true, because $F_1(x) = \varphi(F_1) \neq \varphi_\pi(F_1) = F_1(y)$. Thus, at least one of the homomorphisms φ or φ_π is not an evaluation at some point of $\mathbb{D}$.

Note that the the homomorphism "generated by evaluation at (a_n)" is a character on $\mathcal{P}_s(\ell_1)$ too, but we do not know whether this character is continuous in the topology of uniform convergence on bounded sets.

3.3. The Symmetric Convolution

Recall that in [3] the convolution operation "$*$" for elements φ, θ in the spectrum, $\mathcal{M}_b(X)$, of $\mathcal{H}_b(X)$, is defined by

$$(\varphi * \theta)(f) = \varphi(\theta(f(\cdot + x))), \text{ where } f \in \mathcal{H}_b(X). \tag{3.3}$$

In [6] we have introduced the analogous convolution in our symmetric setting.

It is easy to see that if f is a symmetric function on ℓ_p, then, in general, $f(\cdot + y)$ is not symmetric for a fixed y. However, it is possible to introduce an analogue of the translation operator which preserves the space of symmetric functions on ℓ_p.

Definition 3.7. [6] Let $x, y \in \ell_p$, $x = (x_1, x_2, \dots,)$ and $y = (y_1, y_2, \dots,)$. We define the *intertwining* of x and y, $x \bullet y \in \ell_p$, according to

$$x \bullet y = (x_1, y_1, x_2, y_2, \dots,).$$

Let us indicate some elementary properties of the intertwining.

Proposition 3.8. [6] *Given $x, y \in \ell_p$ the following assertions hold.*

(1) *If $x = \sigma_1(u)$ and $y = \sigma_2(v)$, $\sigma_1, \sigma_2 \in \mathcal{G}$, then $x \bullet y = \sigma(u \bullet v)$ for some $\sigma \in \mathcal{G}$.*
(2) $\|x \bullet y\|^p = \|x\|^p + \|y\|^p$.
(3) *$F_n(x \bullet y) = F_n(x) + F_n(y)$ for every $n \ge p$.*

Proposition 3.9. [6] *If $f(x) \in \mathcal{H}_{bs}(\ell_p)$, then $f(x \bullet y) \in \mathcal{H}_{bs}(\ell_p)$ for every fixed $y \in \ell_p$.*

Proof. Note that $x \bullet y = x \bullet 0 + 0 \bullet y$ and that the map $x \mapsto x \bullet 0$ is linear. Thus the map $x \mapsto x \bullet y$ is analytic and maps bounded sets into bounded sets, and so is its composition with f. Moreover, $f(x \bullet y)$ is obviously symmetric. Hence it belongs to $\mathcal{H}_{bs}(\ell_p)$. □

The mapping $f \mapsto T^s_y(f)$ where $T^s_y(f)(x) = f(x \bullet y)$ will be referred as to the intertwining operator. Observe that $T^s_x \circ T^s_y = T^s_{x\bullet y} = T^s_y \circ T^s_x$: Indeed, $[T^s_x \circ T^s_y](f)(z) = T^s_x[T^s_y(f)](z) = T^s_y(f)(z \bullet x) = f((z \bullet x) \bullet y)) = f(z \bullet (x \bullet y))$, since f is symmetric.

Proposition 3.10. [6] *For every $y \in \ell_p$, the intertwining operator T^s_y is a continuous endomorphism of $\mathcal{H}_{bs}(\ell_p)$.*

Proof. Evidently, T^s_y is linear and multiplicative. Let x belong to ℓ_p and $\|x\| \le r$. Then $\|x \bullet y\| \le \sqrt[p]{r^p + \|y\|^p}$ and

$$|T^s_y f(x)| \le \sup_{\|z\| \le \sqrt[p]{r^p+\|y\|^p}} |f(z)| = \|f\|_{\sqrt[p]{r^p+\|y\|^p}}. \tag{3.4}$$

So T^s_y is continuous. □

Using the intertwining operator we can introduce a symmetric convolution on $\mathcal{H}_{bs}(\ell_p)'$. For any θ in $\mathcal{H}_{bs}(\ell_p)'$, according to (3.4) the radius function $R(\theta \circ T^s_y) \le \sqrt[p]{R(\theta)^p + \|y\|^p}$. Then arguing as in [3, 6.1. Theorem] , it turns out that for fixed $f \in \mathcal{H}_{bs}(\ell_p)$ the function $y \mapsto \theta \circ T^s_y(f)$ also belongs to $\mathcal{H}_{bs}(\ell_p)$.

Definition 3.11. For any ϕ and θ in $\mathcal{H}_{bs}(\ell_p)'$, their *symmetric convolution* is defined according to

$$(\phi \star \theta)(f) = \phi(y \mapsto \theta(T^s_y f)).$$

Corollary 3.12. [6] *If $\phi, \theta \in \mathcal{M}_{bs}(\ell_p)$, then $\phi \star \theta \in \mathcal{M}_{bs}(\ell_p)$.*

Proof. The multiplicativity of T^s_y yields that $\phi \star \theta$ is a character. Using inequality (3.4), we obtain that

$$R(\phi \star \theta) \le \sqrt[p]{R(\phi)^p + R(\theta)^p}.$$

Hence $\phi \star \theta \in \mathcal{M}_{bs}(\ell_p)$. □

Theorem 3.13. [7] *a) For every $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ the following holds:*

$$(\varphi \star \theta)(F_k) = \varphi(F_k) + \theta(F_k). \tag{3.5}$$

b) The semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$ is commutative, the evaluation at 0, δ_0, is its identity and the cancelation law holds.

Proof. Observe that for each element F_k in the algebraic basis of polynomials, $\{F_k\}$, we have

$$(\theta \star F_k)(x) = \theta(T^s_x(F_k)) = \theta(F_k(x) + F_k) = F_k(x) + \theta(F_k).$$

Therefore,

$$(\varphi \star \theta)(F_k) = \varphi(F_k + \theta(F_k)) = \varphi(F_k) + \theta(F_k).$$

To check that the convolution is commutative, that is, $\phi \star \theta = \theta \star \phi$, it suffices to prove it for symmetric polynomials, hence for the basis $\{F_k\}$. Bearing in mind (3.5) and also by exchanging parameters $(\theta \star \varphi)(F_k) = \theta(F_k) + \varphi(F_k) = (\varphi \star \theta)(F_k)$ as we wanted.

It also follows from (3.5) that the cancelation rule is valid for this convolution: If $\varphi \star \theta = \psi \star \theta$, then $\varphi(F_k) + \theta(F_k) = \psi(F_k) + \theta(F_k)$, hence $\varphi(F_k) = \psi(F_k)$, and thus, $\varphi = \psi$. □

Example 3.14. [7] *There exist nontrivial elements in the semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$ that are invertible*:

In [1, Example 3.1] it was constructed a continuous homomorphism $\varphi = \Psi_1$ on the uniform algebra $A_{us}(B_{\ell_p})$ such that $\varphi(F_p) = 1$ and $\varphi(F_i) = 0$ for all $i > p$. In a similar way, given $\lambda \in \mathbb{C}$ we can construct a continuous homomorphism Ψ_λ on the uniform algebra $A_{us}(|\lambda|B_{\ell_p})$ such that $\Psi_\lambda(F_p) = \lambda$ and $\Psi_\lambda(F_i) = 0$ for all $i > p$: It suffices to consider for each $n \in \mathbb{N}$, the element $v_n = \left(\frac{\lambda}{n}\right)^{1/p}(e_1 + \cdots + e_n)$ for which $F_p(v_n) = \lambda$, and $\lim_n F_j(v_n) = 0$. Now, the sequence $\{\delta_{v_n}\}$ has an accumulation point Ψ_λ in the spectrum of $A_{us}(|\lambda|B_{\ell_p})$. We use the notation ψ_λ for the restriction of Ψ_λ to the subalgebra $\mathcal{H}_{bs}(\ell_p)$ of $A_{us}(|\lambda|B_{\ell_p})$. It turns out that $\psi_\lambda \star \psi_{-\lambda} = \delta_0$ since for all elements F_j in the algebraic basis, $(\psi_\lambda \star \psi_{-\lambda})(F_j) = \psi_\lambda(F_j) + \psi_\lambda(F_j) = 0 = \delta_0(F_j)$.

Therefore, we obtain a complex line of invertible elements $\{\psi_\lambda : \lambda \in \mathbb{C}\}$.

As in the non-symmetric case [3] Theorem 5.5, the following holds:

Proposition 3.15. [7] *Every $\varphi \in \mathcal{M}_{bs}(\ell_p)$ lies in a schlicht complex line through δ_0.*

Proof. For every $z \in \mathbb{C}$, consider the composition operator $L_z : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ defined according to $L_z(f)((x_n)) := f((zx_n))$, and then, the restriction L_z^* to $\mathcal{M}_{bs}(\ell_p)$ of its transpose map. Now put $\varphi^z := L_z^*(\varphi) = \varphi \circ L_z$. Observe that $\varphi^z(F_k) = \varphi \circ L_z(F_k) = \varphi((F_k(z\cdot))) = z^k\varphi(F_k)$. Also, $\varphi^0 = \delta_0$.

For each $f \in \mathcal{H}_{bs}(\ell_p)$ the self-map of $\mathbb{C}$ defined according to $z \rightsquigarrow \varphi^z(f)$ is entire by [3] Lemma 5.4.(i). Therefore, the mapping $z \in \mathbb{C} \rightsquigarrow \varphi^z \in \mathcal{M}_{bs}(\ell_p)$ is analytic.

Since $\varphi \neq \delta_0$, the set $\Sigma := \{k \in \mathbb{N} : \varphi(F_k) \neq 0\}$ is non-empty. Let m be the first element of Σ, so that $\varphi(F_m) \neq 0$. Then if $\varphi^z = \varphi^w$, one has $z^m\varphi(F_m) = w^m\varphi(F_m)$, hence $z^m = w^m$. Taking the principal branch of the m^{th} root, the map $\xi \rightsquigarrow \varphi^{\sqrt[m]{\xi}}$ is one-to-one. ☐

Recall that a linear operator $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is said to be a *convolution operator* if there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Let us denote $H_{conv}(\ell_p) := \{T \in L(\mathcal{H}_{bs}(\ell_p)) : T \text{ is a convolution operator}\}$.

Proposition 3.16. [7] *A continuous homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is a convolution operator if, and only if, it commutes with all intertwining operators T_y^s, $y \in \ell_p$.*

Proof.- Assume there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Fix $y \in \ell_p$. Then $[T \circ T_y^s](f)(x) = [T(T_y^s(f))](x) = [\theta \star T_y^s(f)](x) = \theta[T_x^s(T_y^s(f)] = \theta[T_{x\bullet y}^s(f)]$. On the other hand, $[T_y^s \circ T](f)(x) = [T_y^s(Tf)](x) = Tf(x \bullet y) = (\theta \star f)(x \bullet y) = \theta[T_{x\bullet y}^s(f)]$.

Conversely, set $\theta = \delta_0 \circ T$. Clearly, $\theta \in \mathcal{M}_{bs}(\ell_p)$. Let us check that $Tf = \theta \star f$: Indeed, $(\theta \star f)(x) = \theta[T_x^s(f)] = [T(T_x^s(f))](0) = [T_x^s(T(f))](0) = Tf(0 \bullet x) = Tf(x)$. ☐

Consider the mapping Λ defined by $\Lambda(\theta)(f) = \theta \star f$, that is,

$$\begin{array}{rccc} \Lambda : & \mathcal{M}_{bs}(\ell_p) & \to & H_{conv}(\ell_p) \\ & \theta & \mapsto & f \rightsquigarrow \theta \star f \equiv \Lambda(\theta)(f) \end{array}.$$

It is, clearly, bijective. Moreover we obtain a representation of the convolution semigroup

Proposition 3.17. [7] *The mapping Λ is an isomorphism from $(\mathcal{M}_{bs}(\ell_p), \star)$ into $(H_{conv}(\ell_p), \circ)$ where $\circ$ denotes the usual composition operation.*

Proof.- First, notice that using the above proposition,

$$\begin{aligned}\Lambda(\varphi\star\theta)(f)(x) &= [(\varphi\star\theta)\star f](x) = (\varphi\star\theta)(T^s_x f) = \varphi(\theta\star T^s_x f)\\ &= \varphi[\Lambda(\theta)(T^s_x f)] = \varphi[(\Lambda(\theta)\circ T^s_x)(f)] = \varphi[(T^s_x\circ\Lambda(\theta))(f)].\end{aligned}$$

On the other hand,

$$[\Lambda(\varphi)\circ\Lambda(\theta)](f)(x) = \Lambda(\varphi)[\Lambda(\theta)(f)](x) = [\varphi\star\Lambda(\theta)(f)](x) = \varphi[T^s_x(\Lambda(\theta)(f))].$$

Thus the statement follows. □

As a consequence, the homomorphism θ is invertible in $(\mathcal{M}_{bs}(\ell_p),\star)$, if, and only if, the convolution operator $\Lambda(\theta)$ is an algebraic isomorphism. Observe also that for $\psi\in\mathcal{M}_{bs}(\ell_p)$, one has

$$\psi\circ\Lambda(\theta) = \psi\star\theta,$$

because $[\psi\circ\Lambda(\theta)](f) = \psi[\Lambda(\theta)(f)] = \psi(\theta\star f) = (\psi\star\theta)(f)$.

Next we address the question of solving the equation $\varphi = \psi\star\theta$ for given $\varphi,\theta\in\mathcal{M}_{bs}(\ell_p)$. We begin with a general lemma.

Lemma 3.18. [7] *Let A, B be Fréchet algebras and $T: A\to B$ an onto homomorphism. Then T maps (closed) maximal ideals onto (closed) maximal ideals.*

Proof. Since T is onto, it maps ideals in A onto ideals in B. Let $\mathcal{J}\subset A$ be a maximal ideal, we prove that $T(\mathcal{J})$ is a maximal ideal in B : If $\mathcal{I}$ is another ideal with $T(\mathcal{J})\subset\mathcal{I}\subset B$, it turns out that for the ideal $T^{-1}(\mathcal{I})$, $\mathcal{J}\subset T^{-1}(T(\mathcal{J}))\subset T^{-1}(\mathcal{I})$, hence either $\mathcal{J} = T^{-1}(\mathcal{I})$, or $A = T^{-1}(\mathcal{I})$. That is, either $T(\mathcal{J}) = \mathcal{I}$, or $B = \mathcal{I}$.

Let now $\varphi\in\mathcal{M}(A)$ and $\mathcal{J} = Ker(\varphi)$, a closed maximal ideal. Then $T(\mathcal{J})$ is a maximal ideal in B, so there is a character ψ on B such that $Ker(\psi) = T(\mathcal{J})$. Then $Ker(\varphi)\subset Ker(\psi\circ T)$, because if $\varphi(a) = 0$, that is, $a\in\mathcal{J}$, we have $T(a)\in Ker(\psi)$. By the maximality, either $\varphi = \psi\circ T$, or $\psi\circ T = 0$, hence $\psi = 0$. In the former case, ψ is also continuous since being T an open mapping, if (b_n) is a null sequence in B, there is a null sequence $(a_n)\subset A$ such that $T(a_n) = b_n$; thus $\lim_n\psi(b_n) = \lim_n\psi\circ T(a_n) = \lim_n\varphi(a_n) = 0$. □

Remark 3.19. *Let A, B be Fréchet algebras and $T: A\to B$ an onto homomorphism. If $T(Ker(\varphi))$ is a proper ideal, then there is a unique $\psi\in\mathcal{M}(B)$ such that $\varphi = \psi\circ T$.*

Corollary 3.20. [7] *Let $\theta\in\mathcal{M}_{bs}(\ell_p)$. Assume that $\Lambda(\theta)$ is onto. If $\Lambda(\theta)(Ker\varphi)$ is a proper ideal, then the equation $\varphi = \psi\star\theta$ has a unique solution. In case $\Lambda(\theta)(Ker\varphi) = \mathcal{H}_{bs}(\ell_p)$, then the equation $\varphi = \psi\star\theta$ has no solution.*

Proof. The first statement is just an application of the remark, since $\psi\star\theta = \psi\circ\Lambda(\theta) = \varphi$. For the second statement, if some solution ψ exists, then again $\psi\circ\Lambda(\theta) = \psi\star\theta = \varphi$, so $\psi(\mathcal{H}_{bs}(\ell_p)) = (\psi\circ\Lambda(\theta))((Ker\varphi)) = \varphi(Ker\varphi) = 0$. Therefore, then also $\varphi = 0$.

□

3.4. A Weak Polynomial Topology on $\mathcal{M}_{bs}(\ell_p)$ [7]

Let us denote by w_p the topology in $\mathcal{M}_{bs}(\ell_p)$ generated by the following neighborhood basis:

$$U_{\varepsilon,k_1,\ldots,k_n}(\psi) = \{\psi\star\varphi\colon\ \varphi\in\mathcal{M}_{bs}(\ell_p)\quad |\varphi(F_{k_j})|<\varepsilon,\quad j = 1,\ldots,n\}.$$

It is easy to check that the convolution operation is continuous for the w_p topology, since thanks to (3.5),

$$U_{\varepsilon/2,k_1,\ldots,k_n}(\theta)\star U_{\varepsilon/2,k_1,\ldots,k_n}(\psi)\subset U_{\varepsilon,k_1,\ldots,k_n}(\theta\star\psi).$$

We say that a function $f \in \mathcal{H}_{bs}(\ell_p)$ is *finitely generated* if there are a finite number of the basis functions $\{F_k\}$ and an entire function q such that $f = q(F_1, \ldots, F_j)$.

Theorem 3.21. *A function $f \in \mathcal{H}_{bs}(\ell_p)$ is w_p-continuous if and only if it is finitely generated.*

Proof. Clearly, every finitely generated function is w_p-continuous. Let us denote by V_n the finite dimensional subspace in ℓ_p spanned by the basis vectors $\{e_1, \ldots, e_n\}$. First we observe that if there is a positive integer m such that the restriction $f_{|V_n}$ of f to V_n is generated by the restrictions of $F_1, \ldots, F_m$ to V_n for every $n \geq m$, then f is finitely generated. Indeed, for given $n \geq k \geq m$ we can write

$$f_{|V_k}(x) = q_1(F_1(x), \ldots, F_m(x)) \quad \text{and} \quad f_{|V_n}(x) = q_2(F_1(x), \ldots, F_m(x))$$

for some entire functions q_1 and q_2 on $\mathbb{C}^n$. Since

$$\{(F_1(x), \ldots, F_m(x)) \colon x \in V_k\} = \mathbb{C}^m$$

(see e. g. [1]) and $f|_{V_n}$ is an extension of $f|_{V_k}$ we have $q_1(t_1, \ldots, t_n) = q_2(t_1, \ldots, t_n)$. Hence $f(x) = q_1(F_1(x), \ldots, F_m(x))$ on ℓ_p because $f(x)$ coincides with $q_1(F_1(x), \ldots, F_m(x))$ on the dense subset $\bigcup_n V_n$.

Let f be a w_p-continuous function in $\mathcal{H}_{bs}(\ell_p)$. Then f is bounded on a neighborhood $U_{\varepsilon,1,\ldots,m} = \{x \in \ell_p \colon |F_1(x)| < \varepsilon, \ldots, |F_m(x)| < \varepsilon\}$. For a given $n \geq m$ let

$$f|_{V_n}(x) = q(F_1(x), \ldots, F_m(x))$$

be the representation of $f|_{V_n}(x)$ for some entire function q on $\mathbb{C}^n$. Since $\{(F_1(x), \ldots, F_m(x)) \colon x \in V_n\} = \mathbb{C}^m$, $q(t_1, \ldots, t_n)$ must be bounded on the set $\{|t_1| < \varepsilon, \ldots, |t_m| < \varepsilon\}$. The Liouville Theorem implies $q(t_1, \ldots, t_n) = q(t_1, \ldots, t_m, 0 \ldots, 0)$, that is, $f|_{V_n}$ is generated by $F_1, \ldots, F_m$. Since it is true for every n, f is finitely generated. □

For example $f(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{n!}$ is not w_p-continuous.

Proposition 3.22. *w_p is a Hausdorff topology.*

Proof. If $\varphi \neq \psi$, then there is a number k such that

$$|\varphi(F_k) - \psi(F_k)| = \rho > 0.$$

Let $\varepsilon = \rho/3$. Then for every θ_1 and θ_2 in $U_{\varepsilon,k}(0)$,

$$|(\varphi \star \theta_1)(F_k) - (\varphi \star \theta_2)(F_k)| = |(\varphi(F_k) - \psi(F_k)) - (\theta_2(F_k)) - \theta_1(F_k)| \geq \rho/3.$$

□

Proposition 3.23. *On bounded sets of $\mathcal{M}_{bs}(\ell_p)$ the topology w_p is finer than the weak-star topology $w(\mathcal{M}_{bs}(\ell_p), \mathcal{H}_{bs}(\ell_p))$.*

Proof. Since $(\mathcal{M}_{bs}(\ell_p), w_p)$ is a first-countable space, it suffices to verify that for a bounded sequence $(\varphi_i)_i$ which is w_p convergent to some ψ, we have $\lim_i \varphi_i(f) = \psi(f)$ for each $f \in \mathcal{H}_{bs}(\ell_p)$: Indeed, by the Banach-Steinhaus theorem, it is enough to see that $\lim_i \varphi_i(P) = \psi(P)$ for each symmetric polynomial P. Being $\{F_k\}$ an algebraic basis for the symmetric polynomials, this will follow once we check that $\lim_i \varphi_i(F_k) = \psi(F_k)$ for each F_k. To see this, notice that given $\varepsilon > 0$, $\varphi_i \in U_{\varepsilon,k}$ for i large enough, that is, there is θ_i such that $\varphi_i = \psi \star \theta_i$ with $|\theta_i(F_k)| < \varepsilon$. Then, $|\varphi_i(F_k) - \psi(F_k)| = |\theta_i(F_K)| < \varepsilon$ for i large enough. □

Proposition 3.24. *If $(\mathcal{M}_{bs}(\ell_p), \star)$ is a group, then w_p coincides with the weakest topology on $\mathcal{M}_{bs}(\ell_p)$ such that for every polynomial $P \in \mathcal{H}_{bs}(\ell_p)$ the Gelfand extension $\widehat{P}$ is continuous on $\mathcal{M}_{bs}(\ell_p)$.*

Proof. The sets $F_k^{-1}(B(F_k(\psi), \varepsilon))$ generate the weakest topology such that all $\widehat{P}$ are continuous. Let $\theta \in \mathcal{M}_{bs}(\ell_p)$ be such that $|F_k(\theta) - F_k(\psi)| < \varepsilon$. Set $\varphi = \theta \star \psi^{-1}$. Then $|F_k(\varphi)| = |F_k(\theta) - F_k(\psi)| < \varepsilon$ and $\theta = \psi \star \varphi$. □

3.5. Representations of the Convolution Semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$[7]

In this subsection we consider the case $\mathcal{H}_{bs}(\ell_1)$. This algebra admits besides the power series basis $\{F_n\}$, another natural basis that is useful for us: It is given by the sequence $\{G_n\}$ defined by $G_0 = 1$, and

$$G_n(x) = \sum_{k_1 < \cdots < k_n}^{\infty} x_{k_1} \cdots x_{k_n},$$

and we refer to it as the *basis of elementary* symmetric polynomials.

Lemma 3.25. *We have that $\|G_n\| = 1/n!$*

Proof. To calculate the norm, it is enough to deal with vectors in the unit ball of ℓ_1 whose components are non-negative. And we may reduce ourselves to calculate it on L_m the linear span of $\{e_1, \ldots, e_m\}$ for $m \geq n$. We do the calculation in an inductive way over m.

Since $G_{n|_{L_m}}$ is homogeneous, its norm is achieved at points of norm 1. If $m = n$, then G_n is the product $x_1 \cdots x_n$. By using the Lagrange multipliers rule, we deduce that the maximum is attained at points with equal coordinates, that is at $\frac{1}{n}(e_1 + \cdots + e_n)$. Thus $|G_n(\frac{1}{n}, \overset{n}{\ldots}, \frac{1}{n}, 0, \ldots)| = 1/n^n \leq \frac{1}{n!}$.

Now for $m > n$, and $x \in L_m$, we have $G_n(x) = \sum_{k_1 < \cdots < k_n \leq m}^{\infty} x_{k_1} \cdots x_{k_n}$. Again the Lagrange multipliers rule leads to either some of the coordinates vanish or they are all equal, hence they have the same value $\frac{1}{m}$. In the first case, we are led back to some the previous inductive steps, with L_k with $k < m$, so the aimed inequality holds. While in the second one, we have

$$G_n(\frac{1}{m}, \overset{m}{\ldots}, \frac{1}{m}, 0, \ldots) = \binom{m}{n} \frac{1}{m^n} \leq \frac{1}{n!}.$$

Moreover, $\|G_n\| \geq \lim_m \binom{m}{n} \frac{1}{m^n} = \frac{1}{n!}$. This completes the proof. □

Let $\mathbb{C}\{t\}$ be the space of all power series over $\mathbb{C}$. We denote by $\mathcal{F}$ and $\mathcal{G}$ the following maps from $\mathcal{M}_{bs}(\ell_1)$ into $\mathbb{C}\{t\}$

$$\mathcal{F}(\varphi) = \sum_{n=1}^{\infty} t^{n-1} \varphi(F_n) \quad \text{and} \quad \mathcal{G}(\varphi) = \sum_{n=0}^{\infty} t^n \varphi(G_n).$$

Let us recall that every element $\varphi \in \mathcal{M}_{bs}(\ell_1)$ has a radius-function

$$R(\varphi) = \limsup_{n \to \infty} \|\varphi_n\|^{\frac{1}{n}} < \infty,$$

where φ_n is the restriction of φ to the subspace of n-homogeneous polynomials [6].

Proposition 3.26. *The mapping $\varphi \in \mathcal{M}_{bs}(\ell_1) \xrightarrow{\mathcal{G}} \mathcal{G}(\varphi) \in \mathcal{H}(\mathbb{C})$ is one-to-one and ranges into the subspace of entire functions on $\mathbb{C}$ of exponential type. The type of $\mathcal{G}(\varphi)$ is less than or equal to $R(\varphi)$.*

Proof. Using Lemma 3.25,

$$\begin{aligned} \limsup_{n\to\infty} \sqrt[n]{n!|\varphi_n(G_n)|} &\le \limsup_{n\to\infty} \sqrt[n]{n!\|\varphi_n\|\|G_n\|} \\ &= \limsup_{n\to\infty} \sqrt[n]{\|\varphi_n\|} = R(\varphi) < \infty, \end{aligned}$$

hence $\mathcal{G}(\varphi)$ is entire and of exponential type less than or equal to $R(\varphi)$. That $\mathcal{G}$ is one-to-one follows from the fact $\{G_n\}$ is a basis. □

Theorem 3.27. *The following identities hold:*

(1) $\mathcal{F}(\varphi \star \theta) = \mathcal{F}(\varphi) + \mathcal{F}(\theta)$.
(2) $\mathcal{G}(\varphi \star \theta) = \mathcal{G}(\varphi)\mathcal{G}(\theta)$.

Proof. The first statement is a trivial conclusion of the properties of the convolution. To prove the second we observe that

$$G_n(x \bullet y) = \sum_{k=0}^{n} G_k(x)G_{n-k}(y).$$

Thus

$$(\theta \star G_n)(x) = \theta(T_x^s(G_n)) = \theta\Big(\sum_{k=0}^{n} G_k(x)G_{n-k}\Big) = \sum_{k=0}^{n} G_k(x)\theta(G_{n-k}).$$

Therefore,

$$(\varphi \star \theta)(G_n) = \varphi\Big(\sum_{k=0}^{n} G_k(x)\theta(G_{n-k})\Big) = \sum_{k=0}^{n} \varphi(G_k)\theta(G_{n-k}).$$

Hence, being the series absolutely convergent,

$$\begin{aligned} \mathcal{G}(\varphi)\mathcal{G}(\theta) &= \sum_{k=0}^{\infty} t^k\varphi(G_k) \sum_{m=0}^{\infty} t^m\theta(G_m) = \sum_{n=0}^{\infty} \sum_{k+m=n} t^n\varphi(G_k)\theta(G_m) \\ &= \sum_{n=0}^{\infty} t^n \sum_{k+m=n} \varphi(G_k)\theta(G_m) = \sum_{n=0}^{\infty} t^n(\varphi \star \theta)(G_n) = \mathcal{G}(\varphi \star \theta). \end{aligned}$$

□

Example 3.28. Let ψ_λ be as defined in Example 3.14. We know that $\mathcal{F}(\psi_\lambda) = \lambda$. To find $\mathcal{G}(\psi_\lambda)$ note that

$$G_k(v_n) = \Big(\frac{\lambda}{n}\Big)^k \binom{n}{k}, \text{ hence } \varphi(G_k) = \lim_n G_k(v_n) = \frac{\lambda^k}{k!}$$

and so

$$\mathcal{G}(\psi_\lambda)(t) = \lim_{n\to\infty} \sum_{k=0}^{n} (\lambda t)^k \psi_\lambda(G_n) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} = e^{\lambda t}.$$

According to well-known Newton's formula we can write for $x \in \ell_1$,

$$nG_n(x) = F_1(x)G_{n-1}(x) - F_2(x)G_{n-2}(x) + \cdots + (-1)^{n+1}F_n(x). \tag{3.6}$$

Moreover, if ξ is a complex homomorphism (not necessarily continuous) on the space of symmetric polynomials $\mathcal{P}_s(\ell_1)$, then

$$n\xi(G_n) = \xi(F_1)\xi(G_{n-1}) - \xi(F_2)\xi(G_{n-2}) + \cdots + (-1)^{n+1}\xi(F_n). \tag{3.7}$$

Next we point out the limitations of the construction's technique described in 3.14.

Remark 3.29. *Let ξ be a complex homomorphism on $\mathcal{P}_s(\ell_1)$ such that $\xi(F_m) = c \neq 0$ for some $m \geq 2$ and $\xi(F_n) = 0$ for $n \neq m$. Then ξ is not continuous.*

Proof. Using formula (3.7) we can see that

$$\xi(G_{km}) = (-1)^{m+1}\frac{\xi(F_m)\xi(G_{(k-1)m})}{km}$$

and $\xi(G_n) = 0$ if $n \neq km$ for some $k \in \mathbb{N}$. By induction we have

$$\xi(G_{km}) = \frac{\left((-1)^{m+1}c/m\right)^k}{k!}$$

and so

$$\mathcal{G}(\xi)(t) = 1 + \sum_{k=1}^{\infty} \frac{\left((-1)^{m+1}c/m\right)^k}{k!} t^{km} = 1 + \sum_{k=1}^{\infty} \frac{((-1)^{m+1}\frac{ct^m}{m})^k}{k!} = e^{\left((-1)^{m+1}\frac{ct^m}{m}\right)}.$$

Hence $\mathcal{G}(\xi)(t) = e^{-\frac{(-ct)^m}{m}} = e^{-\frac{(-c)^m}{m}t^m}$. Since $m \geq 2$, $\mathcal{G}(\xi)$ is not of exponential type. So if ξ were continuous, it could be extended to an element in $\mathcal{M}_{bs}(\ell_1)$, leading to a contradiction with Proposition 3.26. □

According to the Hadamard Factorization Theorem (see [14, p. 27]) the function of the exponential type $\mathcal{G}(\varphi)(t)$ is of the form

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\Big(1 - \frac{t}{a_k}\Big)e^{t/a_k}, \tag{3.8}$$

where $\{a_k\}$ are the zeros of $\mathcal{G}(\varphi)(t)$. If $\sum |a_k|^{-1} < \infty$, then this representation can be reduced to

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\Big(1 - \frac{t}{a_k}\Big). \tag{3.9}$$

Recall how ψ_λ was defined in Example 3.14.

Proposition 3.30. *If $\varphi \in (\mathcal{M}_{bs}(\ell_1), \star)$ is invertible, then $\varphi = \psi_\lambda$ for some λ. In particular, the semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$ is not a group.*

Proof. If φ is invertible then $\mathcal{G}(\varphi)(t)$ is an invertible entire function of exponential type and so has no zeros. By Hadamard's factorization (3.8) we have that $\mathcal{G}(\varphi)(t) = e^{\lambda t}$ for some complex number λ. Hence $\varphi = \psi_\lambda$ by Proposition 3.26.

The evaluation $\delta_{(1,0\ldots,0,\ldots)}$ does not coincide with any ψ_λ since, for instance, $\psi_\lambda(F_2) = 0 \neq 1 = \delta_{(1,0\ldots,0,\ldots)}(F_2)$. □

Another consequence of our analysis is the following remark.

Corollary 3.31. *Let Φ be a homomorphism of $\mathcal{P}_s(\ell_1)$ to itself such that $\Phi(F_k) = -F_k$ for every k. Then Φ is discontinuous.*

Proof. If Φ is continuous it may be extended to continuous homomorphism $\widetilde{\Phi}$ of $\mathcal{H}_{bs}(\ell_1)$. Then for $x = (1, 0 \ldots, 0, \ldots)$, $\delta_x \star (\delta_x \circ \widetilde{\Phi}) = \delta_0$. However, this is impossible since δ_x is not invertible. □

We close this section by analyzing further the relationship established by the mapping $\mathcal{G}$.

It is known from Combinatorics (see e.g. [15, p. 3, 4]) that

$$\mathcal{G}(\delta_x)(t) = \prod_{k=1}^{\infty}(1 + x_k t) \quad \text{and} \quad \mathcal{F}(\delta_x)(t) = \sum_{k=1}^{\infty} \frac{x_k}{1 - x_k t} \tag{3.10}$$

for every $x \in c_{00}$. Formula (3.10) for $\mathcal{G}(\delta_x)$ is true for every $x \in \ell_1$: Indeed, for fixed t, both the infinite product and $\mathcal{G}(\delta_x)(t)$ are analytic functions on ℓ_1.

Taking into account formula (3.10) we can see that the zeros of $\mathcal{G}(\delta_x)(t)$ are $a_k = -1/x_k$ for $x_k \neq 0$. Conversely, if $f(t)$ is an entire function of exponential type which is equal to the right hand side of (3.9) with $\sum |a_k|^{-1} < \infty$, then for $\varphi \in \mathcal{M}_{bs}(\ell_1)$ given by $\varphi = \psi_\lambda \star \delta_x$, where $x \in \ell_1$, $x_k = -1/a_k$ and ψ_λ is defined in Example 3.14, it turns out that $\mathcal{G}(\varphi)(t) = f(t)$. So we have just to examine entire functions of exponential type with Hadamard canonical product

$$f(t) = \prod_{k=1}^{\infty}\Big(1 - \frac{t}{a_k}\Big)e^{t/a_k} \tag{3.11}$$

with $\sum |a_k|^{-1} = \infty$. Note first that the growth order of $f(t)$ is not greater than 1. According to Borel's theorem [14, p. 30] the series

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{1+d}}$$

converges for every $d > 0$. Let

$$\Delta_f = \limsup_{n\to\infty} \frac{n}{|a_n|}, \qquad \eta_f = \limsup_{r\to\infty} \Big| \sum_{|a_n|<r} \frac{1}{a_n} \Big|$$

and $\gamma_f = \max(\Delta_f, \eta_f)$. Due to Lindelöf's theorem [14, p. 33] the type σ_f of f and γ_f simultaneously are equal either to zero, or to infinity, or to positive numbers. Hence $f(t)$ of the form (3.11) is a function of exponential type if and only if $\sum |a_k|^{-1-d}$ converges for every $d > 0$ and γ_f is finite.

Corollary 3.32. *If a sequence $(x_n) \notin \ell_p$ for some $p > 1$, then there is no $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that*

$$\varphi(F_k) = \sum_{n=1}^{\infty} x_n^k$$

for all k.

Let $x = (x_1, \ldots, x_n, \ldots)$ be a sequence of complex numbers such that $x \in \ell_{1+d}$ for every $d > 0$,

$$\limsup_{n\to\infty} n|x_n| < \infty, \qquad \limsup_{r\to 1} \Big| \sum_{\frac{1}{|x_n|}<r} x_n \Big| < \infty \tag{3.12}$$

and $\lambda \in \mathbb{C}$. Let us denote by $\delta_{(x,\lambda)}$ a homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ of the form

$$\delta_{(x,\lambda)}(F_1) = \lambda, \qquad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k, \quad k > 1.$$

Proposition 3.33. *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then the restriction of φ to $\mathcal{P}_s(\ell_1)$ coincides with $\varphi_{(x,\lambda)}$ for some $\lambda \in \mathbb{C}$ and x satisfying (3.15).*

Proof. Consider the exponential type function $\mathcal{G}(\varphi)$ given by (3.8) and the corresponding sequence $x = (\frac{-1}{a_n})$.

If $x \in \ell_1$, then according to (3.9), $\varphi = \psi_\lambda \star \delta_x$. If $x \notin \ell_1$, then $\mathcal{G}(\varphi)(t) = e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n}$ and, on the other hand, $\mathcal{G}(\varphi)(t) = \sum_{n=0}^{\infty} \varphi(G_n) t^n$.

We have

$$\begin{aligned}\Big(e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n}\Big)'_t &= \lambda e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n} = e^{\lambda t}\Big(- tx_1^2 e^{-tx_1} \prod_{n \neq 1} \Big(1 + tx_n\Big) e^{-tx_n} \\ &- tx_2^2 e^{-tx_2} \prod_{n \neq 2} \Big(1 + tx_n\Big) e^{-tx_n} - \dots\Big) \\ &= \lambda e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n} - te^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n \neq k} \Big(1 + tx_n\Big) e^{-tx_n}\end{aligned}$$

and

$$\Big(e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n}\Big)'\Big|_{t=0} = \lambda.$$

So by the uniqueness of the Taylor coefficients, $\varphi(G_1) = \varphi(F_1) = \lambda$.

Now

$$\begin{aligned}\Big(e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n}\Big)''_t &= \Big(\lambda e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n}\Big)'_t \\ &- \Big(te^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n \neq k} \Big(1 + tx_n\Big) e^{-tx_n}\Big)'_t \\ &= \lambda^2 e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n} - \lambda te^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n \neq k} \Big(1 + tx_n\Big) e^{-tx_n} \\ &- e^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n \neq k} \Big(1 + tx_n\Big) e^{-tx_n} \\ &- t\Big(e^{\lambda t} \sum_{k=1}^{\infty} x_k^2 e^{-tx_k} \prod_{n \neq k} \Big(1 + tx_n\Big) e^{-tx_n}\Big)'_t\end{aligned}$$

and

$$\Big(e^{\lambda t} \prod_{n=1}^{\infty} \Big(1 + tx_n\Big) e^{-tx_n}\Big)''\Big|_{t=0} = \lambda^2 - \sum_{k=1}^{\infty} x_k^2.$$

Then

$$\varphi(G_2) = \frac{\lambda^2 - F_2(x)}{2} = \frac{(\varphi(F_1))^2 - F_2(x)}{2}.$$

On the other hand,

$$\varphi(G_2) = \frac{\varphi(F_1^2) - \varphi(F_2)}{2}$$

and we have

$$\varphi(F_2) = F_2(x).$$

Now using induction we obtain the required result. □

Question 3.34. Does the map $\mathcal{G}$ act onto the space of entire functions of exponential type?

3.6. THE MULTIPLICATIVE CONVOLUTION [8]

Definition 3.35. Let $x, y \in \ell_p$, $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$. We define the *multiplicative intertwining* of x and y, $x \diamond y$, as the resulting sequence of ordering the set $\{x_i y_j : i, j \in \mathbb{N}\}$ with one single index in some fixed order.

Note that for further consideration the order of numbering does not matter.

Proposition 3.36. *For arbitrary $x, y \in \ell_p$ we have*

(1) $x \diamond y \in \ell_p$ *and* $\|x \diamond y\| = \|x\|\|y\|$;
(2) $F_k(x \diamond y) = F_k(x)F_k(y) \ \forall k \geq \lceil p \rceil$.
(3) *If P is an n-homogeneous symmetric polynomial on ℓ_p and y is fixed, then the function $x \mapsto P(x \diamond y)$ is n-homogeneous.*

Proof. It is clear that $\|x \diamond y\|^p = \sum_{i,j} |x_i y_j|^p = \sum_i |x_i|^p \sum_i |y_j|^p = \|x\|^p \|y\|^p$. Also $F_k(x \diamond y) = \sum_{i,j}(x_i y_j)^k = \sum_i x_i^k \sum_j y_j^k = F_k(x)F_k(y)$. Statement (3) follows from the equality $\lambda(x \diamond y) = (\lambda x) \diamond y$. □

Given $y \in \ell_p$, the mapping $x \in \ell_p \overset{\pi_y}{\to} (x \diamond y) \in \ell_p$ is linear and continuous because of Proposition 3.36. Therefore if $f \in \mathcal{H}_{bs}(\ell_p)$, then $f \circ \pi_y \in \mathcal{H}_{bs}(\ell_p)$ because $f \circ \pi_y$ is analytic and bounded on bounded sets and clearly $f(\sigma(x) \diamond y) = f(x \diamond y)$ for every permutation $\sigma \in \mathcal{G}$. Thus if we denote $M_y(f) = f \circ \pi_y$, M_y is a composition operator on $\mathcal{H}_{bs}(\ell_p)$, that we will call the *multiplicative convolution operator*. Notice as well that $M_y = M_{\sigma(y)}$ for every permutation $\sigma \in \mathcal{G}$ and that $M_y(F_k) = F_k(y)F_k \ \forall k \geq \lceil p \rceil$.

Proposition 3.37. *For every $y \in \ell_p$ the multiplicative convolution operator M_y is a continuous homomorphism on $\mathcal{H}_{bs}(\ell_p)$.*

Note that in particular, if f_n is an n-homogeneous continuous polynomial, then $\|M_y(f_n)\| \leq \|f_n\|\|y\|^n$. And also that for $\lambda \in \mathbb{C}$, $M_{\lambda y}(f_n) = \lambda^n M_y(f_n)$, because $\pi_{\lambda y}(x) = \lambda \pi(x)$. Analogously, $M_{y+z}(f_n) = f_n \circ (\pi_y + \pi_z)$, because $\pi_{y+z} = \pi_y + \pi_z$. Therefore the mapping $y \in \ell_p \mapsto M_y(f_n)$ is an n-homogeneous continuous polynomial.

Recall that the *radius function* $R(\phi)$ of a complex homomorphism $\phi \in \mathcal{M}_{bs}(\ell_p)$ is the infimum of all r such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB_{ℓ_p}, that is $|\phi(f)| \leq C_r \|f\|_r$. It is known that

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n},$$

where ϕ_n is the restriction of ϕ to $\mathcal{P}_s(^n\ell_p)$ and $\|\phi_n\|$ is its corresponding norm (see [6]).

Proposition 3.38. *For every $\theta \in \mathcal{H}_{bs}(\ell_p)'$ and every $y \in \ell_p$ the radius-function of the continuous homomorphism $\theta \circ M_y$ satisfies*

$$R(\theta \circ M_y) \leq R(\theta)\|y\|$$

and for fixed $f \in \mathcal{H}_{bs}(\ell_p)$ the function $y \mapsto \theta \circ M_y(f)$ also belongs to $\mathcal{H}_{bs}(\ell_p)$.

Proof. For a given $y \in \ell_p$, let $(\theta \circ M_y)_n$ (respectively, θ_n) be the restriction of $\theta \circ M_y$ (respectively, θ) to the subspace of n-homogeneous symmetric polynomials. Then we have

$$\|(\theta \circ M_y)_n\| = \sup_{\|f_n\| \leq 1} \left|\theta_n\Big(\frac{M_y(f_n)}{\|y\|^n}\Big)\right| \|y\|^n \leq \|\theta_n\| \|y\|^n.$$

So

$$R(\theta \circ M_y) \leq \limsup_{n \to \infty} (\|\theta_n\| \|y\|^n)^{1/n} = R(\theta)\|y\|.$$

Since the terms in the Taylor series of the function $y \mapsto \theta \circ M_y(f)$ are $y \mapsto \theta \circ M_y(f_n)$, where (f_n) are the terms in the Taylor series of f, the formula above proves the second statement. □

Using the multiplicative convolution operator we can introduce a multiplicative convolution on $\mathcal{H}_{bs}(\ell_p)'$.

Definition 3.39. Let $f \in \mathcal{H}_{bs}(\ell_p)$ and $\theta \in \mathcal{H}_{bs}(\ell_p)'$. The *multiplicative convolution* $\theta \Diamond f$ is defined as

$$(\theta \Diamond f)(x) = \theta[M_x(f)] \text{ for every } x \in \ell_p .$$

We have by Proposition 3.38, that $\theta \Diamond f \in \mathcal{H}_{bs}(\ell_p)$.

Definition 3.40. For arbitrary $\varphi, \theta \in \mathcal{H}_{bs}(\ell_p)'$ we define their *multiplicative convolution* $\varphi \Diamond \theta$ according to

$$(\varphi \Diamond \theta)(f) = \varphi(\theta \Diamond f) \text{ for every } f \in \mathcal{H}_{bs}(\ell_p).$$

For the evaluation homomorphism at y, δ_y, observe that

$$(\delta_y \Diamond f)(x) = \delta_y(M_x(f)) = (f \circ \pi_x)(y) = f(\pi_x(y)) = f(x \diamond y) = f(\pi_y(x)) = M_y(f)(x).$$

Hence, $\delta_x \Diamond \delta_y = \delta_{x \diamond y}$.

Proposition 3.41. *If $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$, then $\varphi \Diamond \theta \in \mathcal{M}_{bs}(\ell_p)$.*

Proof. From the multiplicativity of M_y it follows that $\varphi \Diamond \theta$ is a character. Using arguments as in Proposition 3.38, we have that

$$R(\varphi \Diamond \theta) \leq R(\varphi) R(\theta).$$

Hence $\varphi \Diamond \theta \in \mathcal{M}_{bs}(\ell_p)$. □

Theorem 3.42.

$$1. \textit{If } \varphi, \theta \in \mathcal{M}_{bs}(\ell_p), \textit{ then } (\varphi \Diamond \theta)(F_k) = \varphi(F_k)\theta(F_k) \quad \forall k \geq \lceil p \rceil. \tag{3.13}$$

2. *The semigroup $(\mathcal{M}_{bs}(\ell_p), \Diamond)$ is commutative and the evaluation at $x_0 = (1, 0, 0, \ldots)$, δ_{x_0}, is its identity.*

Proof. Let us take firstly $x, y \in \ell_p$ and $\delta_x, \delta_y \in \mathcal{M}_{bs}(\ell_p)$ the corresponding point evaluation homomorphisms. Then $(\delta_x \Diamond \delta_y)(F_k) = F_k(x \diamond y) = \sum x_i^k y_j^k = F_k(x) F_k(y)$.

Now let $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$. Then

$$(\theta \Diamond F_k)(x) = \theta(M_x(F_k)) = \theta(F_k(x) F_k) = F_k(x)\theta(F_k).$$

So,

$$(\varphi \Diamond \theta)(F_k) = \varphi(F_k \theta(F_k)) = \varphi(F_k)\theta(F_k).$$

Exchanging parameters in (3.13) we get that

$$(\theta \Diamond \varphi)(F_k) = \theta(F_k)\varphi(F_k) = (\varphi \Diamond \theta)(F_k),$$

whence it follows that the multiplicative convolution is commutative for F_k. Since every symmetric polynomial is an algebraic combination of polynomials F_k and each function of $\mathcal{H}_{bs}(\ell_p)$ is uniformly approximated by symmetric polynomials, then the convolution operation is commutative. Analogously, $\Diamond$ is associative since

$$(\psi\Diamond(\varphi\Diamond\theta))\,(F_k) = \psi(F_k)\varphi(F_k)\theta(F_k) = ((\psi\Diamond\varphi)\Diamond\theta))(F_k).$$

Also from (3.13) it follows that the cancelation rule holds and δ_{x_0}, where $x_0 = (1,0,0,\ldots)$, is the identity. □

In [7] it was constructed a family $\{\psi_\lambda : \lambda \in \mathbb{C}\}$ of elements of the set $\mathcal{M}_{bs}(\ell_p)$ such that $\psi_\lambda(F_p) = \lambda$ and $\psi_\lambda(F_k) = 0$ for $k > p$. Let us recall the construction: Consider for each $n \in \mathbb{N}$, the element $v_n = \left(\frac{\lambda}{n}\right)^{1/p}(e_1 + \cdots + e_n)$ for which $F_p(v_n) = \lambda$, and $\lim_n F_j(v_n) = 0$ for $j > p$. Now, the sequence $\{\delta_{v_n}\}$ has an accumulation point ψ_λ in the spectrum for the pointwise convergence topology for which $\psi_\lambda(F_k) = 0$ for $k > p$ that prevents ψ_λ from being invertible because of (3.13).

Remark 3.43. The semigroup $(\mathcal{M}_{bs}(\ell_p), \Diamond)$ is not a group.

Recall that for any $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ and $f \in \mathcal{H}_{bs}(\ell_p)$, the symmetric convolution $\varphi \star \theta$ was defined in [6] as follows:

$$(\varphi \star \theta)(f) = \varphi(T^s_y(f)),$$

where $T^s_y(f)(x) = f(x \bullet y)$.

Proposition 3.44. *For arbitrary $\theta, \varphi, \psi \in \mathcal{M}_{bs}(\ell_p)$ the following equality holds:*

$$\theta\Diamond(\varphi \star \psi) = (\theta\Diamond\varphi) \star (\theta\Diamond\psi).$$

Proof. Indeed, using Theorem 3.42 and [7, Thm 1.5], we obtain that

$$\begin{aligned}((\theta\Diamond\varphi) \star (\theta\Diamond\psi))(F_k) &= (\theta\Diamond\varphi)(F_k) + (\theta\Diamond\psi)(F_k) = \theta(F_k)\varphi(F_k) + \theta(F_k)\psi(F_k) \\ &= \theta(F_k)(\varphi(F_k) + \psi(F_k)) = \theta(F_k)(\varphi \star \psi)(F_k) \\ &= \theta\Diamond(\varphi \star \psi)(F_k).\end{aligned}$$

□

Corollary 3.45. *The set $(\mathcal{M}_{bs}(\ell_p), \Diamond, \star)$ is a commutative semi-ring with identity.*

A linear operator $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is called a *multiplicative convolution operator* if there exists $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta\Diamond f$.

Proposition 3.46. *A continuous homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is a multiplicative convolution operator if and only if it commutes with all multiplicative operators M_y, $y \in \ell_p$.*

Proof. Suppose that there exists $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta\Diamond f$. Fix $y \in \ell_p$. Then

$$[T \circ M_y](f)(x) = [T(M_y(f))](x) = [\theta\Diamond M_y(f)](x) = \theta[M_x(M_y(f)] = \theta[M_{x\diamond y}(f)].$$

On the other hand,

$$[M_y \circ T](f)(x) = [M_y(Tf)](x) = Tf(x \diamond y) = (\theta\Diamond f)(x \diamond y) = \theta[M_{x\diamond y}(f)].$$

Conversely, for $x_0 = (1,0,0,\ldots)$ we put $\theta = \delta_{x_0} \circ T$. Clearly, $\theta \in \mathcal{M}_{bs}(\ell_p)$. Let us check that $Tf = \theta\Diamond f$. Indeed, $(\theta\Diamond f)(x) = \theta[M_x(f)] = [T(M_x(f))](x_0) = [M_x(T(f))](x_0) = Tf(x_0 \diamond x) = Tf(x)$. □

Theorem 3.47. *A homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ such that $T(F_k) = a_k F_k,\ \ k \geq \lceil p \rceil$, is continuous if and only if there exists $\varphi \in \mathcal{M}_{bs}(\ell_p)$ such that $\varphi(F_k) = a_k,\ \ k \geq \lceil p \rceil$.*

Proof. Let $\varphi \in \mathcal{M}_{bs}(\ell_p)$ with $\varphi(F_k) = a_k$. Then

$$(\varphi \Diamond F_k)(x) = \varphi(M_x(F_k)) = \varphi(F_k F_k(x)) = a_k F_k(x).$$

Thus if $Tf = \varphi \Diamond f$, T defines a continuous homomorphism and $T(F_k) = a_k F_k$.

Conversely, if such homomorphism T is continuous, then clearly T commutes with all M_y. By Proposition 3.46 it has the form $T(f) = \varphi \Diamond f$ for some $\varphi \in \mathcal{M}_{bs}(\ell_p)$. Thus, $T(F_k) = \varphi(F_k)F_k(x) = a_k F_k$, hence $\varphi(F_k) = a_k$. □

Proposition 3.48. *The identity is the only operator on $\mathcal{H}_{bs}(\ell_p)$ that is both a convolution and a multiplicative convolution operator.*

Proof. Let $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ be such an operator. Then there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$ and T commutes with all M_y. In particular we have for all polynomials $F_k,\ \ k \geq \lceil p \rceil$, that

$$M_y(TF_k) = M_y(\theta \star F_k) = M_y(\theta(F_k) + F_k) = \theta(F_k) + M_y(F_k) = \theta(F_k) + F_k(y)F_k \quad \text{and}$$

$$T(M_y(F_k)) = T(F_k(y)F_k) = F_k(y)\theta \star F_k = F_k(y)(\theta(F_k) + F_k) \quad \text{coincide.}$$

Hence $\theta(F_k) = F_k(y)\theta(F_k)$, that leads to $\theta(F_k) = 0$, that in turn shows that $\theta = \delta_0$, or in other words, $T = Id$. □

3.7. THE CASE OF ℓ_1 [8]

In this section we consider the algebra $\mathcal{H}_{bs}(\ell_1)$. In addition to the basis $\{F_n\}$, this algebra has a different natural basis that is given by the sequence $\{G_n\}$:

$$G_n(x) = \sum_{k_1 < \cdots < k_n}^{\infty} x_{k_1} \cdots x_{k_n}$$

and $G_0 := 1$.

According to [7] Lemma 3.1, $\|G_n\| = \frac{1}{n!}$, so it follows that for every $t \in \mathbb{C}$, the function $\sum_{n=0}^{\infty} t^n G_n \in \mathcal{H}_{bs}(\ell_1)$ and that such series converges uniformly on bounded subsets of ℓ_1. Thus if $\varphi \in \mathcal{M}_{bs}(\ell_1)$,

$$\mathcal{G}(\varphi)(t) = \varphi\Big(\sum_{n=0}^{\infty} t^n G_n\Big) = \sum_{n=0}^{\infty} t^n \varphi(G_n)$$

is well defined and as it was shown in [7, Proposition 3.2], the mapping

$$\varphi \in \mathcal{M}_{bs}(\ell_1) \xrightarrow{\mathcal{G}} \mathcal{G}(\varphi) \in H(\mathbb{C})$$

is one-to-one and ranges into the subspace of entire functions of exponential (finite) type. Whether $\mathcal{G}$ is an onto mapping was an open question there that we answer negatively here, see Corollary 3.52, using the multiplicative convolution we are dealing with.

Observe that for every $a \in \mathbb{C}$,

$$\Big(\delta_{(a,0,0,\ldots)} \Diamond \sum_{n=0}^{\infty} t^n G_n\Big)(x) = M_x(\sum_{n=0}^{\infty} t^n G_n)(a,0,0,\ldots) = (\sum_{n=0}^{\infty} t^n G_n)(x \diamond (a,0,0,\ldots))$$

$$= \sum_{n=0}^{\infty} t^n G_n(ax) = \sum_{n=0}^{\infty} t^n a^n G_n(x).$$

Therefore,

$$\mathcal{G}(\varphi \Diamond \delta_{(a,0,0,\dots)})(t) = \varphi(\sum_{n=0}^{\infty} t^n a^n G_n) = \sum_{n=0}^{\infty} t^n a^n \varphi(G_n).$$

According to [7, Theorem 1.6 (a)] , $\delta_{(a,0,0,\dots)} \star \delta_{(b,0,0,\dots)} = \delta_{(a,b,0,0,\dots)}$, consequently using Proposition 3.44 and [7, Theorem 3.3 (2)] ,

$$\mathcal{G}(\varphi \Diamond \delta_{(a,b,0,0,\dots)})(t) = \mathcal{G}\Big((\varphi \Diamond \delta_{(a,0,0,\dots)}) \star (\varphi \Diamond \delta_{(b,0,0,\dots)})\Big)(t) = \mathcal{G}(\varphi \Diamond \delta_{(a,0,0,\dots)})(t)\mathcal{G}(\varphi \Diamond \delta_{(b,0,0,\dots)})(t) =$$

$$\sum_{n=0}^{\infty} t^n a^n \varphi(G_n) \cdot \sum_{n=0}^{\infty} t^n b^n \varphi(G_n).$$

Therefore,

$$\mathcal{G}(\varphi \Diamond \delta_{(x_1,x_2,\dots,x_m,0,\dots)})(t) = \prod_{k=1}^{m} \sum_{n=0}^{\infty} t^n x_k^n \varphi(G_n).$$

Further since the sequence $\Big(\delta_{(x_1,x_2,\dots,x_m,0,\dots)}\Big)_m$ is pointwise convergent to $\delta_{(x_1,x_2,\dots,x_m\dots)}$ in $M_{bs}(\ell_1)$ we have, bearing in mind the commutativity of $\Diamond$, that the sequence $\Big(\varphi \Diamond \delta_{(x_1,x_2,\dots,x_m,0,\dots)}\Big)_m$ is pointwise convergent to $\varphi \Diamond \delta_{(x_1,x_2,\dots,x_m\dots)}$. Thus

$$\mathcal{G}(\varphi \Diamond \delta_x)(t) = \prod_{k=1}^{\infty} \sum_{n=0}^{\infty} t^n x_k^n \varphi(G_n) \quad \text{for } x = (x_1, x_2, \dots, x_m \dots) \in \ell_1. \tag{3.14}$$

For the mentioned above family $\{\psi_\lambda : \lambda \in \mathbb{C}\}$, it was shown in [7] that $\mathcal{G}(\psi_\lambda)(t) = e^{\lambda t}$. Further, it is easy to see that

(1) $\psi_\lambda \Diamond \varphi(F_1) = \lambda \varphi(F_1)$.
(2) $\psi_\lambda \Diamond \varphi(F_k) = 0, \quad k > 1$.
(3) $\mathcal{G}(\psi_\lambda \Diamond \varphi) = e^{\lambda \varphi(F_1) t}$.

The following theorem might be of interest in Function Theory.

Theorem 3.49. *Let $g(t)$ and $h(t)$ be entire functions of exponential type of one complex variable such that $g(0) = h(0) = 1$. Let $\{a_n\}$ be zeros of $g(t)$ with $\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$ and let $\{b_n\}$ be zeros of $h(t)$ with $\sum_{n=1}^{\infty} \frac{1}{|b_n|} < \infty$. Then there exists a function of exponential type $u(t)$ with zeros $\{a_n b_m\}_{n,m}$, which can be represented as*

$$u(t) = \prod_{k=1}^{\infty} \sum_{n=0}^{\infty} \Big(-\frac{1}{a_k}\Big)^n h_n(t) = \prod_{k=1}^{\infty} \sum_{n=0}^{\infty} \Big(-\frac{1}{b_k}\Big)^n g_n(t).$$

Proof. By [7], $g(t) = \mathcal{G}(\delta_x)(t)$ and $h(t) = \mathcal{G}(\delta_y)(t)$, where $x, y \in \ell_1$, $x_n = -\frac{1}{a_n}$, $y_n = -\frac{1}{b_n}$. So $u(t) = \mathcal{G}(\delta_x \Diamond \delta_y)(t)$ and using (3.14) we obtain the statement of the theorem. □

Let $x = (x_1, \dots, x_n, \dots)$ be a sequence of complex numbers such that $x \in \ell_{1+d}$ for every $d > 0$,

$$\limsup_{n\to\infty} n|x_n| < \infty, \qquad \limsup_{r\to\infty} \Big| \sum_{\frac{1}{|x_n|}<r} x_n \Big| < \infty \tag{3.15}$$

(think for instance of $x_n = \frac{(-1)^n}{n}$) and $\lambda \in \mathbb{C}$. Let us denote by $\delta_{(x,\lambda)}$ a homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ of the form

$$\delta_{(x,\lambda)}(F_1) = \lambda, \qquad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k, \quad k > 1.$$

Recall that according to [14, p. 17], $\limsup_{n\to\infty} n|x_n|$ coincides with the so-called *upper density* of the sequence $(\frac{1}{x_n})$ that is defined by $\limsup_{r\to\infty} \frac{\mathbf{n}(r)}{r}$, where $\mathbf{n}(r)$ denotes the *counting number* of $(\frac{1}{x_n})$, that is, the number of terms of the sequence with absolute value not greater than r.

Proposition 3.50. [7, Proposition 3.9] *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then the restriction of φ to $\mathcal{P}_s(\ell_1)$ coincides with $\delta_{(x,\lambda)}$ for some $\lambda \in \mathbb{C}$ and x satisfying (3.15).*

Actually, thanks to [1, Theorem 1.3] such sequence x is unique up to permutation.

Theorem 3.51. *There is no continuous character of the form $\delta_{(v,\lambda)}$ in the space $\mathcal{M}_{bs}(\ell_1)$, where*

$$v = \left\{c_1, \frac{c_2}{2}, \dots, \frac{c_n}{n}, \dots\right\},$$

and $|c_k| = 1$ for each k.

Proof. Assume otherwise, i.e., $\delta_{(v,\lambda)}$ is the restriction of some $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then by (3.13),

$$(\varphi\Diamond\varphi)(F_k) = \varphi(F_k)^2 = \left(\sum_{n=1}^{\infty} v_n^k\right)^2 = \left(\sum_{n=1}^{\infty} v_n^k\right)\left(\sum_{m=1}^{\infty} v_m^k\right) = \sum_{n,m=1}^{\infty} (v_n v_m)^k.$$

Therefore the sequence $(v_n v_m)_{n,m} = v \diamond v := s$, is, up to permutation, the one appearing in Proposition 3.50, so it must satisfy condition (3.15), that is, the sequence of the inverses has finite upper density.

Denote by $d(m)$ the number of divisors of a positive integer m. Then in the sequence $|s|$ of absolute values each number with absolute value $1/m$ can be found $d(m)$ times. So $|s|$ can be rearranged, if necessary, in the form

$$\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots, \underbrace{\frac{1}{m}, \dots \frac{1}{m}}_{d(m)}, \dots\right).$$

In particular, the index of the last entry of the element with absolute value $\frac{1}{m}$ is $\sum_{n=1}^{m} d(n)$. Hence for the sequence of the inverses and their counting number $\mathbf{n}(m)$, we have $\mathbf{n}(m) = \sum_{n=1}^{m} d(n)$. From Number Theory [2, Theorem 3.3] it is known that

$$\sum_{n=1}^{m} d(n) = m \ln m + 2(\gamma - 1)m + O(\sqrt{m}),$$

where γ is the Euler constant. So we are led to a contradiction because

$$\limsup_{m\to\infty} \frac{\mathbf{n}(m)}{m} \geq \limsup_{m\to\infty} \frac{m \ln m}{m} = \limsup_{m\to\infty} \ln m = \infty.$$

□

Corollary 3.52. *There is a function of exponential type $g(t)$ for which there is no character $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\mathcal{G}(\varphi)(t) = g(t)$.*

Proof. It is enough to take a function of exponential (finite) type whose zeros are the elements of the sequence

$$\left\{\frac{1}{v_n}\right\} = \{-1, 2, \dots (-1)^n n, \dots\}.$$

Such is, for example, the function

$$g(t) = \prod_1^\infty \left(1 + (-1)^n \frac{t}{n}\right) \exp\left((-1)^n \frac{t}{n}\right).$$

□

Every $\varphi \in \mathcal{M}_{bs}(\ell_1)$ is determined by the sequence $(\varphi(F_m))$, that verifies the inequality $\limsup_n |\varphi(F_m)|^{1/m} \leq R(\varphi)$ because $\|F_m\| \leq 1$. As a byproduct of Theorem 3.51, we notice that the condition $\limsup_m |a_m|^{1/m} < +\infty$, does not guarantee that there is $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\varphi(F_m) = a_m$: Indeed, let $a_m = \sum_n \frac{1}{n^m}$ for $m > 1$ and arbitrary a_1. Then the sequence (a_m) is bounded, so $\limsup_m |a_m|^{1/m} \leq 1$, and if there existed $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\varphi(F_m) = a_m$, it would mean that for the sequence $x := (\frac{1}{n})$, $\varphi(F_m) = \sum_n \frac{1}{n^m}$, so $\delta_{(x,a_1)} = \varphi_{|_{P_s(\ell_1)}}$.

Question 3.53. Can each element of $\mathcal{M}_{bs}(\ell_1)$ be represented as an entire function of exponential type with zeros $\{a_n\}_{n=1}^\infty$ such that either $\{a_n\} = \emptyset$ or $\sum_{n=1}^\infty \frac{1}{|a_n|} < \infty$?

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Стаття містить огляд основних результатів про спектри алгебр симетричних голоморфних функцій і алгебр симетричних аналітичних функцій обмеженого типу на банахових просторах.

Ключові слова: поліноми і аналітичні функції на банахових просторах, симетричні поліноми, спектри алгебр.