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Research on the convergence of some types of functional branched continued fractions

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An analysis of research on the problem of convergence of various types of functional branched continued fractions has been carried out. Branched continued fractions with N branching branches and branched continued fractions with independent variables are considered. The definition and, in our opinion, characteristic criteria of convergence of multidimensional generalizations of C-, S-, g-, J-fractions are given, both for branched continued fractions of the general form with N branching branches and branched continued fractions with independent variables. Such multidimensional generalizations of continued fractions arise, in particular, in the development of various classes of hypergeometric functions of several variables, in particular, the functions of Appel, Lauricella, Horn, etc.

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Introduction

In the analytical theory of branched continued fractions, by analogy with continued fractions [48, 50, 55], various types of functional branched continued fractions (BCF) are considered. They are used to construct rational approximations of various classes of analytical functions of several variables. To obtain these expansions, the principle of correspondence between multiple power series and functional BCF is used [7, 17, 19, 23, 27, 31, 34, 40, 49, 51, 52], as well as some specific properties of the functions expressed in the presence of certain recurrence relations. This is characteristic of hypergeometric functions, in particular the functions of Appel, Lauricella, Horn, etc. The latter approach is the most effective for constructing the expansions of these functions or their ratios in BCF [1–5, 20–23, 42, 46]. It is here that original, non-standard constructions of BCF often arise, which are the subject of separate studies (see, for example, [2]).

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1 Classification of functional continued fractions

A functional continued fraction is a continued fraction of the form

$$b_0(z) + \cfrac{a_1(z)}{b_1(z) + \cfrac{a_2(z)}{b_2(z) + \ddots}}, \tag{1}$$

where $b_j(z)$, $j \geq 0$, $a_i(z)$, $i \geq 1$, are some functions defined in the domain G, $G \subset \mathbb{C}$. For convenience, we will denote the same continued fraction (1) with the symbol

$$b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \cdots$$

or

$$b_0(z) + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k(z)}{b_k(z)}.$$

Most often, the elements of the continued fraction (1) are polynomials.

In the theory of continued fractions, different types of functional continued fractions are considered. These include the following ones (see [48,50,55]).

Regular C-fractions are continued fractions of the form

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \frac{c_k z}{1}, \tag{2}$$

where $c_k \in \mathbb{C} \setminus \{0\}$, $k \geq 1$, $z \in \mathbb{C}$.

S-fractions are of the form (2), where all $c_k > 0$, $k \geq 1$.

g-fractions are continued fractions of the form

$$s_0 \left(1 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{g_k(1-g_{k-1})z}{1} \right)^{-1}, \tag{3}$$

where $s_0 > 0$, $0 < g_k < 1$, $k \geq 1$, $g_0 = 0$, $z \in \mathbb{C}$.

J-fractions are of the form

$$\frac{1}{b_1 + z} - \frac{a_1^2}{b_1 + z} - \frac{a_2^2}{b_2 + z} - \cdots, \tag{4}$$

where b_n, a_n, $n \geq 1$, are complex constants with $a_n \neq 0$, $n \geq 1$, $z \in \mathbb{C}$.

Note that the g-fractions are the most studied. The continued fraction (3) converges in the domain $|\arg(z+1)| < \pi$ to the function $f(z)$ holomorphic in this domain. H.S. Wall [55] established that every non-rational function $f(z)$, analytic in the domain $|\arg(z+1)| < \pi$ and such that $\mathrm{Re}(\sqrt{1+z}f(z)) > 0$ (here the principal branch of the square root is assumed), is represented by a g-fraction of the form (3). W.B. Gragg [45] gave an estimate of the rate of convergence of such continued fractions. At the same time, he significantly used the π-fraction introduced by him. The even part of a π-fraction is g-fraction.

At an early stage of the development of the theory of g-fractions, the question of convergence of continued fractions of the form

$$\frac{g_1}{1} + \frac{(1-g_1)g_2x_2}{1} + \frac{(1-g_2)g_3x_3}{1} + \cdots, \tag{5}$$

and

$$\frac{1}{1+}\frac{g_1x_1}{1}\,\frac{}{+}\,\frac{(1-g_1)g_2x_2}{1}\,\frac{}{+}\,\frac{(1-g_2)g_3x_3}{1}\,\frac{}{+}\,\ldots, \tag{6}$$

where $x_k \in \mathbb{C}$, $k \geq 1$, are complex independent variables, $0 \leq g_k < 1$, $k \geq 1$, or $0 < g_k \leq 1$, $k \geq 1$, was investigated in the works of W.T. Scott and H.S. Wall, J.F. Peydon and H.S. Wall, E.B. Van Vleck, O. Perron, A. Pringsheim and H. Koch (see references and analysis of these results in [55]).

Theorem of Stieltjes [54] is a classic criteria of the convergence of S-fractions.

Theorem 1. *Let* (2) *be an S-fraction, where all* $c_k > 0$, $k \geq 1$. *Then:*

(A) the sequences of even $\{f_{2n}\}$ *and odd* $\{f_{2n+1}\}$ *approximants converge to functions holomorphic in the domain*

$$G = \{z \in \mathbb{C} : \ |\arg z| < \pi\} \tag{7}$$

and, in addition, converge uniformly on every compact subset of (7)*;*

(B) the S-fraction (2) *converges to function holomorphic in G if and only if the series* $\sum_{k=1}^{\infty} d_k$ *diverges, where* $c_k = 1/(d_{k-1}d_k)$, $k \geq 1$, $d_0 = 1$*;*

(C) if the S-fraction (2) *converges at a single point in G, then it converges at all points in G to a holomorphic function in G;*

(D) if there exists a constant $M > 0$ *such that* $|c_k| \leq M$, $k \geq 1$, *then the S-fraction* (2) *converges to function holomorphic in G.*

For C-fractions, the convergence criteria of Van Vleck, when the limit $\lim_{n\to\infty} c_n = c$ exists [48, 50, 55], as well as cardioid theorem, when c_k, $k \geq 1$, is taken from the parabolic domain, and the variable z belongs to the cardioid [55], are characteristic.

Close to J-fractions are the so-called *positive definite continued fractions*

$$\frac{1}{b_1 + z_1 + \underset{k=2}{\overset{\infty}{\mathrm{D}}} \frac{a_{k-1}^2}{b_k + z_k}}, \tag{8}$$

where a_k, b_k, $k \geq 1$, are complex constants, z_k, $k \geq 1$, are complex variables. A continued fraction (8) is called *positive definite* if the quadratic forms

$$\sum_{k=1}^{n} \beta_k \xi_k^2 - 2\sum_{k=1}^{n-1} \alpha_k \xi_k \xi_{k-1}, \quad n \geq 2,$$

where $\beta_k = \operatorname{Im}(b_k)$, $\alpha_k = \operatorname{Im}(a_k)$, $k \geq 1$, are non-negative definite.

A detailed analysis of positive definite fractions and related J-fractions is made in book [55]. In particular, bounded J-fractions are studied, when there is a constant M such that $|a_k| \leq M/3$, $|b_k| \leq M/3$, $k \geq 1$. Then the J-fraction converges in the region $|z| > M$. A J-fraction is said to be real if a_k, b_k, $k \geq 1$, are real constants. All zeros of the denominators of the approximants of a real J-fraction are real. If such a continued fraction converges at a point z_0, $z_0 \notin \mathbb{R}$, then it converges in the domain $\operatorname{Im} z > 0$ or $\operatorname{Im} z < 0$ to the holomorphic function in this domain [55].

2 Some types of functional branched continued fractions

Branched continued fractions are analogous to continued fractions for functions of several variables. Their research was initiated by V.Ya. Skorobohatko [53], and was developed in the works of his students P.I. Bodnarchuk, D.I. Bodnar, M.O. Nedashkovskyi, M.S. Siavavko, Kh.Yo. Kuchminska and their students.

Let N be a fixed natural number and

$$I_k = \{i(k) = (i_1, i_2, \dots, i_k) : \ 1 \le i_p \le N, \ p = \overline{1,k}\}, \quad k \ge 1, \quad I = \bigcup_{k=1}^{\infty} I_k,$$

be the sets of multiindices.

Let $\mathbf{z} = (z_1, z_2, \dots, z_N) \in \mathbf{C}^N$. If the functions $b_0(\mathbf{z})$, $a_{i(k)}(\mathbf{z})$, $b_{i(k)}(\mathbf{z})$, $i(k) \in I$, are defined in some domain $D \subset \mathbb{C}^N$, then the *functional BCF with N branching branches* is called the BCF of the form

$$b_0(\mathbf{z}) + \sum_{i_1=1}^{N} \cfrac{a_{i(1)}(\mathbf{z})}{b_{i(1)}(\mathbf{z}) + \sum\limits_{i_2=1}^{N} \cfrac{a_{i(2)}(\mathbf{z})}{b_{i(2)}(\mathbf{z}) + \ddots}},$$

for the compact notation of which we will use the symbols

$$b_0(\mathbf{z}) + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}(\mathbf{z})}{b_{i(k)}(\mathbf{z})} \tag{9}$$

and

$$b_0(\mathbf{z}) + \sum_{i_1=1}^{N} \frac{a_{i(1)}(\mathbf{z})}{b_{i(1)}(\mathbf{z})} + \sum_{i_2=1}^{N} \frac{a_{i(2)}(\mathbf{z})}{b_{i(2)}(\mathbf{z})} + \cdots.$$

Let

$$J_k = \{i(k) = (i_1, i_2, \dots, i_k) : \ 1 \le i_k \le i_{k-1} \le i_0, \ i_0 = N\}, \quad k \ge 1, \quad J = \bigcup_{k=1}^{\infty} J_k,$$

be the sets of multiindices.

If the functions $b_0(\mathbf{z})$, $a_{i(k)}(\mathbf{z})$, $b_{i(k)}(\mathbf{z})$, $i(k) \in I$, are defined in some domain $G \subset \mathbb{C}^N$, then the BCF with independent variables is called the BCF of the form

$$b_0(\mathbf{z}) + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}(\mathbf{z})}{b_{i(k)}(\mathbf{z})}. \tag{10}$$

Let us consider the criteria of convergence of some types of functional BCF with N branching branches and functional BCF with independent variables.

The following approach is used to study the convergence of functional BCF types. There are considered two sets: the coefficients are taken from one set and the variables are taken from the other. Such sets are parabolic, circular, angular regions or their subsets. At the same time, criteria of convergence of numerical BCF or their modifications are used [6,8,9,11–15,33].

As in the case of continued fractions, the most studied are multidimensional g-fractions. The first, in particular, were the subject of R.I. Dmytryshyn's PhD thesis [40].

A *multidimensional g-fraction with N branching branches* is called a functional BCF of the form

$$s_0\left(1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{g_{i(k)}(1-g_{i(k-1)})z_{i_k}}{1}\right)^{-1}, \tag{11}$$

where $s_0>0$, $0<g_{i(k)}<1$, $i(k)\in I$, $g_{i(0)}=g_0=0$, $\mathbf{z}\in\mathbb{C}^N$.

At the first stage of study of BCF (11) by analogy with continued fractions (5) and (6), the independent variables z_{i_k}, $1\le k\le N$, in (11) were replaced by independent variables $z_{i(k)}$, $i(k)\in I$. Multidimensional analogs of theorems of A. Pringsheim, H. Koch, E.B. Van Vleck, O. Perron, H.S. Wall [10,15,18,40,41] were established, multidimensional chain sequences were investigated. R.I. Dmytryshyn defined and applied *multidimensional π-fractions*

$$\frac{\pi_0}{1+\sum_{i=1}^N z_i}-\sum_{i_1=1}^{N}\frac{z_{i_1}}{1}+\frac{\pi_{i(1)}}{1+\sum_{i=1}^N z_i}-\sum_{i_2=1}^{N}\frac{z_{i_2}}{1}+\frac{\pi_{i(2)}}{1+\sum_{i=1}^N z_i}-\cdots,$$

where $\pi_{i(k)}>0$, $i(k)\in I$, $\sum_{i=1}^N z_i\neq -1$, $\mathbf{z}\in\mathbb{C}^N$.

It is established that the even part of a multidimensional π-fraction is a multidimensional g-fraction.

Theorem 2 ([18]). *The multidimensional g-fraction* (11) *converges to a function $f(z)$ holomorphic in the domain*

$$Q=\left\{\mathbf{z}\in\mathbb{C}^N:\ \sum_{i=1}^{N}|z_i|<1\right\}$$

and, in addition, the following estimates

$$|f(z)-f_n(z)|\le\frac{s_0(\sum_{i=1}^N|z_i|)^n}{1-T\sum_{i=1}^N|z_i|},\quad n\ge 2,$$

of the truncation error hold, where $f_n(z)$ is the nth approximant of (11)*, and*

$$T=\sup\left\{1-\left(1+\sum_{r=n}^{\infty}\prod_{k=n}^{r}\mu_k\right)^{-1}\right\},\quad \mu_k=\max\left\{\frac{g_{i(k)}}{1-g_{i(k)}},\ i(k)\in I_k\right\},\quad k\ge 1.$$

Theorem 3 ([41]). *The multidimensional g-fraction* (11) *converges to a function holomorphic in the domain*

$$P=\bigcup_{\alpha\in(-\pi/2,\pi/2)}P_\alpha, \tag{12}$$

where

$$P_\alpha=\left\{\mathbf{z}\in\mathbb{C}^N:\ \sum_{n=1}^{N}|z_n|-\mathrm{Re}(z_ne^{-2i\alpha})<2\cos^2\alpha\right\}.$$

In the work [41], an example of the expansion of function

$$\frac{\ln(1+z_1+z_2)}{z_1+z_2}$$

into a two-dimensional g-fraction of the form (11) is constructed.

A *multidimensional g-fraction with independent variables* is called a BCF of the form

$$s_0\left(1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{g_{i(k)}(1-g_{i(k-1)})z_{i_k}}{1}\right)^{-1}, \tag{13}$$

where $s_0>0$, $0<g_{i(k)}<1$, $i(k)\in J$, $g_{i(0)}=g_0=0$, $\mathbf{z}\in\mathbb{C}^N$, and a *multidimensional π-fraction with independent variables* is a BCF of the form

$$\frac{\pi_0}{1+\sum_{i_1=1}^{N}z_{i_1}}+\sum_{i_1=1}^{N}\frac{-z_{i_1}}{1}+\frac{\pi_{i(1)}}{1+\sum_{i_2=1}^{i_1}z_{i_1}}+\sum_{i_2=1}^{i_1}\frac{-z_{i_2}}{1}+\frac{\pi_{i(2)}}{1+\sum_{i_3=1}^{i_2}z_{i_3}}+\ldots,$$

where $\pi_{i(k)}>0$, $i(k)\in J$.

BCF (13) is an even part of this fraction and

$$g_{i(k)}=\frac{\pi_{i(k)}}{1+\pi_{i(k)}},\quad i(k)\in J_k,\ k\geq 1.$$

Theorem 4 ([36]). *The multidimensional g-fraction* (13) *converges to a function holomorphic in the domain* (12) *and, in addition, converges uniformly on every compact subset of this domain.*

For BCF (13), multidimensional generalizations of theorems of W.T. Scott and H.S. Wall, J.F. Peydon and H.S. Wall, E.B. Van Vleck, O. Perron, A. Pringsheim and H. Koch have also been established, estimates of the rate of convergence on some bounded subsets of P_α have been found [10, 36, 41].

In the work [37], an example of the expansion of function of two variables

$$\frac{\ln(1+(z_2/z_1^2)\ln^2(1+z_1)}{(z_2/z_1)\ln(1+z_1)}$$

into a two-dimensional g-fraction with independent variables is constructed.

A *multidimensional regular C-fraction with N branching branches* is called BCF

$$b_0+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{c_{i(k)}z_{i_k}}{1}, \tag{14}$$

or its reciprocal, where $c_{i(k)}\in\mathbb{C}$, $c_{i(k)}\neq 0$, $i(k)\in I$, $\mathbf{z}\in\mathbb{C}^N$.

Let $\beta_k=\max\{|c_{i(k)}|:\ i(k)\in I_k\}$, $k\geq 1$.

Theorem 5 ([15]). *If for the multidimensional regular C-fraction* (14) $\lim_{k\to\infty}\beta_k=0$, *then for an arbitrary positive M there exists a number n dependent on M such that the tails of the BCF* (14), *namely*

$$\left(1+\mathop{\mathrm{D}}_{k=n}^{\infty}\sum_{i_k=1}^{N}\frac{c_{i(k)}z_{i_k}}{1}\right)^{-1},\quad i(n-1)\in I_{n-1}, \tag{15}$$

converge uniformly to a function holomorphic in the polydisc $\{|z_k|<M:\ k=\overline{1,N}\}$.

Theorem 6 ([15]). *Let* (14) *be a multidimensional regular C-fraction such that each set* $A_m = \{c_{i(k),m} : i(k) \in I\}$ *has one limit point* α_m*, that is,* $\lim_{k\to\infty} c_{i(k),m} = \alpha_m$, $m = \overline{1,N}$, *and let* $P = P_1 \times P_2 \times \ldots P_N$*, where*

$$P_m = \left\{ w \in \mathbb{C} : \ |w| - \mathrm{Re}(w \exp(\arg \alpha_m - 2\gamma)) < \frac{\cos^2 \gamma}{2N|\alpha_m|} \right\}, \quad m = \overline{1,N},$$

γ *is an arbitrary number such that* $|\gamma| < \pi/2$*. Then for an arbitrary compact* K, $K \subset P$*, there exists a number* n*, which depends on* K*, such that the BCF* (14) *converges uniformly on* K *to a function holomorphic in some region* Ω *such that* $K \subset \Omega \subset P$.

A BCF of the form

$$\left(b_0 + z_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{-a_{i(k)}^2}{b_{i(k)} + z_{i_k}} \right)^{-1}, \tag{16}$$

where $b_0, a_{i(k)}, b_{i(k)} \in \mathbb{C}$, $i(k) \in I$, $\mathbf{z}' = (z_0, z_1, z_2, \ldots, z_N) \in \mathbb{C}^{N+1}$, is called a *multidimensional J-fraction with N branching branches*.

Let $\beta_{i(k)} = \mathrm{Im}\, b_{i(k)}$, $\alpha_{i(k)} = \mathrm{Im}\, a_{i(k)}$, $i(k) \in I$. BCF (16) is positive definite if the quadratic forms

$$\sum_{k=0}^{n} \sum_{i_1, i_2, \ldots, i_k = 1}^{N} \beta_{i(k)} \xi_{i(k)}^2 - \sum_{k=1}^{n} \sum_{i_1, i_2, \ldots, i_k = 1}^{N} \alpha_{i(k)} \xi_{i(k)} \xi_{i(k-1)},$$

where $\xi_{i(0)} = \xi_0$, are non-negative definite.

If (16) is positive definite, then its denominators $B_n(\mathbf{z}')$, $n \geq 1$, are different from zero in the domain $\mathrm{Im}\, z_k > 0$, $k = \overline{0,N}$, that is, the approximants take finite values.

If there is a real positive constant M such that

$$|a_{i(k)}| \leq \frac{M}{1 + 2\sqrt{N}}, \quad |b_{i(k)}| \leq \frac{M}{1 + 2\sqrt{N}}, \quad i(k) \in I,$$

then the BCF (16) is called bounded, and the minimum possible number M is called the limit of the multidimensional J-fraction (16).

Theorem 7 ([15,16]). *The multidimensional bounded J-fraction* (16) *with the limit* M *converges uniformly if* $|z_k| \geq M$, $k = \overline{0,N}$.

A multidimensional J-fraction (16) is called real if $b_0, a_{i(k)}, b_{i(k)}$, $i(k) \in I$, are real numbers.

In the multidimensional real J-fraction (16), all approximants of $f_n(\mathbf{z}')$, $n \geq 1$, are functions holomorphic in the domain $\mathrm{Im}\, z_k > 0$, $k = \overline{0,N}$, or $\mathrm{Im}\, z_k < 0$, $k = \overline{0,N}$. In addition, it is positive definite.

Theorem 8 ([15, 16]). *The multidimensional real J-fraction* (16) *converges uniformly on every compact subset* $\mathbb{C}^{N+1}$ *whose distance to the set*

$$\mathrm{Im}\, z_k = 0, \quad |\,\mathrm{Re}\, z_k| \leq \frac{M}{1 + 2\sqrt{N}}, \quad k = \overline{0,N},$$

is positive.

Theorem 9 ([15]). *BCF*
$$1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{-a_{i(k)}^2 z_{i_{k-1}}z_{i_k}}{1},$$
where $a_{i(k)}\in\mathbb{R}$, $i(k)\in I$, *such that*
$$a_{i(k)}^2\le\frac{1}{N}(1-g_{i(k-1)})g_{i(k)},\quad i(k)\in I,$$
where $g_{i(0)}=0$, $0\le g_{i(k)}\le 1$, $i(k)\in I$, *converges uniformly on every compact set whose distance to the set* $\operatorname{Im} z_i=0$, $|\operatorname{Re} z_i|\ge 1$, $i=\overline{0,N}$, *is positive.*

A *multidimensional regular C-fraction with independent variables* is called BCF
$$\sum_{i_1=1}^{N}\frac{c_{i(1)}z_{i_1}}{1}+\mathop{\mathrm{D}}_{k=2}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{c_{i(k)}z_{i_k}}{1},\tag{17}$$
where $c_{i(k)}\in\mathbb{C}\setminus\{0\}$, $i(k)\in J$.

For the BCF (17), in particular, the conditions of convergence, when the coefficients $c_{i(k)}$, $i(k)\in J$, lie on the same beam, were investigated.

Theorem 10 ([34]). *Let the coefficients* $c_{i(k)}$, $i(k)\in J_k$, $k\ge 2$, *of BCF* (17) *lie on the same beam* $\arg c_{i(k)}=\varphi$, $i(k)\in J_k$, $k\ge 2$, $-\pi<\varphi<\pi$, *and*
$$\sum_{i_{k+1}=1}^{i_k}\frac{l_{i_{k+1}}|c_{i(k+1)}|}{(1-g_{i(k+1)})(1+\cos\varphi)}\le g_{i(k)},\quad i(k)\in J_k,\ k\ge 1,$$
where l_k, $1\le k\le N$, *are the positive numbers,* $0<g_{i(k)}<1$, $i(k)\in J$. *Then BCF* (17) *converges to a function holomorphic in the domain*
$$D_{l(N),M}=\Big\{\mathbf{z}\in\mathbb{C}^N:\ |z_k|-\operatorname{Re} z_k<l_k,\ |z_k|<M,\ 1\le k\le N\Big\}$$
for each constant $M>0$, *and, in addition, converges uniformly on every compact subset of* $D_{l(N),M}$.

Theorem 11 ([34]). *Let the coefficients* $c_{i(k)}$, $i(k)\in J_k$, $k\ge 2$, *of BCF* (17) *lie on the same beam* $\arg c_{i(k)}=\varphi$, $i(k)\in J_k$, $k\ge 2$, $-\pi<\varphi<\pi$, *and*
$$|c_{i(k)}|\le g_{i(k-1)}(1-g_{i(k)})\frac{1+\cos\varphi}{l_{i_k}},\quad i(k)\in J_k,\ k\ge 2,$$
where l_k, $1\le k\le N$, *are the positive numbers,* $0<g_{i(k)}<1$, $i(k)\in J$. *Then BCF* (17) *converges to a function holomorphic in the domain*
$$O_{l(N),M}=\Big\{\mathbf{z}\in\mathbb{C}^N:\ \sum_{k=1}^{N}\frac{|z_k|-\operatorname{Re} z_k}{l_k}<1,\ \sum_{k=1}^{N}|z_k|<M\Big\}$$
for each constant $M>0$, *and, in addition, converges uniformly on every compact subset of* $O_{l(N),M}$.

Let us consider the multidimensional S-fraction with independent variables (17), where $c_{i(k)} > 0$, $i(k) \in J_k$, $k \geq 2$, that is, $c_{i(k)}$, $i(k) \in J_k$, $k \geq 2$, lie on the same beam $\arg c_{i(k)} = 0$, $i(k) \in J_k$, $k \geq 2$.

Theorem 12 ([25, 26, 35]). *Let the coefficients $c_{i(k)}$, $i(k) \in J_k$, $k \geq 2$, of the multidimensional S-fraction with independent variables* (17) *satisfy the following conditions*

$$\sum_{i_k=1}^{i_{k-1}} c_{i(k)} l_{i_k} \leq 1, \quad i(k-1) \in J_{k-1},\ k \geq 2,$$

where l_k, $1 \leq k \leq N$, are the positive numbers. Then the multidimensional S-fraction with independent variables (17) *converges to a function holomorphic in the domain*

$$P_{l(N),M} = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N : \ \frac{|z_k| - \operatorname{Re}(z_k e^{-2i\alpha})}{l_k \cos^2 \alpha} < \frac{1}{2},\ |z_k| < M,\ 1 \leq k \leq N \right\}$$

for each constant $M > 0$, and, in addition, converges uniformly on every compact subset of $P_{l(N),M}$.

In the case $N = 1$, when the multidimensional S-fraction with independent variables is transformed into simple S-fraction, it follows from this theorem that if $c_k l \leq 1$, $k \geq 2$, where l is a positive constant, then the S-fraction (2) converges to a function holomorphic in the domain

$$H_{l,M} = \left\{ z \in \mathbb{C} : \ \left| \arg \left(z + \frac{l}{4} \right) \right| < \pi,\ |z| < M \right\}$$

for each constant $M > 0$. Comparing this result with Theorem 1, we see that a new convergence criterion for S-fraction (2) is obtained.

Theorem 13 ([25, 26, 35]). *Let the coefficients $c_{i(k)}$, $i(k) \in J_k$, $k \geq 2$, of the multidimensional S-fraction with independent variables* (17) *satisfy the following conditions*

$$c_{i(k)} \leq q_{i(k)}^{i_k} q_{i(k-1)}^{i_{k-1}} (1 - q_{i(k-1)}), \quad i(k) \in J_k,\ k \geq 2,$$

where $0 < q_{i(k)} < 1$, $i(k) \in J_k$, $k \geq 1$. Then BCF (17) *converges to a function holomorphic in the domain*

$$P_M = \bigcup_{\alpha \in (-\pi/2, \pi/2)} P_{M,\alpha},$$

where

$$P_{M,\alpha} = \{ \mathbf{z} \in \mathbb{C}^N : \ |z_k| - \operatorname{Re} z_k e^{-2i\alpha} < 2 \cos^2 \alpha,\ |z_k| < M,\ 1 \leq k \leq N \},$$

and, in addition, converges uniformly on every compact subset of P_M.

Estimates of the rate of convergence have been established for multidimensional S-fractions with independent variables.

For the multidimensional J-fraction with independent variables

$$\left(b_0 + z_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)} + z_{i_k}} \right)^{-1}, \tag{18}$$

where b_0, $a_{i(k)}$, $b_{i(k)}$, $i(k) \in J$, are complex constants, a multidimensional analogue of the Tron's parabolic theorem is established.

Theorem 14 ([8]). *Let the coefficients of BCF* (18) *satisfy the conditions* $a_{i(k)} \in P_{i_{k-1}}(a,\alpha)$, $b_0, b_{i(k)} \in H(b,\alpha)$, $i(k) \in J$, *where*

$$P_{i_{k-1}}(a,\alpha) = \left\{ w \in \mathbb{C}: \ |w| - \operatorname{Re}(we^{-2i\alpha}) \le \frac{a^2\cos^2\alpha}{2i_{k-1}}, \ |w| \le M \right\}$$
$$H(b,\alpha) = \{ w \in \mathbb{C}: \ \operatorname{Re}(we^{-i\alpha}) \ge b\cos\alpha \},$$

where $i(k) \in J$, $a > 0$, $b \in \mathbb{R}$, $M > 0$, $|\alpha| < \pi/2$. *Then BCF* (18) *converges to a function holomorphic in the domain*

$$D = \{ \mathbf{z}' \in \mathbb{C}^{N+1}: \ z_s \in H_0(a-b,\alpha), \ s = \overline{0,N} \},$$

where

$$H_0(a-b,\alpha) = \{ w \in \mathbb{C}: \ \operatorname{Re}(we^{-i\alpha}) > (a-b)\cos\alpha \},$$

and, in addition, converges uniformly on every compact subset of D.

Papers [32,33,38,39] investigate positive definite BCF with independent variables, multidimensional J-fractions with independent variables of the form

$$\sum_{i_1=1}^{N} \frac{-c_{i(1)}^2}{d_{i(1)} + z_{i_1}} + \mathop{\mathrm{D}}_{k=2}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{-c_{i(k)}^2}{d_{i(k)} + z_{i_k}}, \tag{19}$$

where $c_{i(k)} \in \mathbb{C} \setminus \{0\}$, $d_{i(k)} \in \mathbb{C}$, $i(k) \in J$.

Theorem 15 ([39]). *Let the coefficients* $c_{i(k)}$, $i(k) \in J$, *of multidimensional J-fractions with independent variables* (19) *satisfy inequalities*

$$\sum_{i_1=1}^{N} \frac{(\operatorname{Im} c_{i(1)})^2}{l_{i_1} g_0 (1 - g_{i(1)})} \le 1 - \varepsilon, \quad \sum_{i_k=1}^{i_{k-1}} \frac{(\operatorname{Im} c_{i(k)})^2}{l_{i_{k-1}} l_{i_k} g_{i(k-1)} (1 - g_{i(k)})} \le 1 - \varepsilon, \quad i(k) \in J_k, \ k \ge 2,$$

where $0 < \varepsilon < 1$, l_k, $1 \le k \le N$, *are the positive numbers,* $g_0 \ge 0$, $0 \le g_{i(k)} < 1 - \varepsilon$, $i(k) \in J$, *and the coefficients* $d_{i(k)}$, $i(k) \in J$, *such that* $\operatorname{Re} d_{i(k)} = 0$, $\operatorname{Im} d_{i(k)} \ge 0$, $i(k) \in J$. *Then:*

(A) for all $\mathbf{z}$ *from the set*

$$P_{i(N),d,\varepsilon} = \left\{ \mathbf{z} \in \mathbb{C}^N : \frac{l_k}{\operatorname{Re}(d - iz_k)} < 1, \ |\arg(d - iz_k)| < \frac{\pi}{2(1+\varepsilon)}, \ 1 \le k \le N \right\},$$

where $d = \inf\{\operatorname{Im} d_{i(k)} : i(k) \in J\}$, *the sequences of even and odd approximants of* (19) *converge to holomorphic functions* $g(\mathbf{z})$ *and* $h(\mathbf{z})$ *and, in addition, converge uniformly on every compact subset of* $\operatorname{Int} P_{i(N),d,\varepsilon}$;

(B) BCF (19) *converges to holomorphic function* $f(\mathbf{z})$ *in the domain* $\operatorname{Int} P_{i(N),d,\varepsilon}$ *if series*

$$\sum_{k=1}^{\infty} \max\left(\frac{1}{|c_{i(k)}^2|}, \ i(k) \in J \right)$$

diverges, and, in addition, converge uniformly on every compact subset of $\operatorname{Int} P_{i(N),d,\varepsilon}$.

Some types of functional branched continued fractions with N branching branches and branched continued fractions with independent variables, which, in particular, arise when special functions are represented [1–5, 28–30, 43, 44, 47], are considered. An analysis of the study of the convergence of such fractions was conducted. We do not claim the completeness of the results of such studies obtained by various authors. The conditions of convergence of different types of BCF are formulated, which, from our point of view, correspond to certain trends and directions of development of the analytical theory of BCF.

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Боднар Д.І., Боднар О.С., Дмитришин М.В., Попов М.М., Марцінків М.В., Саламаха О.Б. *Дослідження збіжності деяких типів функціональних гіллястих ланцюгових дробів* // Карпатські матем. публ. — 2024. — Т.16, №2. — C. 448–460.

Проведено аналіз досліджень питання збіжності різних типів функціональних гіллястих ланцюгових дробів. Розглядаються гіллясті ланцюгові дроби з N гілками розгалуження та гіллясті ланцюгові дроби з нерівнозначними змінними. Дано означення і сформульовані, на наш погляд, характерні ознаки збіжності багатовимірних узагальнень C-, S-, g-, J-дробів як для гіллястих ланцюгових дробів загального вигляду з N гілками розгалуження так і гіллястих ланцюгових дробів з нерівнозначними змінними. Такі багатовимірні узагальнення неперервних дробів зустрічаються, зокрема, при розвиненні різних класів гіпергеометричних функцій багатьох змінних, зокрема, функцій Аппеля, Лаурічелли, Горна тощо.

Ключові слова і фрази: гіллястий ланцюговий дріб, збіжність, апроксимація раціональними функціями.