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Approximation characteristics of the Nikol'skii-Besov-type classes of periodic functions of several variables in the space $B_{q,1}$

Fedunyk-Yaremchuk O.V.[1,✉], Hembars'ka S.B.[1], Romanyuk I.A.[1], Zaderei P.V.[2]

We obtained the exact order estimates of approximation of periodic functions of several variables from the Nikol'skii-Besov-type classes $B^{\Omega}_{p,\theta}$ by using their step hyperbolic Fourier sums in the space $B_{q,1}$. The norm in this space is stronger than the L_q-norm. In the considered situations, approximations by the mentioned Fourier sums realize the orders of the best approximations by polynomials with "numbers" of harmonics from the step hyperbolic cross. We also established the exact order estimates of the Kolmogorov, linear and trigonometric widths of classes $B^{\Omega}_{p,\theta}$ in the space $B_{q,1}$ for certain relations between the parameters p and q.

Key words and phrases: Nikol'skii-Besov-type class, step hyperbolic Fourier sum, best approximation, widht.

[1] Lesya Ukrainka Volyn National University, 9 Bankova str., 43025, Lutsk, Ukraine
[2] National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", 37 Beresteiskyi av., 03056, Kyiv, Ukraine
✉ Corresponding author
E-mail: fedunyk.o.v@gmail.com (Fedunyk-Yaremchuk O.V.), gembarskaya72@gmail.com (Hembars'ka S.B.), iromaniuk@ukr.net (Romanyuk I.A.), zaderey.p.v@ukr.net (Zaderei P.V.)

Introduction

In this paper, we continue to study the approximation characteristics of the Nikol'skii-Besov-type classes $B^{\Omega}_{p,\theta}$ of periodic functions of several variables in the space $B_{q,1}$, $1 < q < \infty$, which norm is stronger than the L_q-norm.

As noted in works [3, 7–10, 12, 14, 16, 21, 27, 33–35, 37, 38, 43], the motivation to study the approximation characteristics of functional classes $B^{r}_{p,\theta}$ and their generalizations $B^{\Omega}_{p,\theta}$ in the spaces $B_{q,1}$, $q \in \{1, \infty\}$, was the fact, that the questions of their orders in the spaces L_1 and L_∞ still remain open in some important cases. We also have a similar situation in the spaces L_q, $1 < q < \infty$ (see [5, 23, 44]). In this regard, we note that in the one-dimensional case the considered approximation characteristics of the classes $B^{r}_{p,\theta}$ and $B^{\omega}_{p,\theta}$ in the space $B_{q,1}$, $1 < q < \infty$, were studied in works [11, 36]. This will be discussed more detailed in the comments to the obtained results.

The paper consists of two parts. In the first part, we obtained the exact order estimates of approximation of functions from the classes $B^{\Omega}_{p,\theta}$ in the space $B_{q,1}$, $1 < q < \infty$, by their step hyperbolic Fourier sums. In addition, the orders of the best approximations of the mentioned classes of functions by polynomials with "numbers" of harmonics from the step hyperbolic cross are also established.

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The second part of the paper is devoted to obtaining the exact order estimates of the Kolmogorov, linear, and trigonometric widths of the classes $B^{\Omega}_{p,\theta}$ in the space $B_{q,1}$ for certain relations between the parameters p and q. It was found that in the considered situations the orders of all mentioned widths are realized by their approximations by step hyperbolic Fourier sums that contain the necessary number of harmonics.

1 Functional classes $B^{\Omega}_{p,\theta}$ and spaces $B_{q,1}$

Let $\mathbb{R}^d$ be a d-dimensional space with the elements $x = (x_1, \ldots, x_d)$. Denote by $(x, y) = x_1y_1 + \ldots + x_dy_d$ a scalar product of the elements $x, y \in \mathbb{R}^d$. Let $L_p(\mathbb{T}^d)$, $\mathbb{T}^d = \prod_{j=1}^{d}[0, 2\pi)$, be the space of functions f that are 2π-periodic in each variable and for which

$$\|f\|_{L_p(\mathbb{T}^d)} := \|f\|_p := \left((2\pi)^{-d}\int_{\mathbb{T}^d}|f(x)|^p\,dx\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{L_\infty(\mathbb{T}^d)} := \|f\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbb{T}^d}|f(x)| < \infty.$$

In what follows, we assume that $f \in L_p(\mathbb{T}^d)$ satisfy the condition

$$\int_0^{2\pi} f(x)dx_j = 0, \quad j = \overline{1, d},$$

and the set of such functions will be denoted as $L^0_p(\mathbb{T}^d)$. In addition, for convenience, we will use the notation L_p instead of $L_p(\mathbb{T}^d)$ and, accordingly, L^0_p instead $L^0_p(\mathbb{T}^d)$.

Let us define the lth difference of the function $f \in L^0_p$, $1 \le p \le \infty$, with the increment h_j of the variable x_j according to the formula

$$\Delta^l_{h_j}f(x) = \sum_{n=0}^{l}(-1)^{l-n}C^n_l f(x_1, \ldots, x_{j-1}, x_j + nh_j, x_{j+1}, \ldots, x_d).$$

For $f \in L^0_p, 1 \le p \le \infty$, $h = (h_1, \ldots, h_d)$ and $t \in \mathbb{R}^d_+$, let us introduce the mixed lth difference

$$\Delta^l_h f(x) = \Delta^l_{h_1}\ldots\Delta^l_{h_d}f(x) = \Delta^l_{h_d}(\ldots(\Delta^l_{h_1}f(x)))$$

and define the mixed modulus of continuity of lth order by

$$\Omega_l(f, t)_p = \sup_{\substack{|h_j|\le t_j\\ j=\overline{1,d}}}\|\Delta^l_h f(\cdot)\|_p.$$

Let $\Omega(t) = \Omega(t_1, t_2, \ldots, t_d)$ be a given function of the type of mixed modulus of continuity of the lth order. This means that the function $\Omega(t)$ satisfies the following conditions:

1) $\Omega(t) > 0$, $t_j > 0$, $j = \overline{1, d}$, and $\Omega(t) = 0$ if $\prod_{j=1}^{d} t_j = 0$;

2) $\Omega(t)$ increases in each variable;

3) $\Omega(m_1t_1, m_2t_2, \ldots, m_dt_d) \le (\prod_{j=1}^{d} m_j)^l\Omega(t)$, $m_j \in \mathbb{N}$, $j = \overline{1, d}$;

4) $\Omega(t)$ is continuous at $t_j \ge 0$, $j = \overline{1, d}$.

Following S.N. Bernstein [4], a single-variable function $\varphi(\tau)$ will be called almost increasing (almost decreasing) on $[a,b]$, if there is a constant $C_1 > 0$ independent of τ_1 and τ_2 such that $\varphi(\tau_1) \le C_1\varphi(\tau_2)$, $a \le \tau_1 \le \tau_2 \le b$, in the "almost increasing" case and, accordingly, $\varphi(\tau_1) \ge C_2\varphi(\tau_2)$, $a \le \tau_1 \le \tau_2 \le b$, $C_2 > 0$, in the "almost decreasing" one.

We assume that the function $\Omega(t)$, $t \in \mathbb{R}^d_+$, also satisfies the following conditions (S^α) and (S_l), which are refferend to as the Bari-Stechkin conditions [2,41].

A single-variable function $\varphi(\tau) \ge 0$, $\tau \in [0,1]$, satisfies the condition (S^α) if $\varphi(\tau)/\tau^\alpha$ almost increases at a certain $\alpha > 0$.

A function $\varphi(\tau) \ge 0$, $\tau \in [0,1]$, satisfies condition (S_l) if $\varphi(\tau)/\tau^\gamma$ almost decreases at a certain $0 < \gamma < l$, $l \in \mathbb{N}$.

In the case $d > 1$, they say that $\Omega(t)$, $t \in \mathbb{R}^d_+$, satisfies those conditions in each variable t_j at fixed t_i, $i \ne j$.

Now let us define the functional classes $B^\Omega_{p,\theta}$ ($B^\omega_{p,\theta}$ in the one-dimensional case), which were considered by S. Youngshen and W. Heping [47].

Let $1 \le p, \theta \le \infty$, and a function $\Omega(t)$ be of the type of mixed modulus of continuity of the lth order, which satisfies conditions 1) – 4), (S^α) and (S_l).

Define the classes

$$B^\Omega_{p,\theta} := \{f \in L^0_p :\ \|f\|_{B^\Omega_{p,\theta}} \le 1\},$$

where

$$\|f\|_{B^\Omega_{p,\theta}} = \left\{ \int_{\mathbb{T}^d} \left(\frac{\Omega_l(f,t)_p}{\Omega(t)}\right)^\theta \prod_{j=1}^d \frac{dt_j}{t_j} \right\}^{1/\theta}, \quad 1 \le \theta < \infty, \quad \|f\|_{B^\Omega_{p,\infty}} = \sup_{t \in \mathbb{R}^d_+} \frac{\Omega_l(f,t)_p}{\Omega(t)}.$$

Note that if $r = (r_1, \dots, r_d)$, $0 < r_j < l$, $j = \overline{1,d}$, and $\Omega(t) = \prod_{j=1}^d t_j^{r_j}$, then the classes $B^\Omega_{p,\theta}$ coincide with the analogs of Besov classes $B^r_{p,\theta}$, which were considered in the works [1, 18]. Furthermore, if $\theta = \infty$, the classes $B^r_{p,\infty} \equiv H^r_p$ are analogs of the Nikol'skii classes [19]. The classes $B^\Omega_{p,\infty} \equiv H^\Omega_p$ were considered by N.N. Pustovoitov [22].

In the following considerations, it will be convenient to use a slightly different definition of the $B^\Omega_{p,\theta}$ classes. For this purpose, recall the concept of order relation.

For two non-negative sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ the relation (the order inequality) $a_n \ll b_n$ means that there exists a constant $C_3 > 0$ that is independent of n and such that $a_n \le C_3 b_n$. The relation $a_n \asymp b_n$ is equivalent to $a_n \ll b_n$ and $b_n \ll a_n$.

Let us put every vector $s \in \mathbb{N}^d$ in correspondence with a set

$$\rho(s) := \{k \in \mathbb{Z}^d :\ 2^{s_j-1} \le |k_j| < 2^{s_j},\ j = \overline{1,d}\}$$

and, for $f \in L^0_p, 1 < p < \infty$, put

$$\delta_s(f) := \delta_s(f,x) = \sum_{k \in \rho(s)} \widehat{f}(k) e^{i(k,x)},$$

where $\widehat{f}(k) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(t) e^{-i(k,t)}\, dt$ are the Fourier coefficients of the function f.

Hence, for $f \in B^\Omega_{p,\theta}$, $1 < p < \infty$, $1 \le \theta \le \infty$, where $\Omega(t)$ is a given function of the type of mixed modulus of continuity of the lth order, which satisfies conditions 1)–4), (S^α) and (S_l) the relation

$$\|f\|_{B^\Omega_{p,\theta}} \asymp \begin{cases} \left(\sum_{s \in \mathbb{N}^d} \Omega^{-\theta}(2^{-s}) \|\delta_s(f)\|_p^\theta\right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup_{s \in \mathbb{N}^d} \|\delta_s(f)\|_p / \Omega(2^{-s}), & \theta = \infty, \end{cases} \tag{1}$$

is valid. Hereafter, $\Omega(2^{-s}) = \Omega(2^{-s_1}, 2^{-s_2}, \dots, 2^{-s_d})$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$.

Note that the case $1 \le \theta < \infty$ in (1) was considered in [47], and the case $\theta = \infty$ in [22].

For the norms of functions from the classes $B^{\Omega}_{p,\theta}$, it is possible to write mappings similar to (1) with slightly modified "blocks" $\delta_s(f)$ in the cases $p = 1$ and $p = \infty$.

Let $V_m(t), m \in \mathbb{N}, t \in \mathbb{R}$, denote the de la Vall'ee-Poussin kernel

$$V_m(t) = 1 + 2\sum_{k=1}^{m} \cos kt + 2\sum_{k=m+1}^{2m-1} \frac{2m-k}{m} \cos kt.$$

We put every vector $s \in \mathbb{N}^d$ in correspondence with the polynomial

$$A_s(x) := \prod_{j=1}^{d} (V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j)), \quad x \in \mathbb{R}^d,$$

and, for $f \in L^0_p, 1 \le p \le \infty$, put $A_s(f) := A_s(f, x) = (f * A_s)(x)$, where $*$ is the convolution operation. Then the following relation

$$\|f\|_{B^{\Omega}_{p,\theta}} \asymp \begin{cases} \left(\sum_{s \in \mathbb{N}^d} \Omega^{-\theta}(2^{-s}) \|A_s(f)\|_p^{\theta} \right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup_{s \in \mathbb{N}^d} \|A_s(f)\|_p / \Omega(2^{-s}), & \theta = \infty, \end{cases} \tag{2}$$

is valid.

Note that the case $1 \le \theta < \infty$ in (2) was considered in work [40], and the case $\theta = \infty$ in [22].

Below we will consider the classes $B^{\Omega}_{p,\theta}$ (or $B^{\omega}_{p,\theta}$ if $d = 1$), which are determined by a function of the type of mixed modulus of continuity of the lth order possessing a special form, namely,

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j \right), \tag{3}$$

where $\omega(\tau)$ is a given single-variable function of the type modulus of continuity of the lth order, which satisfies conditions (S^{α}) and (S_l).

It is clear that for $\Omega(t)$ in form (3), the properties 1)−4) of the function of the type of mixed modulus of continuity of the lth order, as well as conditions (S^{α}) and (S_l), are obeyed. Therefore, mappings (1) and (2) presented above are valid for the norms of functions from the $B^{\Omega}_{p,\theta}$ classes.

Define the norm $\|\cdot\|_{B_{q,1}}$ in the subspaces $B_{q,1} \subset L^0_q, 1 \le q \le \infty$, by $\|t\|_{B_{q,1}} := \sum_{s \in \mathbb{N}^d} \|A_s(t)\|_q$ (the sum contains a finite number of terms), where t is a trigonometric polynomial with respect to the trigonometric system $\{e^{i(k,x)}\}_{k \in \mathbb{Z}^d}$.

Similarly we define the norm for functions $f \in L^0_q$, such that the series $\sum_{s \in \mathbb{N}^d} \|A_s(f)\|_q$ is convergent, namely

$$\|f\|_{B_{q,1}} := \sum_{s \in \mathbb{N}^d} \|A_s(f)\|_q, \quad 1 \le q \le \infty.$$

Note, that in the case $1 < q < \infty$ the relation $\|f\|_{B_{q,1}} \asymp \sum_{s \in \mathbb{N}^d} \|\delta_s(f)\|_q$ holds.

For $f \in B_{q,1}, 1 \le q \le \infty$, the following relations

$$\|f\|_q \ll \|f\|_{B_{q,1}}, \quad \|f\|_{B_{1,1}} \ll \|f\|_{B_{q,1}} \ll \|f\|_{B_{\infty,1}}$$

hold.

2 Approximation by the step hyperbolic Fourier sums and best approximation

Let us define the approximation characteristics that will be investigated in this part of the paper.

For $n \in \mathbb{N}$ and $s \in \mathbb{N}^d$ we set $Q_n := \bigcup_{\|s\|_1 < n} \rho(s)$, where $\|s\|_1 = s_1 + \ldots + s_d$.

The set Q_n is called the step hyperbolic cross. For the number of elements of the set Q_n, the following equality $|Q_n| \asymp 2^n n^{d-1}$ holds (see, e.g., [5, Ch.2, §2.3]).

Next, we will consider the sets of trigonometric polynomials

$$T(Q_n) := \Big\{ t : t(x) = \sum_{k \in Q_n} c_k e^{i(k,x)}, c_k \in \mathbb{C}, x \in \mathbb{R}^d \Big\}$$

and for $f \in L_1^0(\mathbb{T}^d)$ we put $S_{Q_n}(f) := S_{Q_n}(f, x) = \sum_{k \in Q_n} \widehat{f}(k) e^{i(k,x)}$, $x \in \mathbb{R}^d$.

Polynomials $S_{Q_n}(f)$ are called step hyperbolic Fourier sums of the function f. According to the notation considered above, $S_{Q_n}(f)$ can be expressed in the form

$$S_{Q_n}(f) := \sum_{\|s\|_1 < n} \delta_s(f) = \sum_{\|s\|_1 < n} \delta_s(f, x).$$

Taking into account the sets of trigonometric polynomials defined above, consider the following approximation characteristics.

Let X be a normed functional space with the norm $\|\cdot\|_X$. For $f \in X$ we define by $E_{Q_n}(f)_X := \inf_{t \in T(Q_n)} \|f - t\|_X$ the best approximation of the function f by polynomials from the set $T(Q_n)$. Accordingly, for the functional class $F \subset X$ we set

$$E_{Q_n}(F)_X := \sup_{f \in F} E_{Q_n}(f)_X. \tag{4}$$

In addition to values (4), we will consider approximation characteristics

$$\mathcal{E}_{Q_n}(F)_X := \sup_{f \in F} \mathcal{E}_{Q_n}(f)_X := \sup_{f \in F} \|f - S_{Q_n}(f)\|_X. \tag{5}$$

The quantities (4) and (5) for the Nikol'skii-Besov classes $B^r_{p,\theta}$ and their generalizations $B^\Omega_{p,\theta}$ in the space $X = L_q$ were investigated in many papers (see, e.g., [22, 24, 25, 39, 40, 47]). A more detailed bibliography can be found in the monographs [5, 23, 42, 44].

In this part of the work, we will obtain the exact order estimates of quantities (4) and (5) for the classes $B^\Omega_{p,\theta}$ in the space $B_{q,1}$, $1 < p, q < \infty$.

In the following considerations, the relations between the quantities $E_{Q_n}(f)_{B_{q,1}}$, $\mathcal{E}_{Q_n}(f)_{B_{q,1}}$, $1 < q < \infty$, will be significantly used.

Let $\mathbf{S}_{Q_n}$ be the Fourier operator, which puts in correspondence to a function $f \in B_{q,1}$, $1 < q < \infty$, its step hyperbolic Fourier sum $S_{Q_n}(f)$, i.e. $\mathbf{S}_{Q_n} f := S_{Q_n}(f)$.

It is easy to see that the norm of the operator $\mathbf{S}_{Q_n}$ that maps $B_{q,1}$ into $B_{q,1}$, $1 < q < \infty$, (denoted as $\|\mathbf{S}_{Q_n}\|_{B_{q,1} \to B_{q,1}}$) is bounded.

According to the definition of the norm of the operator, we have

$$\begin{aligned} \|\mathbf{S}_{Q_n}\|_{B_{q,1} \to B_{q,1}} &= \sup_{\|f\|_{B_{q,1}} \le 1} \|S_{Q_n}(f)\|_{B_{q,1}} \asymp \sup_{\|f\|_{B_{q,1}} \le 1} \sum_{s \in \mathbb{N}^d} \|\delta_s(S_{Q_n}(f))\|_q \\ &= \sup_{\|f\|_{B_{q,1}} \le 1} \sum_{\|s\|_1 < n} \|\delta_s(f)\|_q \le \sup_{\|f\|_{B_{q,1}} \le 1} \sum_{s \in \mathbb{N}^d} \|\delta_s(f)\|_q \le 1. \end{aligned} \tag{6}$$

Next, let $t^* \in T(Q_n)$ be the polynomial of the best approximation of the function $f \in B_{q,1}$. Then, taking into account that $S_{Q_n}(t^*) = t^*$ and using (6), we can write

$$\begin{aligned}
\mathcal{E}_{Q_n}(f)_{B_{q,1}} &= \|f - S_{Q_n}(f)\|_{B_{q,1}} = \|f - t^* + t^* - S_{Q_n}(f)\|_{B_{q,1}} \\
&\leq \|f - t^*\|_{B_{q,1}} + \|t^* - S_{Q_n}(f)\|_{B_{q,1}} = \|f - t^*\|_{B_{q,1}} + \|S_{Q_n}(f) - t^*\|_{B_{q,1}} \\
&= \|f - t^*\|_{B_{q,1}} + \|S_{Q_n}(f - t^*)\|_{B_{q,1}} \leq E_{Q_n}(f)_{B_{q,1}} + \|\mathbf{S}_{Q_n}\|_{B_{q,1}\to B_{q,1}}\|f - t^*\|_{B_{q,1}} \\
&\leq E_{Q_n}(f)_{B_{q,1}} + C_4 E_{Q_n}(f)_{B_{q,1}} \ll E_{Q_n}(f)_{B_{q,1}}.
\end{aligned} \tag{7}$$

Directly from the definitions, for $f \in B_{q,1}$ we have

$$E_{Q_n}(f)_{B_{q,1}} \leq \mathcal{E}_{Q_n}(f)_{B_{q,1}}. \tag{8}$$

Hence, comparing (7) and (8) we get $E_{Q_n}(f)_{B_{q,1}} \asymp \mathcal{E}_{Q_n}(f)_{B_{q,1}}$, $1 < q < \infty$.

Let us proceed to the formulation and proof of the obtained results.

Theorem 1. *Let $d \geq 2$, $1 < p < \infty$, $1 \leq \theta \leq \infty$, and $\Omega(t) = \omega(\prod_{j=1}^{d} t_j)$, where $\omega(\tau)$ satisfies condition (S^α) with some $\alpha > 0$ and condition (S_l). Then the following relations hold:*

$$\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}} \asymp E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}} \asymp \omega(2^{-n})n^{(d-1)(1-1/\theta)}. \tag{9}$$

Proof. Let us obtain the upper estimate in (9) for the quantity $\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}}$, from which, according to the definitions (4) and (5), the necessary estimate for the best approximation of $E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}}$ will follow.

Therefore, for an arbitrary function $f \in B^{\Omega}_{p,\theta}$, taking into account the definition of the norm in the space $B_{p,1}$, we can write

$$\begin{aligned}
\|f - S_{Q_n}(f)\|_{B_{p,1}} &= \Big\| \sum_{\|s\|_1 > n} \delta_s(f) \Big\|_{B_{p,1}} = \sum_{s \in \mathbb{N}^d} \Big\| A_s * \sum_{\substack{s' \in \mathbb{N}^d \\ \|s'\|_1 \geq n+1}} \delta_{s'}(f) \Big\|_p \\
&\leq \sum_{\|s\|_1 \geq n-d} \Big\| A_s * \sum_{\|s-s'\|_\infty \leq 1} \delta_{s'}(f) \Big\|_p \leq \sum_{\|s\|_1 \geq n-d} \|A_s\|_1 \Big\| \sum_{\|s-s'\|_\infty \leq 1} \delta_{s'}(f) \Big\|_p = I_1.
\end{aligned} \tag{10}$$

To further estimate the quantity I_1, we note that according to the relation $\|V_{2^s}\|_p \asymp 2^{s(1-1/p)}$, $1 \leq p \leq \infty$, $s \in \mathbb{N}$ (see, e.g., [42, Ch.1, §1]), we have

$$\|A_s\|_1 = \|V_{2^s} - V_{2^{s-1}}\|_1 \leq \|V_{2^s}\|_1 + \|V_{2^{s-1}}\|_1 \leq C_5,$$

and therefore for $s \in \mathbb{N}^d$ we get

$$\|A_s\|_1 \leq C_6. \tag{11}$$

Thus, taking into account (11) we can write

$$I_1 \ll \sum_{\|s\|_1 \geq n-d} \Big\| \sum_{\|s-s'\|_\infty \leq 1} \delta_{s'}(f) \Big\|_p \leq \sum_{\|s\|_1 \geq n-2d} \|\delta_{s'}(f)\|_p = I_2. \tag{12}$$

Next, we consider several cases depending on the values of parameter θ.

1. Let $\theta \in (1, \infty)$. Applying the Hölder inequality, we obtain

$$\begin{aligned}
I_2 &\leq \Big(\sum_{\|s\|_1 \geq n-2d} \omega^{-\theta}(2^{-\|s\|_1}) \|\delta_s(f)\|_p^\theta \Big)^{1/\theta} \Big(\sum_{\|s\|_1 \geq n-2d} \omega^{\theta'}(2^{-\|s\|_1}) \Big)^{1/\theta'} \\
&\ll \|f\|_{B^{\Omega}_{p,\theta}} \Big(\sum_{\|s\|_1 \geq n-2d} \omega^{\theta'}(2^{-\|s\|_1}) \Big)^{1/\theta'} \ll \Big(\sum_{\|s\|_1 \geq n-2d} \omega^{\theta'}(2^{-\|s\|_1}) \Big)^{1/\theta'} = I_3,
\end{aligned} \tag{13}$$

where $1/\theta + 1/\theta' = 1$.

We denote $n - 2d = m$. Taking into account that

$$\frac{\omega(2^{-\|s\|_1})}{2^{-\alpha\|s\|_1}} \leq C_7 \frac{\omega(2^{-m})}{2^{-\alpha m}}, \quad C_7 > 0, \quad \|s\|_1 \geq m, \tag{14}$$

the estimation (13) can be continued as follows

$$\begin{aligned} I_3 &\ll \frac{\omega(2^{-m})}{2^{-\alpha m}} \Bigg(\sum_{\|s\|_1 \geq m} 2^{-\|s\|_1 \alpha\theta'} \Bigg)^{1/\theta'} = \frac{\omega(2^{-m})}{2^{-\alpha m}} \Bigg(\sum_{j \geq m} 2^{-j\alpha\theta'} \sum_{\|s\|_1 = j} 1 \Bigg)^{1/\theta'} \\ &\asymp \frac{\omega(2^{-m})}{2^{-\alpha m}} \Bigg(\sum_{j \geq m} 2^{-j\alpha\theta'} j^{d-1} \Bigg)^{1/\theta'} \ll \omega(2^{-m}) m^{(d-1)(1-1/\theta)}. \end{aligned} \tag{15}$$

2. In the case $\theta = \infty$, we may write

$$\begin{aligned} I_2 &\leq \sup_{s:\|s\|_1 \geq m} \frac{\|\delta_s(f)\|_p}{\omega(2^{-\|s\|_1})} \sum_{\|s\|_1 \geq m} \omega(2^{-\|s\|_1}) \ll \|f\|_{B^{\Omega}_{p,\infty}} \sum_{\|s\|_1 \geq m} \omega(2^{-\|s\|_1}) \\ &\ll \sum_{\|s\|_1 \geq m} \omega(2^{-\|s\|_1}) \ll \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{\|s\|_1 \geq m} 2^{-\alpha\|s\|_1} = \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{j \geq m} 2^{-j\alpha} \sum_{\|s\|_1 = j} 1 \\ &\asymp \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{j \geq m} 2^{-j\alpha} j^{d-1} \ll \omega(2^{-m}) m^{d-1}. \end{aligned} \tag{16}$$

3. If $\theta = 1$, then taking into account (14), the estimate of the quantity I_2 is as follows

$$\begin{aligned} I_2 &\leq \sup_{s:\|s\|_1 \geq m} \omega(2^{-\|s\|_1}) \sum_{\|s\|_1 \geq m} \omega^{-1}(2^{-\|s\|_1}) \|\delta_s(f)\|_p \\ &\ll \sup_{s:\|s\|_1 \geq m} \omega(2^{-\|s\|_1}) \|f\|_{B^{\Omega}_{p,1}} \leq \sup_{s:\|s\|_1 \geq m} \omega(2^{-\|s\|_1}) \ll \omega(2^{-m}). \end{aligned} \tag{17}$$

Thus, combining (10), (12), (13), (15)–(17), we obtain the required upper estimate for the quantity $\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}}$.

The lower estimates in (9) follow from the estimate of the corresponding approximation characteristic in the space $B_{1,1}$ obtained in [13].

Theorem A ([13]). *Let $d \geq 2$, $1 \leq p, \theta \leq \infty$, and $\Omega(t) = \omega(\prod_{j=1}^d t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 0$ and condition (S_l). Then the following estimate*

$$E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{1,1}} \asymp \omega(2^{-n}) n^{(d-1)(1-1/\theta)} \tag{18}$$

is true.

Hence, using the estimate (18), we can write

$$\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}} \geq E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}} \geq E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{1,1}} \asymp \omega(2^{-n}) n^{(d-1)(1-1/\theta)}.$$

□

Let us comment on the obtained result by quoting a well-known statement concerning quantity $\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_p$.

Theorem B ([47]). *Let $d \geq 2$, $1 \leq p < \infty$, $1 \leq \theta \leq \infty$, and $\Omega(t) = \omega(\prod_{j=1}^d t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 0$ and condition (S_l). Then the following estimate*

$$\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_p \asymp \omega(2^{-n}) n^{(d-1)(1/p^* - 1/\theta)_+} \tag{19}$$

is true, where $1/p^* = \max\{1/p; 1/2\}$, $a_+ = \max\{a; 0\}$.

Remark 1. *Analyzing the proof of Theorem B, we can argued that the estimate* (19) *is also valid for the best approximation* $E_{Q_n}(B^{\Omega}_{p,\theta})_p$.

Thus, comparing the results of the Theorem 1, Theorem B and taking into account the Remark 1, we come to the following conclusion.

Except for the case $\theta = 1$, the corresponding approximation characteristics of the classes $B^{\Omega}_{p,\theta}$ in the spaces $B_{p,1}$ and $L_p, 1 < p < \infty$, are different in order. As for the one-dimensional case, it was established in [11] that the considered approximation characteristics of the classes $B^{\omega}_{p,\theta}$, $1 < p < \infty, 1 \le \theta \le \infty$, in the spaces $B_{p,1}$ and L_p have the same orders.

To obtain the orders of quantities $\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}}$ and $\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}}$ for $1 < p < q < \infty$, we formulate an auxiliary statement.

Theorem C. *Let* $t(x) = \sum_{|k_j| \le n_j} c_k e^{i(k,x)}$, *where* $x \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$, $n \in \mathbb{N}^d$, $c_k \in \mathbb{C}$. *Then for* $1 \le p < q \le \infty$ *the following inequality holds:*

$$\|t\|_q \le 2^d \left(\prod_{j=1}^{d} n_j \right)^{1/p-1/q} \|t\|_p. \tag{20}$$

Inequality (20) was established by S.M. Nikol'skii [20] and referred to as the inequality for different metrics.

Theorem 2. *Let* $d \ge 2$, $1 < p < q < \infty$, $1 \le \theta \le \infty$, *and* $\Omega(t) = \omega(\prod_{j=1}^{d} t_j)$, *where* $\omega(\tau)$ *satisfies condition* (S^{α}) *with some* $\alpha > 1/p - 1/q$ *and condition* (S_l). *Then the following relations hold:*

$$\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp \omega(2^{-n}) 2^{n(1/p-1/q)} n^{(d-1)(1-1/\theta)}. \tag{21}$$

Proof. Let us prove the upper estimate in (21) for the quantity $\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}}$. For this purpose, we will use the same considerations as when establishing the estimate of the quantity I_2. Thus, also using the inequality (20), we can write

$$\|f - S_{Q_n}(f)\|_{B_{q,1}} \ll \sum_{\|s\|_1 \ge n-2d} \|\delta_s(f)\|_q \ll \sum_{\|s\|_1 \ge n-2d} 2^{\|s\|_1 (1/p-1/q)} \|\delta_s(f)\|_p = I_4. \tag{22}$$

Next, we will consider three cases.

1. Let $\theta \in (1, \infty)$. Applying the Hölder inequality and taking into account that $\alpha > 1/p - 1/q$, we have

$$\begin{aligned} I_4 &\le \left(\sum_{\|s\|_1 \ge n-2d} \omega^{-\theta}(2^{-\|s\|_1}) \|\delta_s(f)\|_p^{\theta} \right)^{1/\theta} \left(\sum_{\|s\|_1 \ge n-2d} \omega^{\theta'}(2^{-\|s\|_1}) 2^{\|s\|_1 \theta'(1/p-1/q)} \right)^{1/\theta'} \\ &\ll \|f\|_{B^{\Omega}_{p,\theta}} \left(\sum_{\|s\|_1 \ge n-2d} \omega^{\theta'}(2^{-\|s\|_1}) 2^{\|s\|_1 \theta'(1/p-1/q)} \right)^{1/\theta'} \\ &\le \left(\sum_{\|s\|_1 \ge n-2d} \omega^{\theta'}(2^{-\|s\|_1}) 2^{\|s\|_1 \theta'(1/p-1/q)} \right)^{1/\theta'} \\ &\ll \frac{\omega(2^{-m})}{2^{-\alpha m}} \left(\sum_{\|s\|_1 \ge m} 2^{-\|s\|_1 \alpha \theta'} 2^{\|s\|_1 \theta'(1/p-1/q)} \right)^{1/\theta'} \\ &= \frac{\omega(2^{-m})}{2^{-\alpha m}} \left(\sum_{j \ge m} 2^{-j\theta'(\alpha - 1/p + 1/q)} j^{d-1} \right)^{1/\theta'} \\ &\ll \omega(2^{-m}) 2^{m(1/p-1/q)} m^{(d-1)(1-1/\theta)}. \end{aligned} \tag{23}$$

2. In the case $\theta = \infty$ for the quantity I_4 we get

$$\begin{aligned} I_4 &\le \sup_{s:\|s\|_1\ge m} \frac{\|\delta_s(f)\|_p}{\omega(2^{-\|s\|_1})} \sum_{\|s\|_1\ge m} \omega(2^{-\|s\|_1})2^{\|s\|_1(1/p-1/q)} \ll \sum_{\|s\|_1\ge m} \omega(2^{-\|s\|_1})2^{\|s\|_1(1/p-1/q)} \\ &\ll \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{\|s\|_1\ge m} 2^{-\|s\|_1(\alpha-1/p+1/q)} = \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{j\ge m} 2^{-j(\alpha-1/p+1/q)} \sum_{\|s\|_1=j} 1 \\ &\asymp \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{j\ge m} 2^{-j(\alpha-1/p+1/q)} j^{d-1} \ll \omega(2^{-m})2^{m(1/p-1/q)} m^{d-1}. \end{aligned} \tag{24}$$

3. If $\theta = 1$, then the estimate of the quantity I_4 is as follows

$$\begin{aligned} I_4 &\le \sup_{s:\|s\|_1\ge m} \omega(2^{-\|s\|_1})2^{\|s\|_1(1/p-1/q)} \sum_{\|s\|_1\ge m} \omega^{-1}(2^{-\|s\|_1})\|\delta_s(f)\|_p \\ &\ll \sup_{s:\|s\|_1\ge m} \omega(2^{-\|s\|_1})2^{\|s\|_1(1/p-1/q)} \ll \frac{\omega(2^{-m})}{2^{-\alpha m}} \sup_{s:\|s\|_1\ge m} 2^{-\|s\|_1(\alpha-1/p+1/q)} \\ &\asymp \omega(2^{-m})2^{m(1/p-1/q)}. \end{aligned} \tag{25}$$

Therefore, combining (21)–(25), we get the upper estimates in (21).

To establish the lower estimates in (21), we construct extremal functions that realize the orders of the obtained upper estimates.

Consider the functions

$$\begin{aligned} g_1(x) &= C_8\omega(2^{-n})2^{-n(1-1/p)}n^{-(d-1)/\theta}d_n(x), \quad 1 \le \theta < \infty, \quad C_8 > 0, \\ g_2(x) &= C_9\omega(2^{-n})2^{-n(1-1/p)}d_n(x), \quad \theta = \infty, \quad C_9 > 0, \end{aligned}$$

where $d_n(x) = \sum_{s:\|s\|_1=n} \sum_{k\in\rho(s)} e^{i(k,x)}$.

Let us show that $g_1 \in B^{\Omega}_{p,\theta}, 1 \le \theta < \infty$, and $g_2 \in B^{\Omega}_{p,\infty}$ with a certain choice of constants C_8 and C_9.

First, let $1 \le \theta < \infty$. Then, according to the definition of the norm in the space $B^{\Omega}_{p,\theta}$, we can write

$$\begin{aligned} \|g_1\|_{B^{\Omega}_{p,\theta}} &\asymp \Bigg(\sum_{\|s\|_1=n} \omega^{-\theta}(2^{-\|s\|_1})\|\delta_s(g_1)\|_p^{\theta} \Bigg)^{1/\theta} \\ &\asymp \omega(2^{-n})2^{-n(1-1/p)}n^{-(d-1)/\theta} \Bigg(\sum_{\|s\|_1=n} \omega^{-\theta}(2^{-\|s\|_1})\|\delta_s(d_n)\|_p^{\theta} \Bigg)^{1/\theta} \\ &= 2^{-n(1-1/p)}n^{-(d-1)/\theta} \Bigg(\sum_{\|s\|_1=n} \|\delta_s(d_n)\|_p^{\theta} \Bigg)^{1/\theta} = I_5. \end{aligned} \tag{26}$$

Next, using the the well-known relation

$$\Bigg\| \sum_{k=-m}^{m} e^{ikx} \Bigg\|_p \asymp m^{1-1/p}, \quad 1 < p < \infty, \quad x \in \mathbb{R},$$

(see, e.g., [40, Ch.1, §1]), we have

$$\|\delta_s(d_n)\|_p \asymp 2^{\|s\|_1(1-1/p)}. \tag{27}$$

Then, taking into account (26) and (27), we proceed the estimation of the quantity I_5 as follows

$$\begin{aligned} I_5 &\asymp 2^{-n(1-1/p)} n^{-(d-1)/\theta} \Bigg(\sum_{\|s\|_1=n} 2^{\|s\|_1(1-1/p)\theta} \Bigg)^{1/\theta} \\ &= n^{-(d-1)/\theta} \Bigg(\sum_{\|s\|_1=n} 1 \Bigg)^{1/\theta} \asymp n^{-(d-1)/\theta} n^{(d-1)/\theta} = 1. \end{aligned} \tag{28}$$

Hence, from (27) and (28) it follows that $g_1 \in B^{\Omega}_{p,\theta}$, $1 \le \theta < \infty$, with the corresponding constant $C_8 > 0$.

Now let $\theta = \infty$. Then for the function g_2 we have

$$\|g_2\|_{B^{\Omega}_{p,\theta}} \asymp \sup_{s\in\mathbb{N}^d} \frac{\|\delta_s(g_2)\|_p}{\omega(2^{-\|s\|_1})} \asymp \omega(2^{-n}) 2^{-n(1-1/p)} \sup_{\|s\|_1=n} \frac{\|\delta_s(d_n)\|_p}{\omega(2^{-\|s\|_1})} \asymp 1.$$

Thus, $g_2 \in B^{\Omega}_{p,\infty}$ with with some constant $C_9 > 0$.

Now, taking into account that $S_{Q_n}(g_1) = 0$ and using the relation (27), for $1 \le \theta < \infty$ we can write

$$\begin{aligned} \mathcal{E}_{Q_n}(g_1)_{B_{q,1}} = \|g_1\|_{B_{q,1}} &\asymp \omega(2^{-n}) 2^{-n(1-1/p)} n^{-(d-1)/\theta} \sum_{\|s\|_1=n} 2^{\|s\|_1(1-1/q)} \\ &\asymp \omega(2^{-n}) 2^{-n(1-1/p)} n^{-(d-1)/\theta} 2^{n(1-1/q)} n^{d-1} \\ &= \omega(2^{-n}) 2^{n(1/p-1/q)} n^{(d-1)/(1-1/\theta)}. \end{aligned} \tag{29}$$

Similarly, for the function g_2, given that $S_{Q_n}(g_2) = 0$, we get

$$\mathcal{E}_{Q_n}(g_2)_{B_{q,1}} = \|g_2\|_{B_{q,1}} \asymp \omega(2^{-n}) 2^{n(1/p-1/q)} n^{d-1}. \tag{30}$$

Hence, from (29), (30) we obtain the required lower estimates for the quantities $E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}}$ and $\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}}$. $\square$

To compare the obtained result with estimates of the corresponding approximation characteristics in the space L_q, we formulate a well-known statement about this space.

Theorem D ([47]). *Let $d \ge 2$, $1 < p < q < \infty$, $1 \le \theta \le \infty$, and $\Omega(t) = \omega(\prod_{j=1}^d t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 1/p - 1/q$ and condition (S_l). Then the following relations*

$$E_{Q_n}(B^{\Omega}_{p,\theta})_q \asymp \mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_q \asymp \omega(2^{-n}) 2^{n(1/p-1/q)} n^{(d-1)(1/q-1/\theta)_+} \tag{31}$$

hold, where $a_+ = \max\{a; 0\}$.

Thus, comparing the estimates (21) with (31), we observe differences (except for the case $\theta = 1$) in the orders of the corresponding approximation characteristics of the classes $B^{\Omega}_{p,\theta}$ in the spaces $B_{p,1}$ and L_q. Note also that these differences have multidimensional specificity, as they are not observed at $d = 1$ (see [11]).

At the end of this part of the work, we present one more result that is relevant to the case $1 \le q < p \le \infty$.

Theorem 3. *Let $d \ge 2$, $1 \le q < p \le \infty$, $1 \le \theta \le \infty$, and $\Omega(t) = \omega(\prod_{j=1}^d t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 0$ and condition (S_l). Then the following relations hold:*

$$\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp \omega(2^{-n}) n^{(d-1)(1-1/\theta)}. \tag{32}$$

Proof. The upper estimates in (32) follow from the Theorem 1. Indeed, according to the relation $\|\cdot\|_{B_{1,1}} \ll \|\cdot\|_{B_{q,1}}$, $1 \le q < \infty$, it is enough to obtain the corresponding estimates in the case $1 < q < p \le \infty$. In addition, taking into account the embeddings $B^{\Omega}_{\infty,\theta} \subset B^{\Omega}_{p,\theta} \subset B^{\Omega}_{q,\theta}$, it is enough to consider the case $1 < p = q < \infty$, using the estimates of the approximation characteristics established in the Theorem 1.

The lower estimates in (32) are a consequence of the relation (18), i.e.

$$\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \ge E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \ge E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{1,1}} \asymp \omega(2^{-n})n^{(d-1)(1-1/\theta)}.$$

$\square$

Further we compare the result of Theorem 3 with the estimates of the corresponding quantities in the space L_q.

Theorem E ([39])**.** *Let $d \ge 2$, $1 < q < p < \infty$, $p \ge 2$, $1 \le \theta \le \infty$, and $\Omega(t) = \omega(\prod_{j=1}^{d} t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 1/p - 1/q$ and condition (S_l). Then the following relations*

$$E_{Q_n}(B^{\Omega}_{p,\theta})_q \asymp \mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_q \asymp \omega(2^{-n})n^{(d-1)(1/2-1/\theta)_+}$$

hold, where $a_+ = \max\{a;0\}$.

Under the conditions of Theorem E on the parameters p and q, the orders of the corresponding quantities (except for the case $\theta = 1$) are different in the spaces L_q and $B_{q,1}$. In addition, it is important to note that Theorem 3 also covers such values parameters p and q for which the corresponding quantities in the space L_q remain unexplored. As for the one-dimensional case, the similar approximation characteristics of the classes $B^{\omega}_{p,\theta}$ in the space $B_{q,1}$ were studied in the paper [11] and at the same time no differences were found in their orders compared to the L_q space.

3 Widths

In this part of the work, we will establish the exact order estimates of the Kolmogorov, linear, and trigonometric widths of the classes $B^{\omega}_{p,\theta}$ in the space L_q for the cases $1 < p = q < \infty$ and $1 \le q < p \le \infty$. At the same time, it will be shown that the orders of these approximation characteristics are realized by the subspaces of trigonometric polynomials from the set $T(Q_n)$ that contain the necessary number of harmonics.

Let W be a centrally symetric set in the normalized space X. Then the quantity

$$d_M(W,X) := \inf_{L_M} \sup_{w\in W} \inf_{u\in L_M} \|w-u\|_X,$$

where $L_M \subset X$ is a subspace with dimension M, is called a Kolmogorov width. The width $d_M(W,X)$ was introduced by A.N. Kolmogorov [17].

The linear width of a set W in the space X is called the quantity

$$\lambda_M(W,X) := \inf_{A} \sup_{w\in W} \|w-Aw\|_X,$$

where the infimum is taken over all linear operators A which act in X and are such that the dimension of the set of their values is at most M. The width $\lambda_M(W,X)$ was introduced by V.M. Tikhomirov [45].

It what follows, we define the approximattion characteristics that was introduced by R.S. Ismagilov [15]. Let $X = L_q$ or $X = B_{q,1}$, $1 \le q \le \infty$, and $F \subset W$ be a functional class.

Trigonometric width of the class F in the space X (the notation $d_M^{\top}(F, X)$) is defined by the formula

$$d_M^{\top}(F, X) := \inf_{\{k^j\}_{j=1}^M} \sup_{f \in F} \inf_{\{c_j\}_{j=1}^M} \left\| f(\cdot) - \sum_{j=1}^{M} c_j e^{i(k^j, \cdot)} \right\|_X,$$

where $\{k^j\}_{j=1}^M$ is a set of vectors $k^j = (k_1^j, \dots, k_d^j)$, $j = \overline{1, M}$, from the integer grid $\mathbb{Z}^d$, c_j are arbitrary complex numbers. According to the introduced definitions of widths, the following relations

$$d_M(F, X) \le \lambda_M(F, X), \quad d_M(F, X) \le d_M^{\top}(F, X)$$

hold.

The history of research of defined widths of classes $B^r_{p,\theta}$ and $B^{\Omega}_{p,\theta}$ in the Lebesgue spaces L_q, $1 \le q \le \infty$, can be found in works [6,13,24–26,28–32,39,40,47] and monographs [5,23,42,44,46], and in the spaces $B_{q,1}$, $q \in \{1, \infty\}$, respectively, in works [3,7,12,14,16,27,33–35,37,38,43].

Let us formulate the known statement that we will use.

Theorem F ([13]). *Let $1 \le p, \theta \le \infty$, $\Omega(t) = \omega(\prod_{j=1}^d t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 0$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^{\infty}$ of natural numbers such that the relation $M \asymp 2^n n^{d-1}$ holds, the following estimates are valid:*

$$d_M(B^{\Omega}_{p,\theta}, B_{1,1}) \asymp \lambda_M(B^{\Omega}_{p,\theta}, B_{1,1}) \asymp d_M^{\top}(B^{\Omega}_{p,\theta}, B_{1,1}) \asymp \omega(2^{-n}) n^{(d-1)(1-1/\theta)}.$$

Let us proceed to the formulation and proof of the obtained results.

Theorem 4. *Let $d \ge 2$, $1 < p < \infty$, $1 \le \theta \le \infty$, and $\Omega(t) = \omega(\prod_{j=1}^d t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 0$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^{\infty}$ of natural numbers such that the relation $M \asymp 2^n n^{d-1}$ holds, the following estimates are valid:*

$$d_M(B^{\Omega}_{p,\theta}, B_{p,1}) \asymp \lambda_M(B^{\Omega}_{p,\theta}, B_{p,1}) \asymp d_M^{\top}(B^{\Omega}_{p,\theta}, B_{p,1}) \asymp \omega(2^{-n}) n^{(d-1)(1-1/\theta)}. \tag{33}$$

Proof. The upper estimates in (33) for each of the widths are a consequence of the Theorem 1.

Choosing the number $n \in \mathbb{N}$, $n \ge 2d$, such that it satisfies the relation $M \asymp 2^n n^{d-1}$, and using the estimate of quantity $\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}}$ (see [37]), we can write

$$d_M(B^{\Omega}_{p,\theta}, B_{p,1}) \ll \mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{p,1}} \asymp \omega(2^{-n}) n^{(d-1)(1-1/\theta)}.$$

Similarly, we can obtain upper estimates for the linear and trigonometric widths.

The lower estimates in (33) are a consequence of Theorem F and the relation

$$\| \cdot \|_{B_{q,1}} \gg \| \cdot \|_{B_{1,1}}, \quad 1 < q < \infty.$$

□

To compare the estimates (33) with the corresponding results in the space L_p, let us formulate the known statement.

Theorem G ([47]). *Let $d \ge 1$, $1 < p < \infty$, $1 \le \theta \le \infty$, and $\Omega(t) = \omega(\prod_{j=1}^d t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 0$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^{\infty}$ of natural numbers such that the relation $M \asymp 2^n n^{d-1}$ holds, the following estimates*

$$d_M(B^{\Omega}_{p,\theta}, L_p) \asymp \lambda_M(B^{\Omega}_{p,\theta}, L_p) \asymp d_M^{\top}(B^{\Omega}_{p,\theta}, L_p) \asymp \omega(2^{-n}) n^{(d-1)(1/p^* - 1/\theta)_+} \tag{34}$$

are valid, where $1/p^* = \max\{1/p; 1/2\}$, $a_+ = \max\{a; 0\}$.

Thus, comparing (33) and (34) we observe that for $d \geq 2$ and $\theta \neq 1$ estimates of the corresponding approximation characteristics of classes $B^{\Omega}_{p,\theta}$ in the spaces $B_{p,1}$ and L_p are different in order.

Note that in the one-dimensional case, the mentioned differences in estimates of the corresponding approximation characteristics are not observed [11].

Theorem 5. *Let $d \geq 2$, $1 \leq q < p \leq \infty$, $1 \leq \theta \leq \infty$, and $\Omega(t) = \omega(\prod_{j=1}^{d} t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 0$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^{\infty}$ of natural numbers such that the relation $M \asymp 2^n n^{d-1}$ holds, the following estimates*

$$d_M(B^{\Omega}_{p,\theta}, B_{q,1}) \asymp \lambda_M(B^{\Omega}_{p,\theta}, B_{q,1}) \asymp d^{\top}_M(B^{\Omega}_{p,\theta}, B_{q,1}) \asymp \omega(2^{-n}) n^{(d-1)(1-1/\theta)} \tag{35}$$

are valid.

Proof. The upper estimates in (35) follow from the Theorem 3 under the condition $M \asymp 2^n n^{d-1}$. The result on corresponding lower estimates is a corollary of the Theorem F and the relation $\|\cdot\|_{B_{q,1}} \gg \|\cdot\|_{B_{1,1}}, 1 < q < \infty$. □

To comment on the obtained in Theorem 5 result, we recall the corresponding statement in the space L_q.

Theorem H ([39])**.** *Let $d \geq 1, 1 < q \leq 2 < p < \infty$, $2 \leq \theta \leq \infty$, or $2 < q < p < \infty$, $1 \leq \theta \leq \infty$, and $\Omega(t) = \omega(\prod_{j=1}^{d} t_j)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 0$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^{\infty}$ of natural numbers such that the relation $M \asymp 2^n n^{d-1}$ holds, the following estimates*

$$d_M(B^{\Omega}_{p,\theta}, L_q) \asymp \lambda_M(B^{\Omega}_{p,\theta}, L_q) \asymp d^{\top}_M(B^{\Omega}_{p,\theta}, L_q) \asymp \omega(2^{-n}) n^{(d-1)(1/2-1/\theta)_+} \tag{36}$$

are valid, where $a_+ = \max\{a; 0\}$.

Thus, analyzing the results of the Theorem 5 and Theorem H, we note the following.

First, the Theorem 5 contains a number of values of the parameters p, q, θ, for which the Kolmogorov widths of the classes $B^{\Omega}_{p,\theta}$ in the space L_q remain unexplored. This applies to the following cases: $1 \leq q < p \leq 2, 1 \leq \theta \leq \infty; 1 \leq q \leq 2 < p \leq \infty, 1 \leq \theta < 2$.

Second, in the multidimensional case ($d \geq 2$) the estimates (35) and (36) (except for cases $2 < q < p < \infty$, $\theta = 1$) differ in order. The situation will be different if $d = 1$. As follows from the results of the work [11] for all values of parameters p, q, θ, which are considered in Theorem H (at $d = 1$), corresponding approximation characteristics in the spaces $B_{q,1}$ and L_q have the same orders.

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Одержано точні за порядком оцінки наближення періодичних функцій багатьох змінних із класів $B^{\Omega}_{p,\theta}$ типу Нікольського-Бєсова їхніми східчастими гіперболічними сумами Фур'є у просторі $B_{q,1}$. Норма у цьому просторі є більш сильною, ніж L_q-норма. В розглянутих ситуаціях наближення згаданими сумами Фур'є реалізують порядки найкращих наближень поліномами з "номерами" гармонік зі східчастого гіперболічного хреста. Встановлено також точні за порядком оцінки колмогоровських, лінійних та тригонометричних поперечників класів $B^{\Omega}_{p,\theta}$ у просторі $B_{q,1}$ для деяких співвідношень між параметрами p і q.

Ключові слова і фрази: клас типу Нікольського-Бєсова, східчасто-гіперболічна сума Фур'є, найкраще наближення, поперечник.