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Widths and entropy numbers of the classes of periodic functions of one and several variables in the space $B_{q,1}$

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Exact-order estimates are obtained for the entropy numbers and several types of widths (Kolmogorov, linear, trigonometric and orthowidth) for the Sobolev and Nikol'skii-Besov classes of one and several variables in the space $B_{q,1}$, $1 < q < \infty$. It is shown, that in the multivariate case, in contrast to the univariate, the obtained estimates differ in order from the corresponding estimates in the space L_q.

Key words and phrases: Sobolev class, Nikol'skii-Besov class, entropy number, width.

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Introduction

In the paper, the exact-order estimates are obtained for the entropy numbers and several types of widths (Kolmogorov, linear, trigonometric and orthowidth) for the Sobolev $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and Nikol'skii-Besov $B^{\boldsymbol{r}}_{p,\theta}$ classes of one and several variables in the space $B_{q,1}$, $1 < q < \infty$, the norm in which is stronger than the L_q-norm. As it was indicated in [6,8,9,21–27], a motivation to investigate different approximation characteristics on the mentioned function classes and their generalizations in the space $B_{q,1}$, $q \in \{1, \infty\}$, was the fact, that in some cases the question on the exact-order estimates of these characteristics in the space L_q, $q \in \{1, \infty\}$, still remains open. The analogical situation is observed in the Lebesque spaces L_q for $1 < q < \infty$ (see [5,20]).

The paper consists of two parts.

In the first part, the main attention is focused on getting the exact-order estimates for the Kolmogorov widths and entropy numbers of the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ of periodic functions of one and several variables in the space $B_{q,1}$ for some relations between the parameters p and q. As the consequences from the obtained and known before results, we get the orders for the linear and trigonometric widths of the mentioned functional classes in the spaces $B_{q,1}$. As a complement, we obtain the exact-order estimates of the entropy numbers and Kolmogorov widths of the Nikol'skii classes $H^{\boldsymbol{r}}_p$, $2 \le p \le \infty$, of periodic multivariate functions ($d \ge 2$) in the space $B_{\infty,1}$. Note, that in the space L_∞ for $d > 2$ the order of these characteristics of the classes $H^{\boldsymbol{r}}_p$, $1 \le p \le \infty$, still remains unknown (see [5, Open Problems 4.2, 6.3]).

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The second part of the paper is devoted to obtaining the exact-order estimates for the orthowidths of the classes $B^{\boldsymbol{r}}_{p,\theta}$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, as well as close to them characteristics in the spaces $B_{q,1}$, $1 < q < \infty$.

The obtained results complement and generalize the known statements for the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and $B^{\boldsymbol{r}}_{p,\theta}$, that were earlier proved in the spaces L_q for $1 \le q \le \infty$ and $B_{q,1}$, $q \in \{1, \infty\}$, in the papers [2,3,6–10,13,21–27,30]. Here one can find a more detailed bibliography.

1 Definitions of the functional classes and spaces $B_{q,1}$

Let $\mathbb{R}^d$, $d \ge 1$, be an Euclidean space with elements $\boldsymbol{x} = (x_1, \dots, x_d)$ and the scalar product $(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + \dots + x_d y_d$. By $L_p := L_p(\mathbb{T}^d)$, $\mathbb{T}^d = \prod_{j=1}^{d} [0, 2\pi)$, $1 \le p \le \infty$, we denote the space of functions $f(\boldsymbol{x})$, which are 2π-periodic in each variable and for which

$$\|f\|_p := \|f\|_{L_p} = \Big((2\pi)^{-d} \int_{\mathbb{T}^d} |f(\boldsymbol{x})|^p \, d\boldsymbol{x} \Big)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_\infty := \|f\|_{L_\infty} = \operatorname{ess\,sup}_{\boldsymbol{x} \in \mathbb{T}^d} |f(\boldsymbol{x})| < \infty, \quad p = \infty.$$

We further restrict ourself in considering only those functions $f \in L_p$, that satisfy the condition

$$\int_0^{2\pi} f(\boldsymbol{x}) dx_j = 0 \ \text{ a.e.}, \quad j = 1, \dots, d.$$

The respective set is denoted by L^0_p.

First, we define the functional class $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, which is investigated in the paper.

Let $F_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha})$ be a multidimensional analogue of the Bernoulli kernel, i.e. for $\boldsymbol{r}, \boldsymbol{\alpha} \in \mathbb{R}^d$, $r_j > 0$, $j = 1, \dots, d$, $\boldsymbol{x} \in \mathbb{T}^d$ let

$$F_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha}) := 2^d \sum_{\boldsymbol{k} \in \mathbb{N}^d} \prod_{j=1}^{d} k_j^{-r_j} \cos\Big(k_j x_j - \frac{\alpha_j \pi}{2}\Big).$$

Then by $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, $1 \le p \le \infty$, we denote the class of functions f of the form

$$f(\boldsymbol{x}) = \varphi(\cdot) * F_{\boldsymbol{r}}(\cdot, \boldsymbol{\alpha}) = (2\pi)^{-d} \int_{\mathbb{T}^d} \varphi(\boldsymbol{y}) F_{\boldsymbol{r}}(\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{\alpha}) d\boldsymbol{y},$$

where $\varphi \in L^0_p$, $\|\varphi\|_p \le 1$, and $*$ is an operation of convolution.

In what follows, we define the Nikol'skii-Besov functional classes $B^{\boldsymbol{r}}_{p,\theta}$. In will be convenient for us to use the corresponding characterisations in terms of dyadic decompositions of the Fourier transform (see [16, Remark 2.1]).

Let $V_l(t)$, $t \in \mathbb{R}$, $l \in \mathbb{N}$, denotes the de la Vallée-Poussin kernel of the form

$$V_l(t) := 1 + 2 \sum_{k=1}^{l} \cos kt + 2 \sum_{k=l+1}^{2l-1} \Big(1 - \frac{k-l}{l}\Big) \cos kt,$$

where for $l = 1$ we assume that the third term equals to zero.

We associate each vector $\boldsymbol{s} \in \mathbb{N}^d$ with the polynomial

$$A_{\boldsymbol{s}}(\boldsymbol{x}) := \prod_{j=1}^{d} (V_{2^{s_j}}(x_j) - V_{2^{s_j - 1}}(x_j))$$

and for $f \in L^0_p$, $1 \le p \le \infty$, set

$$A_{\boldsymbol{s}}(f) := A_{\boldsymbol{s}}(f, \boldsymbol{x}) := (f * A_{\boldsymbol{s}})(\boldsymbol{x}).$$

Then, for $1 \le p \le \infty$, $\boldsymbol{r} \in \mathbb{R}^d$, $r_j > 0$, $j = 1, \dots, d$, the classes $B^{\boldsymbol{r}}_{p,\theta}$ can be defined as follows

$$B^{\boldsymbol{r}}_{p,\theta} := \left\{ f\colon\ \|f\|_{B^{\boldsymbol{r}}_{p,\theta}} := \left(\sum_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \, \|A_{\boldsymbol{s}}(f)\|_p^{\theta} \right)^{1/\theta} \le 1 \right\}, \quad 1 \le \theta < \infty,$$
$$B^{\boldsymbol{r}}_{p,\infty} \equiv H^{\boldsymbol{r}}_p := \left\{ f\colon\ \|f\|_{B^{\boldsymbol{r}}_{p,\infty}} := \sup_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})} \|A_{\boldsymbol{s}}(f)\|_p \le 1 \right\}.$$

Note, in the case $1 < p < \infty$, the norm of the classes $B^{\boldsymbol{r}}_{p,\theta}$ can be equivalently defined in terms of binary "blocks" of the Fourier series of the functions $f \in L^0_p$.

For vectors $\boldsymbol{s} \in \mathbb{N}^d$, we set

$$\rho(\boldsymbol{s}) := \{\boldsymbol{k} \in \mathbb{Z}^d\colon\ 2^{s_j-1} \le |k_j| < 2^{s_j},\ j = 1, \dots, d\}$$

and for $f \in L^0_p$ we denote

$$\delta_{\boldsymbol{s}}(f) := \delta_{\boldsymbol{s}}(f, \boldsymbol{x}) := \sum_{\boldsymbol{k}\in\rho(\boldsymbol{s})} \widehat{f}(\boldsymbol{k}) e^{i(\boldsymbol{k},\boldsymbol{x})},$$

where $\widehat{f}(\boldsymbol{k}) = \int_{\mathbb{T}^d} f(\boldsymbol{t}) e^{-i(\boldsymbol{k},\boldsymbol{t})} d\boldsymbol{t}$ are the Fourier coefficients of the function f.

Hence, let $1 < p < \infty$, $\boldsymbol{r} \in \mathbb{R}^d$, $r_j > 0$, $j = 1, \dots, d$. Then we can define the norm as follows [1, 16]

$$\|f\|_{B^{\boldsymbol{r}}_{p,\theta}} \asymp \left(\sum_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(f)\|_p^{\theta} \right)^{1/\theta}, \quad 1 \le \theta < \infty,$$
$$\|f\|_{B^{\boldsymbol{r}}_{p,\infty}} \asymp \sup_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})} \|\delta_{\boldsymbol{s}}(f)\|_p.$$

Here and in what follows, for positive quantities a and b, we use the notation $a \asymp b$, that means that there exist positive constants C_1, C_2, that do not depend on one essential parameter in the values of a, b, and such that $C_1 a \le b$ (we write $a \ll b$) and $C_2 a \ge b$ (denoted by $a \gg b$). All constants C_i, $i = 1, 2, \dots$, in this paper may depend only on the parameters contained in the definition of the function class, the metric in which we estimate the approximation error, and the dimension of the space $\mathbb{R}^d$.

Concerning the defined above classes, we note the existing embeddings, that hold for all $\boldsymbol{r}, \boldsymbol{\alpha} \in \mathbb{R}^d$, $r_j > 0$, $j = 1, \dots, d$, namely

$$B^{\boldsymbol{r}}_{p,p} \subset W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,2}, \quad 1 < p \le 2,$$
$$B^{\boldsymbol{r}}_{p,2} \subset W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,p}, \quad 2 \le p < \infty,$$
$$W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,\infty} \equiv H^{\boldsymbol{r}}_p, \quad 1 \le p \le \infty.$$

In particular, for $p = \theta = 2$ it holds

$$W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{2,2} \subset W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}.$$

In the following considerations, we assume that coordinates of the vector $\boldsymbol{r} \in \mathbb{R}^d$ in the defined classes are ordered such that $0 < r_1 = r_2 = \cdots = r_\nu < r_{\nu+1} \le \cdots \le r_d$, and also that $\boldsymbol{\gamma} \in \mathbb{R}^d$ is a vector with the coordinates $\gamma_j = r_j/r_1$, $j = 1, \dots, d$. Besides, $\boldsymbol{\gamma}' \in \mathbb{R}^d$, where $\gamma'_j = \gamma_j = 1$ if $j = 1, \dots, \nu$ and $1 < \gamma'_j < \gamma_j$ if $j = \nu + 1, \dots, d$.

For a finite set $\mathfrak{N}$, by $|\mathfrak{N}|$ we denote the number of its elements.

Now we formulate a definition of the norm $\|\cdot\|_{B_{q,1}}$ in the subspaces $B_{q,1}$, $1 \le q \le \infty$, of functions $f \in L_q^0$.

For trigonometric polynomials t with respect to the multiple trigonometric system $\{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, the norm $\|t\|_{B_{q,1}}$ is defined by the formula

$$\|t\|_{B_{q,1}} := \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|A_{\boldsymbol{s}}(t)\|_q.$$

Note, that the sum above contains a finite number of terms.

Similarly we define the norm $\|f\|_{B_{q,1}}$, $1 \le q \le \infty$, for all functions $f \in L_q^0$ such that the series $\sum_{\boldsymbol{s}\in\mathbb{N}^d} \|A_{\boldsymbol{s}}(f)\|_q$ is convergent.

Note, that in the case $1 < q < \infty$ it holds

$$\|f\|_{B_{q,1}} \asymp \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|\delta_{\boldsymbol{s}}(f)\|_q.$$

For $f \in B_{q,1}$, $1 \le q \le \infty$, the following relations hold:

$$\|f\|_q \ll \|f\|_{B_{q,1}}, \quad \|f\|_{B_{1,1}} \ll \|f\|_{B_{q,1}} \ll \|f\|_{B_{\infty,1}}.$$

2 Approximation characteristics and auxiliary statements

Let $\mathcal{X}$ be a Banach space with the norm $\|\cdot\|_{\mathcal{X}}$. For a compact set $A \subset \mathcal{X}$ and $y \in \mathcal{X}$, $R > 0$, we put $B_{\mathcal{X}}(y, R) := \{x \in \mathcal{X} \colon \ \|x - y\|_{\mathcal{X}} \le R\}$, i.e. define the ball $B_{\mathcal{X}}(y, R)$ in $\mathcal{X}$ of radius R centered at the point y.

For $k \in \mathbb{N}$, the quantity (see, e.g., [11])

$$\varepsilon_k(A, \mathcal{X}) := \inf \left\{ \varepsilon > 0 : \exists\, y^1, \dots, y^{2^k} \in \mathcal{X} \colon \ A \subseteq \bigcup_{j=1}^{2^k} B_{\mathcal{X}}(y^j, \varepsilon) \right\}$$

is called the kth entropy number of the set A in the space $\mathcal{X}$.

Let $\mathcal{Y}$ be a normed space with the norm $\|\cdot\|_{\mathcal{Y}}$, $\mathcal{L}_M(\mathcal{Y})$ be a set of subspaces of dimension at most M in the space $\mathcal{Y}$, and W be a centrally-symmetric set in $\mathcal{Y}$.

The quantity

$$d_M(W, \mathcal{Y}) := \inf_{L_M \in \mathcal{L}_M(\mathcal{Y})} \sup_{w\in W} \inf_{u \in L_M} \|w - u\|_{\mathcal{Y}} \tag{1}$$

is called the Kolmogorov M-width of the set W in the space $\mathcal{Y}$.

The width $d_M(W, \mathcal{Y})$ was introduced in 1936 by A.N. Kolmogorov [15].

Let $\mathcal{Y}$ and $\mathcal{Z}$ be normed spaces and $\mathcal{L}(\mathcal{Y}, \mathcal{Z})$ be a set of linear continuous mappings of $\mathcal{Y}$ into $\mathcal{Z}$.

The quantity

$$\lambda_M(W, \mathcal{Y}) := \inf_{\substack{L_M \in \mathcal{L}_M(\mathcal{Y}) \\ \Lambda \in \mathcal{L}(\mathcal{Y}, L_M)}} \sup_{w \in W} \|w - \Lambda w\|_{\mathcal{Y}}$$

is called the linear M-width of the set W in the space $\mathcal{Y}$.

The width $\lambda_M(W, \mathcal{Y})$ was introduced in 1960 by V.M. Tikhomirov [34].

The following approximation characteristic was introduced by R.S. Ismagilov [12]. So, let either $\mathcal{Y} = L_q$ or $\mathcal{Y} = B_{q,1}$, $1 \leq q \leq \infty$, and $F \subset \mathcal{Y}$ be some functional class. The trigonometric M-width of the class F in the space $\mathcal{Y}$ (denoted by $d_M^{\top}(F, \mathcal{Y})$) is defined by the formula

$$d_M^{\top}(F, \mathcal{Y}) := \inf_{\Omega_M} \sup_{f \in F} \inf_{t(\Omega_M; \boldsymbol{x})} \|f(\cdot) - t(\Omega_M; \cdot)\|_{\mathcal{Y}},$$

where

$$t(\Omega_M; \boldsymbol{x}) = \sum_{j=1}^{M} c_j e^{i(\boldsymbol{k}^j, \boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

$\Omega_M = \{\boldsymbol{k}^1, \ldots, \boldsymbol{k}^M\}$ is a set of vectors $\boldsymbol{k}^j \in \mathbb{Z}^d$, c_j be arbitrary complex numbers, $j = 1, \ldots, M$.

Let $\{u_i\}_{i=1}^M$ be an orthonormal in the space L_2 system of functions $u_i \in L_\infty$, $i = 1, \ldots, M$. Each function $f \in L_q$, $1 \leq q \leq \infty$, we put into the correspondence the approximation aggregate of the form $\sum_{i=1}^M (f, u_i) u_i$, i.e. an orthogonal projection of the function f onto the subspace, generated by the system of functions $\{u_i\}_{i=1}^M$.

If $F \subset L_q$, then the quantity

$$d_M^{\perp}(F, L_q) := \inf_{\{u_i\}_{i=1}^M} \sup_{f \in F} \Big\| f - \sum_{i=1}^{M} (f, u_i) u_i \Big\|_q$$

is called the orthowidth (the Fourier widths) of the class F in the space L_q. The width $d_M^{\perp}(F, L_q)$ was introduced by V.N. Temlyakov [28]. Besides, V.N. Temlyakov [29] considered close to the Fourier width quantity $d_M^B(F, L_q)$, which is defined by the formula

$$d_M^B(F, L_q) := \inf_{G \in L_M(B)_q} \sup_{f \in F \cap D(G)} \|f - Gf\|_q. \tag{2}$$

Here $L_M(B)_q$ denotes a set of linear operators, that satisfy the following conditions:

a) the domain $D(G)$ of these operators contains all trigonometric polynomials, and their range is contained in a subset of the space L_q of dimension M;

b) there exists such number $B \geq 1$ that for all vectors $\boldsymbol{k} \in \mathbb{Z}^d$ it holds

$$\|G e^{i(\boldsymbol{k}, \boldsymbol{x})}\|_2 \leq B.$$

Note, that to $L_M(1)_2$ belong operators of an orthogonal projection into the subspaces of dimension M of the space L_2, as well as operators, defined on an orthonormal system of functions by a multiplier determined by such sequence $\{\lambda_l\}_{l \in \mathbb{N}}$ that $|\lambda_l| \leq 1$ for all $l \in \mathbb{N}$.

Let either $\mathcal{Y} = L_q$ or $\mathcal{Y} = B_{q,1}$, $1 \leq q \leq \infty$. Then the approximation characteristics (1), (2) of the classes $F \subset \mathcal{Y}$ relate as follows:

$$\begin{aligned} &d_M(F, \mathcal{Y}) \leq d_M^B(F, \mathcal{Y}) \leq d_M^{\perp}(F, \mathcal{Y}), \\ &d_M(F, \mathcal{Y}) \leq \lambda_M(F, \mathcal{Y}) \leq d_M^{\perp}(F, \mathcal{Y}), \\ &d_M(F, \mathcal{Y}) \leq d_M^{\top}(F, \mathcal{Y}). \end{aligned} \tag{3}$$

Note also, that the quantities (1), (2) on the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and Nikol'skii-Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ in the spaces L_q, $1 \leq q \leq \infty$, were extensively studied. For the corresponding bibliography, we refer to the monographs [5, 20, 29, 31, 33, 35]. Concerning the results of investigation of these quantities in the spaces $B_{q,1}$, see the papers [2, 3, 13, 21–24, 30].

Let us formulate some known statements that we will use in further argumentation. A corollary from one of the B. Carl inequalities [4] is the following statement.

Lemma A ([14,32]). *Let $\mathcal{K}$ be a compact set in a separable Banach space $\mathcal{X}$. Assume that for a pair of numbers (a,b), where either $a>0$, $b\in\mathbb{R}$ or $a=0$, $b<0$, the relations*

$$d_M(\mathcal{K},\mathcal{X}) \ll M^{-a}(\log M)^b, \quad \varepsilon_M(\mathcal{K},\mathcal{X}) \gg M^{-a}(\log M)^b$$

are true. Then it holds

$$\varepsilon_M(\mathcal{K},\mathcal{X}) \asymp d_M(\mathcal{K},\mathcal{X}) \asymp M^{-a}(\log M)^b.$$

Lemma B ([29]). *Let $s\in\mathbb{N}^d$, $\gamma\in\mathbb{R}^d$, $\gamma_j>0$, $j=1,\dots,d$ and $\gamma'\in\mathbb{R}^d$ is such that $\gamma_j=\gamma'_j=1$ for $j=1,\dots,\nu$ and $1<\gamma'_j<\gamma_j$ for $j=\nu+1,\dots,d$. Then for $\alpha>0$ the following estimate holds*

$$\sum_{(s,\gamma')\geq l} 2^{-\alpha(s,\gamma)} \asymp 2^{-\alpha l} l^{\nu-1}.$$

Theorem A ([22]). *Let $d\geq 1$, $2<p<\infty$, $r_1>1/2$. Then for $\boldsymbol{\alpha}\in\mathbb{R}^d$ it holds*

$$\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{\infty,1}) \asymp d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{\infty,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}.$$

Theorem B ([21]). *Let $d\geq 1$, $1<p<\infty$, $r_1>0$. Then for $\boldsymbol{\alpha}\in\mathbb{R}^d$ it holds*

$$\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{1,1}) \asymp d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}.$$

Theorem C. *Let $d\geq 1$, $1<q\leq p<\infty$, $r_1>0$. Then for $\boldsymbol{\alpha}\in\mathbb{R}^d$ it holds*

$$\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q) \asymp d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1}. \tag{4}$$

The history of investigation of the quantity $\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q)$ under the conditions of Theorem C can be found in [5, Theorem 6.2.1]. The estimates of the Kolmogorov widths with corresponding comments are also given in [5, Theorem 4.3.1].

Theorem D. *Let $d\geq 1$, $1<q\leq p<\infty$, $r_1>0$. Then for $\boldsymbol{\alpha}\in\mathbb{R}^d$ it holds*

$$\lambda_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q) \asymp d^{\top}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1}. \tag{5}$$

The estimate of the linear widths $\lambda_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q)$ in (5) with corresponding comments is given in [5, Theorem 4.5.1]. Concerning the trigonometric width $d^{\top}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q)$, we note that its upper estimate is realized by the approximation of functions from the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ by their step hyperbolic Fourier sums [5, Theorem 4.3.5]. The corresponding lower estimate is a corollary from the estimate of the Kolmogorov width (see Theorem C).

Theorem E ([3]). *Let $d\geq 1$, $r_1>1/2$. Then it holds*

$$\varepsilon_M(H^{\boldsymbol{r}}_2, B_{\infty,1}) \asymp d_M(H^{\boldsymbol{r}}_2, B_{\infty,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1}.$$

Theorem F ([21]). *Let $d\geq 1$, $1\leq p\leq\infty$, $1\leq\theta\leq\infty$, $r_1>0$. Then it holds*

$$\varepsilon_M(B^{\boldsymbol{r}}_{p,\theta}, B_{1,1}) \asymp d_M(B^{\boldsymbol{r}}_{p,\theta}, B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1-1/\theta}.$$

Theorem G. *Let $d\geq 1$, $1\leq q\leq p\leq\infty$, $(q,p)\notin\{(1,1),(\infty,\infty)\}$, $1\leq\theta\leq\infty$ and $p^*=\min\{2;p\}$. Then for $r_1>0$ it holds*

$$d^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta}, L_q) \asymp d^{B}_M(B^{\boldsymbol{r}}_{p,\theta}, L_q) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+(1/p^*-1/\theta)_+}, \tag{6}$$

where $a_+=\max\{a,0\}$.

The estimates (6) in the case $1 \leq \theta < \infty$ are obtained in [17–19], and for $\theta = \infty$, i.e. for the classes $H^{\boldsymbol{r}}_p$, in [30].

The corresponding statement for the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ has the following form.

Theorem H ([30]). *Let $d \geq 1$, $1 \leq q \leq p \leq \infty$, $(q,p) \notin \{(1,1),(\infty,\infty)\}$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$d^{\perp}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q) \asymp d^{B}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1}.$$

3 Entropy numbers and widths

The following statement holds.

Theorem 1. *Let $d \geq 2$ and either $r_1 > 0$, $1 < p \leq 2$ or $r_1 > 1/2$, $2 < p < \infty$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{7}$$

Proof. In order to use Lemma A, we first prove the upper estimate for the Kolmogorov width $d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1})$ for $1 < p \leq 2$. It will be convenient for us to prove the needed estimate in a more general case, namely for the classes $B^{\boldsymbol{r}}_{p,2}$. Here we note the embedding $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,2}$, $1 < p \leq 2$.

Hence, let f be an arbitrary function form the class $B^{\boldsymbol{r}}_{p,2}$, $1 < p \leq 2$. Let us consider its approximation polynomial of the form

$$S_{Q^{\gamma'}_n}(f) := S_{Q^{\gamma'}_n}(f, \boldsymbol{x}) := \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} \delta_{\boldsymbol{s}}(f, \boldsymbol{x}),$$

where the number $n \in \mathbb{N}$ is choosen according to the relation $M \asymp 2^n n^{\nu-1}$.

The polynomial $S_{Q^{\gamma'}_n}(f)$ is called the step hyperbolic Fourier sum of the function f, and for its number of harmonics it holds

$$|Q^{\gamma'}_n| = \left| \bigcup_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} \rho(\boldsymbol{s}) \right| \asymp 2^n n^{\nu-1}.$$

Then, by the norm definition of the space $B_{p,1}$, we can write

$$\begin{aligned} \|f - S_{Q^{\gamma'}_n}(f)\|_{B_{p,1}} &= \Big\| \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} \delta_{\boldsymbol{s}}(f) \Big\|_{B_{p,1}} \asymp \sum_{\boldsymbol{s}\in\mathbb{N}^d} \left\| \delta_{\boldsymbol{s}} \left(\sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ (\boldsymbol{s}',\boldsymbol{\gamma}')\geq n}} \delta_{\boldsymbol{s}'}(f) \right) \right\|_p \\ &\leq \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} \|\delta_{\boldsymbol{s}}(f)\|_p =: J_1. \end{aligned} \tag{8}$$

Further, using the Cauchy-Bunyakovsky inequality and Lemma B, we get

$$\begin{aligned} J_1 &\leq \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} 2^{2(\boldsymbol{s},\boldsymbol{r})} \|\delta_{\boldsymbol{s}}(f)\|^2_p \right)^{1/2} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} 2^{-2(\boldsymbol{s},\boldsymbol{r})} \right)^{1/2} \\ &\ll \|f\|_{B^{\boldsymbol{r}}_{p,2}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} 2^{-2r_1(\boldsymbol{s},\boldsymbol{\gamma})} \right)^{\frac{1}{2}} \ll 2^{-nr_1} n^{(\nu-1)/2}. \end{aligned} \tag{9}$$

Hence, taking into account that $M \asymp 2^n n^{\nu-1}$, and the relations (8), (9), we obtain

$$d_M(B^{\boldsymbol{r}}_{p,2}, B_{p,1}) \ll \sup_{f\in B^{\boldsymbol{r}}_{p,2}} \|f - S_{Q_n^{\gamma'}}(f)\|_{B_{p,1}} \ll 2^{-nr_1} n^{(\nu-1)/2} \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \qquad (10)$$

In view of the mentioned above embedding $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,2}$, $1 < p \le 2$, (10) yields the required estimate

$$d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \ll d_M(B^{\boldsymbol{r}}_{p,2}, B_{p,1}) \ll M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}, \quad 1 < p \le 2. \qquad (11)$$

Let $2 < p < \infty$. Then, by the relation $\|\cdot\|_{B_{\infty,1}} \gg \|\cdot\|_{B_{p,1}}$ and Theorem A, we can write

$$d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \ll d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{\infty,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}.$$

Concerning the lower estimate of the entropy numbers $\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1})$, we note that it follows from Theorem B, i.e.

$$\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \gg \varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \qquad (12)$$

To complete the proof, we use Lemma A with respect to the estimates (11), (12), and get (7). Theorem 1 is proved. □

In addition to the obtained above result, let us formulate a statement that concerns the univariate case, where for $2 < p < \infty$ we can weaken the restrictions on the parameter r_1.

Theorem 1′. *Let $d = 1$, $r_1 > 0$, $1 < p < \infty$. Then for $\alpha \in \mathbb{R}$ it holds*

$$\varepsilon_M(W^{r_1}_{p,\alpha}, B_{p,1}) \asymp d_M(W^{r_1}_{p,\alpha}, B_{p,1}) \asymp M^{-r_1}. \qquad (13)$$

Proof. The upper estimate for the width $d_M(W^{r_1}_{p,\alpha}, B_{p,1})$ follows from [25, Corollary 1] as $M \asymp 2^n$, i.e.

$$d_M(W^{r_1}_{p,\alpha}, B_{p,1}) \ll d_M(H^{r_1}_p, B_{p,1}) \ll \sup_{f\in H^{r_1}_p} \Big\| f - \sum_{k=-2^n}^{2^n} \widehat{f}(k) e^{ikx} \Big\|_{B_{p,1}} \asymp 2^{-nr_1} \asymp M^{-r_1}. \qquad (14)$$

The lower estimate for the entropy numbers $\varepsilon_M(W^{r_1}_{p,\alpha}, B_{p,1})$ follows from Theorem C for $\nu = 1$ and the inequality $\|\cdot\|_{B_{p,1}} \gg \|\cdot\|_p$, i.e.

$$\varepsilon_M(W^{r_1}_{p,\alpha}, B_{p,1}) \gg \varepsilon_M(W^{r_1}_{p,\alpha}, L_p) \asymp M^{-r_1}. \qquad (15)$$

Using Lemma A with respect to (14), (15), we get (13). Theorem $1'$ is proved. □

Remark 1. *Comparing the estimate (4) for $p = q$, $d \ge 2$ with (7), we see that under the conditions of Theorem 1 on the parameters p and r_1 the estimates of the respective characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ differ in order by the factor $\log^{(\nu-1)/2} M$. In the univariate case, we have a different situation, namely, the considered approximation characteristics of the classes $W^{r_1}_{p,\alpha}$ in the spaces L_p (Theorem C) and $B_{p,1}$ (Theorem $1'$) coincide in order.*

It is also worth noting another detail, which appeared to be specific for the Kolmogorov width of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space $B_{p,1}$ for $d \ge 2$.

While getting the upper estimate for the quantity $d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1})$, we obtained that in the case $1 < p \le 2$ it is realized by a subspace of trigonometric polynomials with "numbers" of harmonics from the step hyperbolic cross $Q_n^{\gamma'}$, where the numbers n and M relate as $M \asymp 2^n n^{\nu-1}$.

In contrast, in the case $2 < p < \infty$, we are not aware of the M-dimensional subspaces that realize the obtained orders of the Kolmogorov widths of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space $B_{p,1}$. Due to this fact, we recall that in the space L_p the above mentioned subsets of trigonometric polynomials are optimal from the point of view of orders of the Kolmogorov width $d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_p)$ for all $1 < p < \infty$.

In what follows, we formulate two corollaries from Theorems 1 and 1′, which concern the estimates of the linear and trigonometric width of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space $B_{p,1}$.

Corollary 1. *Let $d \geq 2$, $1 < p \leq 2$, $r_1 > 0$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$\lambda_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp d^{\top}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{16}$$

Proof. The upper estimates for both of the width are obtained by an approximation of functions $f \in W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ by trigonometric polynomials $S_{Q_n^{\gamma'}}(f)$ as $M \asymp 2^n n^{\nu-1}$. The corresponding arguments were used while proving Theorem 1.

The lower estimates in (16) follow from the estimate of the Kolmogorov width $d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1})$, $1 \leq p < 2$, obtained in Theorem 1, and the relations (3). Corollary 1 is proved. □

To complement this, let us formulate one more corollary concerning the univariate case, where it appeared possible to cover the relation $2 < p < \infty$.

Corollary 1′. *Let $d = 1$, $1 < p < \infty$, $r_1 > 0$. Then for $\alpha \in \mathbb{R}$ it holds*

$$\lambda_M(W^{r_1}_{p,\alpha}, B_{p,1}) \asymp d^{\top}_M(W^{r_1}_{p,\alpha}, B_{p,1}) \asymp M^{-r_1}. \tag{17}$$

Proof. The upper estimates in (17) are realized by approximation of functions from the classes $H^{r_1}_p$ by the Fourier sums of respective order and the embedding $W^{r_1}_{p,\alpha} \subset H^{r_1}_p$ (see [25, Theorem 1]). The lower estimates for both of the widths follow from the estimate of the Kolmogorov width $d_M(W^{r_1}_{p,\alpha}, B_{p,1})$ obtained in (13) and the relations (3). Corollary 1′ is proved. □

Remark 2. *Comparing Corollaries 1 and 1′ with the corresponding statements in the space L_p (Theorem D), we come to the conclusion that the considered in these spaces approximation characteristics have equal orders only either in the univariate case or for $\nu = 1$.*

Theorem 2. *Let $d \geq 2$, $1 < q < p < \infty$ and either $r_1 > 0$, $1 < q \leq 2$ or $r_1 > 1/2$, $2 < q < \infty$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \asymp d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{18}$$

Proof. As in proof of Theorem 1, in order to use Lemma A, we first get the upper estimates for the width $d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}},\ B_{q,1})$.

For this, let us consider several cases.

a) Let $1 < q < p \leq 2$. Then, taking into account the inequality $\|\cdot\|_{B_{q,1}} \ll \|\cdot\|_{B_{p,1}}$ and using the corresponding estimate from Theorem 1, we can write

$$d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \ll d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{19}$$

b) Let $1 < q \leq 2 < p < \infty$. In view of $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}$ and the estimate (19) in the case $p = 2$, we have

$$d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \leq d_M(W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}, B_{q,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}.$$

c) Let $2 < q < p < \infty$. Since in this case $\|\cdot\|_{B_{q,1}} \ll \|\cdot\|_{B_{p,1}}$, from Theorem 1 we obtain

$$d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \ll d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}.$$

Concerning the lower estimate for the entropy numbers $\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1})$, we note that it follows as a corollary from Theorem B, i.e.

$$\varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \gg \varepsilon_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{20}$$

Hence, using Lemma A with respect to the estimates (19), (20), we get (18). Theorem 2 is proved. □

Further we complement the obtained result by considering the univariate case, where for $2 < q < \infty$ it appeared possible to weaken the restrictions on the parameter r_1.

Theorem 2′. *Let $d = 1$, $1 < q < p < \infty$, $r_1 > 0$. Then for $\alpha \in \mathbb{R}$ it holds*

$$\varepsilon_M(W^{r_1}_{p,\alpha}, B_{q,1}) \asymp d_M(W^{r_1}_{p,\alpha}, B_{q,1}) \asymp M^{-r_1}. \tag{21}$$

Proof. The upper estimate of the width $d_M(W^{r_1}_{p,\alpha}, B_{q,1})$ follows from Theorem 1′ , i.e.

$$d_M(W^{r_1}_{p,\alpha}, B_{q,1}) \ll d_M(W^{r_1}_{p,\alpha}, B_{p,1}) \asymp M^{-r_1}. \tag{22}$$

The lower estimate in (21) for the entropy numbers $\varepsilon_M(W^{r_1}_{p,\alpha}, B_{q,1})$ follows from Theorem B and the relation $\|\cdot\|_{B_{q,1}} \gg \|\cdot\|_{B_{1,1}}$, i.e.

$$\varepsilon_M(W^{r_1}_{p,\alpha}, B_{q,1}) \gg \varepsilon_M(W^{r_1}_{p,\alpha}, B_{1,1}) \asymp M^{-r_1}. \tag{23}$$

Using Lemma A to (22) and (23), we get (21). Theorem 2′ is proved. □

Remark 3. *In the case $q = 1$, the relations* (18) *and* (21) *are obtained in [21] (see Theorem B).*

Remark 4. *Comparing the results of Theorems 2, 2′ and C under the respective values of the parameter r_1, we see that the considered approximation characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the spaces $B_{q,1}$ and L_q for $\nu \neq 1$ differ in order.*

Corollary 2. *Let $d \geq 2$, $1 < q < p < \infty$, $q \leq 2$, $r_1 > 0$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$\lambda_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \asymp d^{\top}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{24}$$

Proof. The upper estimates for both of the width are proved using Corollary 1. Let us consider two cases.

a) Let $1 < q < p \leq 2$. Then, taking into account the inequality $\|\cdot\|_{B_{q,1}} \ll \|\cdot\|_{B_{p,1}}$ and the estimate (16) for the linear width, we obtain

$$\lambda_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \ll \lambda_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+\frac{1}{2}}. \tag{25}$$

The same estimate, by (16), holds also for the trigonometric width, i.e.

$$d^{\top}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \ll d^{\top}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+\frac{1}{2}}. \tag{26}$$

b) Let $1 < q \leq 2 < p < \infty$. Taking into account that in this case $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}$ and the estimates (25) and (26) for $p = 2$, we get the required estimates for the corresponding quantities from above.

The lower estimates in (24) follow from the estimate $d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{1,1})$ of the Kolmogorov width (see Theorem B) and the relations (3). Corollary 2 is proved. □

Let us formulate a corollary for the univariate case, that covers also the case $q > 2$.

Corollary 2′. *Let $d = 1$, $1 < q < p < \infty$, $r_1 > 0$. Then for $\alpha \in \mathbb{R}$ it holds*

$$\lambda_M(W^{r_1}_{p,\alpha}, B_{q,1}) \asymp d^{\top}_M(W^{r_1}_{p,\alpha}, B_{q,1}) \asymp M^{-r_1}. \tag{27}$$

Proof. The upper estimate in (27) follow from (17) and the relation $\|\cdot\|_{B_{q,1}} \ll \|\cdot\|_{B_{p,1}}$. The corresponding lower estimates are the corollaries from the estimate of the Kolmogorov width $d_M(W^{r_1}_{p,\alpha},\, B_{1,1})$ (Theorem B) and the relations (3). Corollary 2′ is proved. □

Remark 5. *Comparing the estimates* (24) *and* (27) *with the result of Theorem D, we see that the corresponding approximation characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ have equal orders in the spaces $B_{q,1}$ and L_q only either in the univariate case or for $\nu = 1$.*

To conclude this part of the paper, let us prove the statement that concerns the Nikol'skii classes $H^{\boldsymbol{r}}_p$ and that extends the result of Theorem E from the classes $H^{\boldsymbol{r}}_2$ into the classes $H^{\boldsymbol{r}}_p$, $2 < p < \infty$.

The following statement holds.

Theorem 3. *Let $d \geq 1$, $r_1 > 1/2$. Then for $2 \leq p \leq \infty$ it holds*

$$\varepsilon_M(H^{\boldsymbol{r}}_p, B_{\infty,1}) \asymp d_M(H^{\boldsymbol{r}}_p, B_{\infty,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1}. \tag{28}$$

Note, that for the case $p = \infty$ the relations (28) were obtained in [22], and for $p = 2$, as it was already mentioned, in Theorem E.

Proof. In view of $H^{\boldsymbol{r}}_p \subset H^{\boldsymbol{r}}_2$, $2 < p < \infty$, using the result of Theorem E, we can write the upper estimate for the Kolmogorov width $d_M(H^{\boldsymbol{r}}_p, B_{\infty,1})$, namely

$$d_M(H^{\boldsymbol{r}}_p, B_{\infty,1}) \leq d_M(H^{\boldsymbol{r}}_2, B_{\infty,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1}. \tag{29}$$

The lower estimate of the entropy numbers $\varepsilon_M(H^{\boldsymbol{r}}_p, B_{\infty,1})$ is a corollary from [30, Theorem 2.2] (see the remark), where it was proved that

$$\varepsilon_M(H^{\boldsymbol{r}}_\infty, B_{1,1}) \gg M^{-r_1}(\log^{\nu-1} M)^{r_1+1}. \tag{30}$$

Hence, since $H^{\boldsymbol{r}}_p \supset H^{\boldsymbol{r}}_\infty$ and $\|\cdot\|_{B_{\infty,1}} \gg \|\cdot\|_{B_{1,1}}$, according to (30) we have

$$\varepsilon_M(H^{\boldsymbol{r}}_p, B_{\infty,1}) \gg \varepsilon_M(H^{\boldsymbol{r}}_\infty, B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1}. \tag{31}$$

To conclude the proof of Theorem 3, we use Lemma A to (29) and (31) and get (28). □

Let us comment on the obtained result.

A statement, that corresponds to Theorem 3 in the space L_∞, is known only for two dimensions: $d = 1$ and $d = 2$. For convenience, let us recall these estimates.

Let $d = 1$. Then the following relations hold

$$\varepsilon_M(H^{r_1}_p, L_\infty) \asymp d_M(H^{r_1}_p, L_\infty) \asymp M^{-r_1}, \quad 2 \leq p \leq \infty, \quad r_1 > \frac{1}{2},$$

that are obtained by using Lemma A to the upper estimate of the Kolmogorov width $d_M(H^{r_1}_p, L_\infty)$ (see [31, Chapter 1, Theorem 4.1]) and the corresponding lower estimate of the entropy numbers $\varepsilon_M(H^{r_1}_\infty, L_1)$ [30, Theorem 2.2].

For $\boldsymbol{r} = (r_1, r_1)$, $r_1 > 1/2$, $2 \le p \le \infty$, it is known that

$$\varepsilon_M(H^{\boldsymbol{r}}_p, L_\infty) \asymp d_M(H^{\boldsymbol{r}}_p, L_\infty) \asymp M^{-r_1}(\log M)^{r_1+1}. \tag{32}$$

The history of obtaining these estimates can be found in [5, Theorems 4.3.14, 6.3.4].

Hence, comparing (28) with (32), we see that for the dimensions $d = 1$ and $d = 2$ the corresponding approximation characteristics of the classes $H^{\boldsymbol{r}}_p$ in the spaces L_∞ and $B_{\infty,1}$ coincide in order.

Remark 6. *A statement, analogical to Theorem 3, for the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, $2 \le p < \infty$, is known and was proved for $p = 2$ in [3], and for $2 < p < \infty$ in [22]. We recalled this result in Theorem A.*

4 Orthowidth and a close to it approximation characteristics

As it was already mentioned in Introduction, in this part of the paper we investigate orthowidths of the Nikol'skii-Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ and Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, as well as close to them approximation characteristics, in the space $B_{q,1}$ for some relations between the parameters p and q.

Theorem 4. *Let $d \ge 1$, $1 < p < \infty$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ it holds*

$$d^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta}, B_{p,1}) \asymp d^B_M(B^{\boldsymbol{r}}_{p,\theta}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1-1/\theta}. \tag{33}$$

Proof. According to (3), it is sufficient to obtain the upper estimate in (33) for the orthowidth $d^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta}, B_{p,1})$ and the lower estimate for the quantity $d^B_M(B^{\boldsymbol{r}}_{p,\theta}, B_{p,1})$. Let us first consider the case $d \ge 2$.

Let the numbers M and n relate as $M \asymp 2^n n^{\nu-1}$. For a function $f \in B^{\boldsymbol{r}}_{p,\theta}$, we consider the approximation polynomial $S_{Q^{\gamma'}_n}(f)$, which, as indicated above, for $M \asymp 2^n n^{\nu-1}$ contains a number of harmonics of order M.

Using similar arguments as in (8), we get that

$$\Big\| f - \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} \delta_{\boldsymbol{s}}(f) \Big\|_{B_{p,1}} \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} \|\delta_{\boldsymbol{s}}(f)\|_p =: J_2. \tag{34}$$

To further estimate the quantity J_2, let us consider two cases.

a) Let $1 \le \theta < \infty$. Then, by the Holder's inequality with the power θ (and its corresponding modification for the case $\theta = 1$), using Lemma B, for $1/\theta + 1/\theta' = 1$ we get

$$\begin{aligned} J_2 &\le \Big(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(f)\|^{\theta}_p \Big)^{1/\theta} \Big(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{-(\boldsymbol{s},\boldsymbol{r})\theta'} \Big)^{1/\theta'} \\ &\ll \|f\|_{B^{\boldsymbol{r}}_{p,\theta}} \Big(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})r_1\theta'} \Big)^{1/\theta'} \ll 2^{-nr_1} n^{(\nu-1)(1-1/\theta)}. \end{aligned}$$

b) Let $\theta = \infty$. Then, by the fact that for $f \in B^{\boldsymbol{r}}_{p,\infty}$ it holds $\|\delta_{\boldsymbol{s}}(f)\|_p \ll 2^{-(\boldsymbol{s},\boldsymbol{r})}$, $\boldsymbol{s} \in \mathbb{N}^d$, and again using Lemma B, we obtain

$$J_2 \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{-(\boldsymbol{s},\boldsymbol{r})} \ll 2^{-nr_1} n^{\nu-1}. \tag{35}$$

Further, combining (34), (35) and taking into account that $M \asymp 2^n n^{\nu-1}$, we get the upper estimate for the orthowidth $d_M^{\perp}(B^{\boldsymbol{r}}_{p,\theta},\, B_{p,1})$, i.e.

$$d_M^{\perp}(B^{\boldsymbol{r}}_{p,\theta}, B_{p,1}) \ll M^{-r_1}(\log^{\nu-1} M)^{r_1+1-1/\theta}. \tag{36}$$

Let further $d = 1$. Then the upper estimate of the orthowidth $d_M^{\perp}(B^{r_1}_{p,\theta},\, B_{p,1})$ is obtained by approximating the functions $f \in B^{r_1}_{p,\theta}$ by their partial Fourier sums [25, Theorem 1]. It takes the form

$$d_M^{\perp}(B^{r_1}_{p,\theta},\, B_{p,1}) \ll \sup_{f\in B^{r_1}_{p,\theta}} \Big\| f(x) - \sum_{k=-M}^{M} \widehat{f}(k) e^{ikx} \Big\|_{B_{p,1}} \asymp M^{-r_1}.$$

To conclude the proof, let us note that the lower estimate for the quantity $d_M^{B}(B^{\boldsymbol{r}}_{p,\theta},\, B_{p,1})$ for $d \geq 1$ follows from Theorem F and the relations

$$d_M^{B}(B^{\boldsymbol{r}}_{p,\theta},\, B_{p,1}) \gg d_M^{B}(B^{\boldsymbol{r}}_{p,\theta},\, B_{1,1}) \geq d_M(B^{\boldsymbol{r}}_{p,\theta},\, B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1-1/\theta}. \tag{37}$$

Hence, combining (36), (37), we derive (33). Theorem 4 is proved. □

Further, applying the obtained in Theorem 4 result, let us get the estimates of the corresponding characteristics for the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$.

Theorem 5. *Let $r_1 > 0$ and either $1 < p < \infty$ for $d = 1$ or $1 < p \leq 2$ for $d \geq 2$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$d_M^{\perp}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp d_M^{B}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{38}$$

Proof. The upper estimates for both of the characteristics are corollaries from Theorem 4.

So, for $d = 1$, putting $\nu = 1$, $\theta = \infty$ in (33), for $1 < p < \infty$ we get

$$d_M^{B}(W^{r_1}_{p,\alpha}, B_{p,1}) \leq d_M^{\perp}(W^{r_1}_{p,\alpha}, B_{p,1}) \ll d_M^{\perp}(H^{r_1}_{p}, B_{p,1}) \asymp M^{-r_1}. \tag{39}$$

In the case $d \geq 2$ and $1 < p \leq 2$, we take into account that $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,2}$ and use the estimate (33) for $\theta = 2$. We obtain $d_M^{\perp}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \ll d_M^{\perp}(B^{\boldsymbol{r}}_{p,2}, B_{p,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}$.

The lower estimate for the quantity $d_M^{B}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1})$ for $d \geq 1$ follows from Theorem B, in view of

$$d_M^{B}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{p,1}) \gg d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{40}$$

Combining (39), (40), we obtain (38). Theorem 5 is proved. □

Remark 7. *Comparing the results of Theorems 4 and 5 with the corresponding statements in the space L_p (Theorems G and H), we conclude the following.*

In the multivariate case ($d \geq 2$), in contrast to the univariate, considered approximation characteristics of both of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and $B^{\boldsymbol{r}}_{p,\theta}$ differ in order in the spaces L_p and $B_{p,1}$ (except the values $\nu = 1$, $\theta = 1$).

Theorem 6. *Let $d \geq 1$, $1 < q < p \leq \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ it holds*

$$d_M^{\perp}(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \asymp d_M^{B}(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1-1/\theta}. \tag{41}$$

Note, that for $q = 1$ the order of the corresponding quantities is obtained in [24].

Proof. The upper estimates for both of the approximation characteristics for $p \neq \infty$ follow from Theorem 4 and the relation $\|\cdot\|_{B_{q,1}} \ll \|\cdot\|_{B_{p,1}}$. In the case $p = \infty$, the corresponding estimates are corollaries from the proven one for $p < \infty$ and the embedding $B^{\boldsymbol{r}}_{\infty,\theta} \subset B^{\boldsymbol{r}}_{p,\theta}$.

The respective lower estimates in (41) follow from Theorem F and the relations

$$d^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \geq d^B_M(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \gg d^B_M(B^{\boldsymbol{r}}_{p,\theta}, B_{1,1}) \geq d_M(B^{\boldsymbol{r}}_{p,\theta}, B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1-1/\theta}.$$

Theorem 6 is proved. □

A corresponding to Theorem 6 statement for the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ has the following form.

Theorem 7. *Let $r_1 > 0$ and either $1 < q < p < \infty$ for $d = 1$ or $1 < q < p < \infty$, $q \leq 2$ for $d \geq 2$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$d^{\perp}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \asymp d^B_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}. \tag{42}$$

Proof. The upper estimates in (42) follow from Theorem 5. So, in the case $d = 1$, we get

$$d^B_M(W^{r_1}_{p,\alpha}, B_{q,1}) \leq d^{\perp}_M(W^{r_1}_{p,\alpha}, B_{q,1}) \ll d^{\perp}_M(W^{r_1}_{p,\alpha}, B_{p,1}) \asymp M^{-r_1}.$$

If $d \geq 2$, then taking into account that $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset W^{\boldsymbol{r}}_{q,\boldsymbol{\alpha}}$ and using the estimates (38), we obtain the relations $d^B_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \leq d^{\perp}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \leq d^{\perp}_M(W^{\boldsymbol{r}}_{q,\boldsymbol{\alpha}}, B_{q,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}$.

The lower estimate of the quantity $d^B_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}},\ B_{q,1})$ follows from Theorem B, i.e. the relations $d^B_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}},\ B_{q,1}) \gg d^B_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}},\ B_{1,1}) \geq d_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}},\ B_{1,1}) \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1/2}$ hold. Theorem 7 is proved. □

Remark 8. *Comparing the results of Theorems 6, 7 with he corresponding statements in the space L_p (see Theorems G, H), we see that in the multivariate case ($d \geq 2$) the considered approximation characteristics in the spaces L_p and $B_{p,1}$ differ in order.*

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Одержано точні за порядком оцінки ентропійних чисел і низки поперечників (колмогоровський, лінійний, тригонометричний та ортопоперечник) класів Соболєва та Нікольського-Бєсова періодичних функцій однієї та багатьох змінних у просторі $B_{q,1}$, $1 < q < \infty$. Виявлено, що у багатовимірному випадку, на противагу одновимірному, встановлені оцінки відрізняються за порядком від відповідних оцінок у L_q-просторі.

Ключові слова і фрази: клас Соболєва, клас Нікольського-Бєсова, ентропійне число, поперечник.