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Inverse free boundary problem for degenerate parabolic equation

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The coefficient inverse problem for a degenerate parabolic equation is studied in a free boundary domain. The degeneration of the equation is caused by time dependent function at the higher order derivative of unknown function. It is assumed that the coefficient at the minor derivative of the equation is a polynomial of the first order for the space variable with two unknown time depended functions. The conditions of existence and uniqueness of the classical solution to such inverse problem are established for the weak degeneration case at the Dirichlet boundary conditions and the values of heat moments as overdetermination conditions.

Key words and phrases: coefficient inverse problem, free boundary problem, weak power degeneration, parabolic equation, minor coefficient.

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Introduction

Actually the considered problem includes three types of problems, namely, coefficient inverse problem, free boundary problem and problem with degeneration for a parabolic equation. Each of these types is studied separately in sufficient detail. Thus, in works [1, 4, 8, 13, 24, 26, 27, 29–32, 34, 36, 38], coefficient inverse problems for parabolic equations without degenerations in a domain with fixed boundaries were considered. Note that among these papers there are coefficient inverse problems with an unknown coefficient of a parabolic equation that depends on time only [4, 8, 13, 27, 30, 32, 34] or on a spatial variable [1, 26, 31, 36, 38] only. The mathematical models of these problems arise in material sciences, heat transfer and transport problems, groundwater pollutant source estimation in cities with large populations, in chemical or biochemical application, where unknown coefficient is interpreted as a reaction term (see [26, 27] and bibliography there). Inverse problems for parabolic equations in a free boundary domains are also the topics of many papers. Such problems with unknown time dependent coefficients were studied in [5, 7, 11, 14, 15, 23, 28, 37].

Problems for parabolic equations with degenerations arise when describing such processes as the movement of liquids and gases in a porous medium, desalination of sea water, the behavior of financial markets, population dynamics etc. Inverse problems for determination of the function $a = a(t)$, $a(t) > 0$, $t \in [0, T]$, in parabolic equation

$$u_t = a(t)t^{\beta}u_{xx} + b(x,t)u_x + c(x,t)u + f(x,t)$$

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were studied in [22, 35] for both cases of weak ($0 < \beta < 1$) and strong ($\beta \geq 1$) degeneration, respectively. Coefficient inverse problems for parabolic equations with degenerations were analyzed also in [2, 3, 9, 12, 16, 33, 39]. Among these problems are those where the unknown coefficients depend on time in parabolic equations with degeneration in the time variable [3, 9, 12, 16], and those where the unknown coefficients depend on the spatial variable in the equation with degeneration in this variable [2, 33, 39]. Note that all mentioned problems were considered in the domain with fixed boundaries. The problems of determining the coefficients depend simultaneously on time and space variables in the parabolic equations with degenerations remain unexplored today. Inverse problems of identification the time dependent coefficient in the degenerate parabolic equations were studied in [10, 17, 18].

In this paper, in a free boundary domain we consider the inverse problem of identification of two time dependent functions in a minor coefficient of a parabolic equation with degeneration. It is assumed that the degeneration of this equation is caused by the power function with respect to time at the higher order derivative of unknown function in the equation. The case of weak degeneration is investigated. The conditions of unique solvability to this problem under the given Dirichlet boundary conditions and the values of heat moments as overdetermination conditions are established. Note that such problems in the domain with fixed boundaries are studied in [6, 19, 20] for the case of weak and strong degeneration with different boundary and overdetermination conditions.

1 Statement of the problem and the main results

In a free boundary domain $\Omega_T = \{(x,t) : 0 < x < h(t), 0 < t < T\}$, where $h = h(t)$ is an unknown function, it is considered an inverse problem for simultaneous determination of the time dependent coefficients $b_1 = b_1(t)$, $b_2 = b_2(t)$ in one-dimensional degenerate parabolic equation

$$u_t = t^\beta a(t) u_{xx} + \big(b_1(t)x + b_2(t)\big) u_x + c(x,t)u + f(x,t) \tag{1}$$

with initial condition

$$u(x,0) = \varphi(x), \quad x \in \big[0, h(0)\big], \tag{2}$$

boundary conditions

$$u(0,t) = \mu_1(t), \quad u\big(h(t),t\big) = \mu_2(t), \quad t \in [0,T], \tag{3}$$

and overdetermination conditions

$$\int_0^{h(t)} u(x,t)dx = \mu_3(t), \quad t \in [0,T], \tag{4}$$

$$\int_0^{h(t)} xu(x,t)dx = \mu_4(t), \quad t \in [0,T], \tag{5}$$

$$\int_0^{h(t)} x^2 u(x,t)dx = \mu_5(t), \quad t \in [0,T]. \tag{6}$$

It is known, that $a = a(t)$ is a strongly positive continuous function and degeneration of the equation (1) is caused by power function t^β. It is studied the case of weak degeneration while $0 < \beta < 1$.

Definition. *A set of functions* $(b_1, b_2, h, u) \in \big(C[0, T_0]\big)^2 \times C^1[0, T_0] \times C^{2,1}(\Omega_{T_0}) \cap C^{1,0}(\overline{\Omega}_{T_0})$, $h(t) > 0$, $t \in [0, T_0]$, *that satisfy the equation* (1) *and conditions* (2) – (6) *point by point for all* $t \leq T_0$ *is called the local solution to the problem* (1) – (6) *at* $T_0 < T$ *and the global solution to this problem at* $T_0 = T$.

Substituting $y = \dfrac{x}{h(t)}$, we reduce (1)–(6) to inverse problem with respect to unknowns $b_1 = b_1(t)$, $b_2 = b_2(t)$, $h = h(t)$, $w = w(y, t)$, where $w(y, t) = u\big(yh(t), t\big)$ in a domain with fixed boundary $Q_T = \big\{(y, t) : 0 < y < 1, 0 < t < T\big\}$, namely

$$w_t = \frac{a(t)t^\beta}{h^2(t)} w_{yy} + \frac{\big(b_1(t)h(t) + h'(t)\big)y + b_2(t)}{h(t)} w_y + c\big(yh(t), t\big)w + f\big(yh(t), t\big), \tag{7}$$

$$w(y, 0) = \varphi\big(yh(0)\big), \quad y \in [0, 1], \tag{8}$$

$$w(0, t) = \mu_1(t), \quad w(1, t) = \mu_2(t), \quad t \in [0, T], \tag{9}$$

$$h(t) \int_0^1 w(y, t)dy = \mu_3(t), \quad t \in [0, T], \tag{10}$$

$$h^2(t) \int_0^1 yw(y, t)dy = \mu_4(t), \quad t \in [0, T], \tag{11}$$

$$h^3(t) \int_0^1 y^2 w(y, t)dy = \mu_5(t), \quad t \in [0, T]. \tag{12}$$

Suppose that the following conditions hold:

(A1) $a, c, f \in C\big([0, \infty) \times [0, T]\big)$, $\varphi \in C[0, \infty)$, $\mu_i \in C^1[0, T]$, $i \in \{1, 2, 3\}$;

(A2) $0 < f_1 \leq f(x, t) \leq f_0$, $-c_1 \leq c(x, t) \leq -c_0 < 0, (x, t) \in [0, \infty) \times [0, T]$, f_0, f_1, c_0, c_1 are some positive constants, $\mu_i(t) > 0$, $t \in [0, T]$, $i \in \{1, 2\}$;

(A3) $\varphi(x) \geq \varphi_0 > 0$, $x \in [0, \infty)$, $\mu_3(t) > 0$, $t \in [0, T]$.

Let us determine the initial position of an unknown boundary. The conditions (2), (4) and (**A1**), (**A3**) yield the existence of an unique solution $h(0) \equiv h_0$ to the equation

$$\int_0^{h_0} \varphi(x)dx = \mu_3(0).$$

Now we can estimate the function $w = w(y, t)$ from below. Assume that the function $w = w(y, t)$ achieves its minimum in $\overline{Q}_T$ at the point (y_0, t_0).

If $t_0 = 0, y_0 \in [0, 1]$, then

$$w(y_0, t_0) = \varphi(y_0 h_0) \geq \min_{y \in [0,1]} \varphi(yh_0) > 0.$$

If $y_0 = 0$ or $y_0 = 1, t_0 \in [0, T]$, then

$$w(y_0, t_0) = \mu_1(t_0) \geq \min_{t \in [0,T]} \mu_1(t) > 0 \qquad \text{or} \qquad w(y_0, t_0) = \mu_2(t_0) \geq \min_{t \in [0,T]} \mu_2(t) > 0,$$

respectively.

For a point $(y_0,t_0)\in(0,1)\times(0,T]$ we have $w_y(y_0,t_0)=0, w_{yy}(y_0,t_0)\geq 0$, $w_t(y_0,t_0)\leq 0$. From the equation (7) we deduce

$$\begin{aligned}0&=w_t(y_0,t_0)-\frac{a(t_0)t_0^{\beta}}{h^2(t_0)}w_{yy}(y_0,t_0)-c\big(y_0h(t_0),t_0\big)w(y_0,t_0)-f\big(y_0h(t_0),t_0\big)\\&\leq -c\big(y_0h(t_0),t_0\big)w(y_0,t_0)-f\big(y_0h(t_0),t_0\big).\end{aligned}$$

It means that

$$w(y_0,t_0)\geq\frac{f\big(y_0h(t_0),t_0\big)}{-c\big(y_0h(t_0),t_0\big)}\geq\frac{f_1}{c_1}>0.$$

As a result we conclude

$$w(y,t)\geq\min\left\{\min_{y\in[0,1]}\varphi(yh_0),\ \min_{t\in[0,T]}\mu_1(t),\ \min_{t\in[0,T]}\mu_2(t),\frac{f_1}{c_1}\right\}\equiv M_0>0,\quad (y,t)\in\overline{Q}_T. \tag{13}$$

Using the equation (10) and the estimation (13), we get

$$h(t)\leq\frac{\max\limits_{t\in[0,T]}\mu_3(t)}{M_0}\equiv H_1<\infty,\quad t\in[0,T]. \tag{14}$$

The function $w=w(y,t)$ can be estimated from above in a similar way

$$w(y,t)\leq\max\left\{\max_{y\in[0,1]}\varphi(yh_0),\ \max_{t\in[0,T]}\mu_1(t),\ \max_{t\in[0,T]}\mu_2(t),\frac{f_0}{c_0}\right\}\equiv M_1<\infty,\quad (y,t)\in\overline{Q}_T. \tag{15}$$

From (10) one can find

$$h(t)\geq\frac{\min\limits_{t\in[0,T]}\mu_3(t)}{M_1}\equiv H_0>0,\quad t\in[0,T]. \tag{16}$$

Theorem 1. *Suppose that the assumptions* **(A1)–(A3)** *and conditions*

(A4) $c,f\in C^{1,0}\big([0,2H_1]\times[0,T]\big)$, $\varphi\in C^3[0,h_0]$, $\mu_4\in C^1[0,T]$;

(A5) $\varphi'(x)>0$ *for* $x\in[0,h_0]$, $\varphi'(h_0-x)-\varphi'(x)>0$, $(h_0-x)\varphi'(x)-x\varphi'(h_0-x)>0$ *for* $x\in\left[0,\frac{h_0}{2}\right)$;

(A6) $\varphi(0)=\mu_1(0)$, $\varphi(h_0)=\mu_2(0)$, $\mu_4(0)=\displaystyle\int_0^{h_0}x\varphi(x)dx$

are satisfied. Then there exists the unique local solution to the problem (1)–(6).

2 Existence of the solution to the problem (1)–(6)

Assume temporary that the functions $b_1=b_1(t), b_2=b_2(t), h=h(t)$ are known. Realizing the substitution

$$w(y,t)=\widetilde{w}(y,t)+\varphi(yh_0)-\varphi(0)+\mu_1(t)+y\big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\big), \tag{17}$$

the direct problem (7)–(9) is reduced to a similar one with respect to the function $\widetilde{w} = \widetilde{w}(y,t)$ with homogeneous initial and boundary conditions

$$\begin{aligned}\widetilde{w}_t =& \frac{a(t)t^\beta}{h^2(t)}\widetilde{w}_{yy} + \frac{\big(b_1(t)h(t)+h'(t)\big)y+b_2(t)}{h(t)}\widetilde{w}_y + c\big(yh(t),t\big)\widetilde{w} + f\big(yh(t),t\big)\\ &- \mu_1'(t) - y\big(\mu_2'(t)-\mu_1'(t)\big) + \frac{h_0^2 a(t)t^\beta}{h^2(t)}\varphi''(yh_0)\\ &+ \frac{\big(b_1(t)h(t)+h'(t)\big)y+b_2(t)}{h(t)}\Big(h_0\varphi'(yh_0)+\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\Big)\\ &+ c\big(yh(t),t\big)\Big(\varphi(yh_0)-\varphi(0)+\mu_1(t)+y\big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\big)\Big),\end{aligned} \tag{18}$$

for $(y,t) \in Q_T$, and

$$\widetilde{w}(y,0) = 0, \quad y \in [0,1], \tag{19}$$

$$\widetilde{w}(0,t) = 0, \quad \widetilde{w}(1,t) = 0, \quad t \in [0,T]. \tag{20}$$

The conditions of the Theorem 1 ensure the existence of the Green function $G = G(y,t,\eta,\tau)$, for the first boundary value problems for the equation

$$\widetilde{w}_t = \frac{a(t)t^\beta}{h^2(t)}\widetilde{w}_{yy} + c\big(yh(t),t\big)\widetilde{w}.$$

With the aid of this function we replace the problem (18)–(20) with the equivalent integro-differential equation

$$\begin{aligned}\widetilde{w}(x,t) = \int_0^t\int_0^1 G(y,t,\eta,\tau)\bigg(&\frac{\big(b_1(\tau)h(\tau)+h'(\tau)\big)\eta+b_2(\tau)}{h(\tau)}\widetilde{w}_\eta(\eta,\tau)\\ &+ f\big(\xi h(\tau),\tau\big) - \mu_1'(\tau) - \eta\big(\mu_2'(\tau)-\mu_1'(\tau)\big) + \frac{h_0^2 a(\tau)\tau^\beta}{h^2(\tau)}\varphi''(\eta h_0)\\ &+ \frac{\big(b_1(\tau)h(\tau)+h'(\tau)\big)\eta+b_2(\tau)}{h(\tau)}\big(h_0\varphi'(\eta h_0)+\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)\big)\\ &+ c\big(\eta h(\tau),\tau\big)\big(\varphi(\eta h_0)-\varphi(0)+\mu_1(\tau)+\eta\big(\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)\big)\big)\bigg)\,d\eta\,d\tau.\end{aligned} \tag{21}$$

Consider $v(y,t) \equiv w_y(y,t)$, $p(t) = h'(t)$. Using (17), (21), the direct problem (7)–(9) is reduced to the system of equations with unknown functions $w = w(y,t)$, $v = v(y,t)$, namely

$$\begin{aligned}w(y,t) =& \varphi(yh_0) - \varphi(0) + \mu_1(t) + y\big(\mu_2(t)-\mu_1(t)-\mu_2(0)+\mu_1(0)\big)\\ &+ \int_0^t\int_0^1 G(y,t,\eta,\tau)\bigg(\frac{\big(b_1(\tau)h(\tau)+p(\tau)\big)\eta+b_2(\tau)}{h(\tau)}v(\eta,\tau)\\ &+ f\big(\eta h(\tau),\tau\big) - \mu_1'(\tau) - \eta\big(\mu_2'(\tau)-\mu_1'(\tau)\big) + \frac{h_0^2 a(\tau)\tau^\beta}{h^2(\tau)}\varphi''(\eta h_0)\\ &+ c\big(\eta h(\tau),\tau\big)\Big(\varphi(\eta h_0)-\varphi(0)+\mu_1(\tau)+\eta\big(\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0)\big)\Big)\bigg)d\eta d\tau\\ \equiv& \varphi(yh_0) + w^*(y,t), \quad (y,t) \in \overline{Q}_T,\end{aligned} \tag{22}$$

$$\begin{aligned} v(y,t) =& h_0\varphi'(yh_0) + \mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0) \\ &+ \int_0^t \int_0^1 G_y(y,t,\eta,\tau)\Big(\frac{\big(b_1(\tau)h(\tau) + p(\tau)\big)\eta + b_2(\tau)}{h(\tau)} v(\eta,\tau) \\ &+ f\big(\eta h(\tau),\tau\big) - \mu_1'(\tau) - \eta\big(\mu_2'(\tau) - \mu_1'(\tau)\big) + \frac{h_0^2 a(\tau)\tau^\beta}{h^2(\tau)}\varphi''(\eta h_0) \\ &+ c(\eta h(\tau),\tau)\Big(\varphi(\eta h_0) - \varphi(0) + \mu_1(\tau) + \eta\big(\mu_2(\tau) - \mu_1(\tau) - \mu_2(0) + \mu_1(0)\big)\Big)\Big) d\eta d\tau \\ \equiv& h_0\varphi'(yh_0) + v^*(y,t), \quad (y,t) \in \overline{Q}_T. \end{aligned} \tag{23}$$

Equation (22) is differentiated with respect to the spatial variable aiming to obtain the equation (23). To study the behavior of the integrals in the right-hand sides of the formulas (22), (23), it is necessary to apply the known estimates of the Green function (see [25, p. 469])

$$\begin{gathered} \left| D_t^r D_y^s G(y,t,\eta,\tau) \right| \le C_1 (t-\tau)^{-\frac{1+2r+s}{2}} \exp\left(-C_2 \frac{(y-\eta)^2}{t-\tau} \right), \\ r \in \{0,1\}, \quad s \in \{0,1,2\}, \quad 2r+s = 1 \quad \text{or} \quad 2r+s=2, \quad \tau < t. \end{gathered} \tag{24}$$

Using (24), we obtain

$$I_1 \equiv \int_0^t \int_0^1 G(y,t,\eta,\tau) d\eta d\tau \le C_3 \int_0^t \int_0^1 \frac{1}{\sqrt{\theta(t)-\theta(\tau)}} \exp\left(-C_2 \frac{(y-\eta)^2}{\theta(t)-\theta(\tau)} \right) d\eta d\tau, \tag{25}$$

$$I_2 \equiv \int_0^t \int_0^1 G_y(y,t,\eta,\tau) d\eta d\tau \le C_4 \int_0^t \int_0^1 \frac{1}{\theta(t)-\theta(\tau)} \exp\left(-C_2 \frac{(y-\eta)^2}{\theta(t)-\theta(\tau)} \right) d\eta d\tau, \tag{26}$$

where $\theta(t) = \int_0^t \sigma^\beta d\sigma = \frac{t^{\beta+1}}{\beta+1}$, and the positive constant C_3, C_4 depend on the problem data. After the substitution $\xi = \frac{\sqrt{C_2}(y-\eta)}{\sqrt{\theta(t)-\theta(\tau)}}$, we get

$$\begin{aligned} &I_1 \le C_5 \int_0^t d\tau \le C_5 t, \\ &I_2 \le C_6 \int_0^t \frac{1}{\sqrt{\theta(t)-\theta(\tau)}} d\tau \le C_7 \int_0^t \frac{1}{\sqrt{t^{\beta+1} - \tau^{\beta+1}}} d\tau \le \frac{C_7}{t^{\beta/2}} \int_0^t \frac{1}{\sqrt{t-\tau}} d\tau \le C_8 t^{\frac{1-\beta}{2}}. \end{aligned}$$

Thus, taking into account the definition of the weak degeneration, we conclude that the integrals in the right-hand sides of the formulas (22), (23) tend to zero as t tends to zero.

Taking into account (13), we rewrite the equation (10) in the form

$$h(t) = \frac{\mu_3(t)}{\int_0^1 w(y,t) dy}, \qquad t \in [0,T]. \tag{27}$$

The equations (10)–(12) are differentiated with respect to t. Using (7)–(12) and solving the obtained system with respect to $p(t)$, $b_1(t)$, $b_2(t)$, we find

$$\begin{aligned} p(t) =& \Big(F_1(t)\big(4h^2(t)\mu_2(t)\mu_4(t) - 4\mu_4^2(t) - 3h(t)\mu_2(t)\mu_5(t) \\ &\qquad - h^3(t)\mu_2(t)\mu_3(t) + 3\mu_3(t)\mu_5(t)\big) \\ &+ F_2(t)\big(h^3(t)\mu_1(t)\mu_2(t) + 3\mu_2(t)\mu_5(t) - 3\mu_1(t)\mu_5(t) \\ &\qquad - 2h(t)\mu_2(t)\mu_4(t) - h^2(t)\mu_2(t)\mu_3(t) + 2\mu_3(t)\mu_4(t)\big) \\ &+ F_3(t)\big(2h(t)\mu_2(t)\mu_3(t) - \mu_3^2(t) - 2\mu_2(t)\mu_4(t) \\ &\qquad - h^2(t)\mu_1(t)\mu_2(t) + 2\mu_1(t)\mu_4(t)\big)\Big)\mu_2^{-1}(t)\Delta^{-1}(t), \end{aligned} \tag{28}$$

$$b_1(t) = \Big(F_1(t)\big(h^2(t)\mu_3(t) - 2h(t)\mu_4(t)\big) + F_2(t)\big(2\mu_4(t) - h^2(t)\mu_1(t)\big) \\ + F_3(t)\big(h(t)\mu_1(t) - \mu_3(t)\big)\Big)\Delta^{-1}(t), \tag{29}$$

$$b_2(t) = \Big(F_1(t)\big(3h(t)\mu_5(t) - 2h^2(t)\mu_4(t)\big) + F_2(t)\big(h^2(t)\mu_3(t) - 3\mu_5(t)\big) \\ + F_3(t)\big(2\mu_4(t) - h(t)\mu_3(t)\big)\Big)\Delta^{-1}(t), \tag{30}$$

where

$$\Delta(t) = \mu_3(t)\big(2h(t)\mu_4(t) - h^2(t)\mu_3(t)\big) + 2\mu_4(t)\big(h^2(t)\mu_1(t) - 2\mu_4(t)\big) \\ + 3\mu_5(t)\big(\mu_3(t) - h(t)\mu_1(t)\big),$$

$$F_1(t) \equiv \mu_3'(t) - \frac{a(t)t^\beta}{h(t)}\big(v(1,t) - v(0,t)\big) - h(t)\int_0^1 \big(c(yh(t),t)w(y,t) + f(yh(t),t)\big)dy,$$

$$F_2(t) \equiv \mu_4'(t) - a(t)t^\beta\big(v(1,t) - \mu_2(t) + \mu_1(t)\big) \\ - h^2(t)\int_0^1 y\big(c(yh(t),t)w(y,t) + f(yh(t),t)\big)dy,$$

$$F_3(t) \equiv \mu_5'(t) - a(t)t^\beta h(t)\left(v(1,t) - 2\mu_2(t) + \frac{2\mu_3(t)}{h(t)}\right) \\ - h^3(t)\int_0^1 y^2\big(c(yh(t),t)w(y,t) + f(yh(t),t)\big)dy.$$

It is easy to verify by direct integration that

$$\Delta(t) = \frac{h^4(t)}{2}\bigg(\int_0^1 (1-y)(1-2y)v(y,t)dy \int_0^1 y(1-y)v(y,t)dy \\ + \int_0^1 (1-y)v(y,t)dy \int_0^1 y(1-y)(2y-1)v(y,t)dy\bigg). \tag{31}$$

Let consider the equation (23). The first term in the right hand side of this equation is greater than zero and the second one is infinitely small at $t \to 0$, i.e. $\lim\limits_{t\to 0} v^*(y,t) = 0$. It means that we can indicate a number t_1, $0 < t_1 \le T$, such that

$$\big|v^*(y,t)\big| \le \frac{h_0 \min_{[0,1]} \varphi'(yh_0)}{2}, \quad (y,t) \in \overline{Q}_{t_1}. \tag{32}$$

Then from the equation (23) one can deduce

$$v(y,t) \ge \frac{h_0 \min_{[0,1]} \varphi'(yh_0)}{2} \equiv M_2 > 0, \quad (y,t) \in \overline{Q}_{t_1}. \tag{33}$$

It yields

$$\int_0^1 (1-y)v(y,t)dy > 0, \qquad \int_0^1 y(1-y)v(y,t)dy > 0, \quad t \in [0,t_1]. \tag{34}$$

Two other integrals from the right-hand side of the formula (31) are represented in the forms

$$\int_0^1 y(1-y)(2y-1)v(y,t)dy = \int_0^{\frac12} y(1-y)(1-2y)\big(v(1-y,t) - v(y,t)\big)dt,$$

$$\int_0^1 (1-y)(1-2y)v(y,t)dy = \int_0^{\frac12} (1-2y)\big((1-y)v(y,t) - yv(1-y,t)\big)dt.$$

On the next step we substitute (23) into these formulas. Note, that in these formulas the sum of all terms, except the first, tends to zero as $t \to 0$. So we can choose a number t_2, $0 < t_2 \leq T$, such that

$$\Big|\int_0^t\int_0^1 \left(G_y(1-y,t,\eta,\tau) - G_y(y,t,\eta,\tau)\right)\bigg(\frac{(b_1(\tau)h(\tau)+p(\tau))\eta + b_2(\tau)}{h(\tau)}v(\eta,\tau) + f(\eta h(\tau),\tau) - \mu_1'(\tau) - \eta(\mu_2'(\tau)-\mu_1'(\tau)) + \frac{h_0^2 a(\tau)\tau^\beta}{h^2(\tau)}\varphi''(\eta h_0) + c(\eta h(\tau),\tau)\left(\varphi(\eta h_0) - \varphi(0) + \mu_1(\tau) + \eta(\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0))\right)\bigg)d\eta d\tau\Big| \leq \frac{h_0 \min_{[0,1]}\left(\varphi'(h_0(1-y)) - \varphi'(h_0 y)\right)}{2}, \quad (y,t)\in\overline{Q}_{t_2},$$

$$\Big|\int_0^t\int_0^1 \left((1-y)G_y(y,t,\eta,\tau) - yG_y(1-y,t,\eta,\tau)\right)\bigg(\frac{(b_1(\tau)h(\tau)+p(\tau))\eta + b_2(\tau)}{h(\tau)}v(\eta,\tau) + f(\eta h(\tau),\tau) - \mu_1'(\tau) - \eta(\mu_2'(\tau)-\mu_1'(\tau)) + \frac{h_0^2 a(\tau)\tau^\beta}{h^2(\tau)}\varphi''(\eta h_0) + c(\eta h(\tau),\tau)\left(\varphi(\eta h_0) - \varphi(0) + \mu_1(\tau) + \eta(\mu_2(\tau)-\mu_1(\tau)-\mu_2(0)+\mu_1(0))\right)\bigg)d\eta d\tau\Big| \leq \frac{h_0 \min_{[0,1]}\left((1-y)\varphi'(yh_0) - y\varphi'(h_0(1-y))\right)}{2}, \quad (y,t)\in\overline{Q}_{t_2}.$$

As a result we obtain

$$\int_0^1 y(1-y)(2y-1)v(y,t)dy \geq \frac{h_0 \min_{[0,1]}\left(\varphi'(h_0(1-y)) - \varphi'(h_0 y)\right)}{2}, \quad (y,t)\in\overline{Q}_{t_2}, \tag{35}$$

$$\int_0^1 (1-y)(1-2y)v(y,t)dy \geq \frac{h_0 \min_{[0,1]}\left((1-y)\varphi'(yh_0) - y\varphi'(h_0(1-y))\right)}{2}, \quad (y,t)\in\overline{Q}_{t_2}. \tag{36}$$

So the condition **(A5)** of the Theorem 1 and (34), (35), (36) guarantee that

$$\Delta(t) \geq \Delta_0 > 0, \quad t\in[0,t_0], \tag{37}$$

where $t_0 = \min\{t_1, t_2\}$.

On the other hand, using (32) in the equation (23), we conclude

$$v(y,t) \leq h_0 \max_{[0,1]}\varphi'(yh_0) + \frac{h_0 \min_{[0,1]}\varphi'(yh_0)}{2} \equiv M_3, \qquad (y,t)\in\overline{Q}_{t_0}. \tag{38}$$

Thus, the inverse problem (7)–(12) is reduced to the equivalent system of equations (22), (23), (27)–(30). We understand the term "equivalence" in the following sense: if a set of functions (b_1, b_2, h, w) is a local solution to the problem (7)–(12), then (w, v, h, p, b_1, b_2) is the continuous solution to the system (22), (23), (27)–(30) in $\overline{Q}_{t_0}$, and, contrary, if $(w, v, h, p, b_1, b_2) \in \left(C(\overline{Q}_{t_0})\right)^2 \times \left(C[0,t_0]\right)^4$, $h(t) > 0$, $t\in[0,t_0]$, is a solution to the system of equations (22), (23), (27)–(30), then (b_1, b_2, h, v) is the local solution to the problem (7)–(12).

The first part of this statement follows from the way of obtaining of (22), (23), (27)–(30). Let us show that the contrary statement is true. Assume that (w, v, h, p, b_1, b_2) is the continuous solution to the system of equations (22), (23), (27)–(30). The condition **(A4)** of the Theorem 1 allows us to differentiate the equation (22) with respect to space variable. Using the uniqueness properties of the solutions to the system of Voltera integral equations of the second kind it is easy to see that $v(y,t) \equiv w_y(y,t)$. This means that $w \in C^{2,1}\left(Q_{t_0}\right) \cap C^{1,0}\left(\overline{Q}_{t_0}\right)$ is a solution to the problem (7)–(9).

Multiplying the equations (28)–(30) by $\mu_2(t)$, $h(t)\mu_2(t) - \mu_3(t)$, $\mu_2(t) - \mu_1(t)$, respectively, and adding them, we find

$$p(t)\mu_2(t) + b_1(t)\big(h(t)\mu_2(t) - \mu_3(t)\big) + b_2(t)\big(\mu_2(t) - \mu_1(t)\big) = F_1(t).$$

From the other hand, differentiating (27) with respect to t, we obtain

$$h'(t)\mu_2(t) + b_1(t)\big(h(t)\mu_2(t) - \mu_3(t)\big) + b_2(t)\big(\mu_2(t) - \mu_1(t)\big) \\ = \mu_3'(t) - \frac{a(t)t^{\beta}}{h(t)}\big(w_y(1,t) - w_y(0,t)\big) - h(t)\int_0^1 \big(c\big(yh(t),t\big)w(y,t) + f\big(yh(t),t\big)\big)dy.$$

Subtract these equalities. Taking into account the definition of function $F_1(t)$ and the fact $v(y,t) \equiv w_y(y,t)$ we conclude $\big(h'(t) - p(t)\big)\mu_2(t) = 0$, $t \in [0, t_0]$.

According to the condition **(A2)**, we have $\mu_2(t) > 0$, $t \in [0, T]$, so $h'(t) \equiv p(t)$, $t \in [0, t_0]$.

Multiplying the equations (28), (29), (30) by $h(t)\mu_2(t)$, $h^2(t)\mu_2(t) - 2\mu_4(t)$, $\mu_2(t)h(t) - \mu_3(t)$, respectively, and adding them, we get

$$p(t)h(t)\mu_2(t) + b_1(t)\big(h^2(t)\mu_2(t) - 2\mu_4(t)\big) + b_2(t)\big(\mu_2(t)h(t) - \mu_3(t)\big) = F_2(t).$$

Using the definition of $F_2(t)$ and (7)–(9), it is represented it in the form

$$2b_1(t)\left(h^2(t)\int_0^1 yw(y,t)dy - \mu_4(t)\right) = \mu_4'(t) - 2h(t)h'(t)\int_0^1 yw(y,t)dy - h^2(t)\int_0^1 yw(y,t)dy,$$

where

$$2b_1(t)\left(h^2(t)\int_0^1 yw(y,t)dy - \mu_4(t)\right) = -\left(h^2(t)\int_0^1 yw(y,t)dy - \mu_4(t)\right)'.$$

Consider $z(t) \equiv h^2(t)\int_0^1 yw(y,t)dy - \mu_4(t)$. Then we have $z'(t) = -2b_1(t)z(t)$. Therefore $z(t) = z(0)e^{-2\int_0^t b_1(\tau)d\tau}$. Since $z(0) = 0$ according to the compatibility conditions, so $z(t) \equiv 0$, that is the condition (11) is fulfilled.

Let us multiply the equations (28), (29) and (30) by $h^2(t)\mu_2(t)$, $h^3(t)\mu_2(t) - 3\mu_5(t)$ and $h^2(t)\mu_2(t) - 2\mu_4(t)$, respectively. Repeating the above procedure, we deduce that the condition (12) is fulfilled. It means that the equivalence of the inverse problem (7)–(12) and the system of equations (22), (23), (27)–(30) is proved.

The system of equations (22), (23), (27)–(30) can be rewritten as the following operator equation

$$\omega = P\omega, \tag{39}$$

where $\omega = (w, v, h, p, b_1, b_2)$ and the operator $P = (P_1, P_2, P_3, P_4, P_5, P_6)$ is defined by the right-hand sides of the equations (22), (23), (27)–(30).

We present the conditions (13), (15), (33), (38) in the form

$$\frac{M_0}{2} \le w(y,t) \le M_1 + \frac{M_0}{2}, \qquad M_2 \le v(y,t) \le M_3, \qquad (y,t) \in \overline{Q}_{t_0},$$

where the numbers M_0, M_1, M_2, M_3 are defined above.

Therefore from the equations (27)–(30) we conclude

$$H_2 \equiv \frac{\min\limits_{t\in[0,T]} \mu_3(t)}{M_1 + \frac{M_0}{2}} \le P_3\omega \le \frac{2\max\limits_{t\in[0,T]} \mu_3(t)}{M_0} = 2H_1, \quad t \in [0,t_0], \tag{40}$$

$$|P_4\omega| \le \frac{C_9\,(1 + M_0 + M_1 + M_3)}{\Delta_0 \min\limits_{[0,T]} \mu_2(t)} \equiv M_4, \quad (y,t) \in \overline{Q}_{t_0}, \tag{41}$$

$$|P_5\omega| \le \frac{C_{10}\,(1 + M_0 + M_1 + M_3)}{\Delta_0} \equiv M_5, \quad (y,t) \in \overline{Q}_{t_0}, \tag{42}$$

$$|P_4\omega| \le \frac{C_{11}\,(1 + M_0 + M_1 + M_3)}{\Delta_0} \equiv M_6, \quad (y,t) \in \overline{Q}_{t_0}, \tag{43}$$

where the constants C_9, C_{10}, C_{11} are defined by the problem data.

Estimations of $P_1\omega, P_2\omega$ are the next:

$$P_1\omega \ge \min_{y\in[0,1]} \varphi(yh_0) + \min_{(y,t)\in\overline{Q}_{t_0}} \Big|w^*(y,t)\Big|, \quad (y,t) \in \overline{Q}_{t_0},$$

$$P_1\omega \le \max_{y\in[0,1]} \varphi(yh_0) + \max_{(y,t)\in\overline{Q}_{t_0}} \Big|w^*(y,t)\Big|, \quad (y,t) \in \overline{Q}_{t_0},$$

$$P_2\omega \ge \min_{y\in[0,1]} h_0\varphi'(yh_0) + \min_{(y,t)\in\overline{Q}_{t_0}} \Big|v^*(y,t)\Big|, \quad (y,t) \in \overline{Q}_{t_0},$$

$$P_2\omega \le \max_{y\in[0,1]} h_0\varphi'(yh_0) + \max_{(y,t)\in\overline{Q}_{t_0}} \Big|v^*(y,t)\Big|, \quad (y,t) \in \overline{Q}_{t_0}.$$

Let us consider

$$\begin{aligned}
\Big|w^*(y,t)\Big| \le{}& \max_{(y,t)\in\overline{Q}_{t_0}} \Big|y\,(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0))\,\Big| \\
&+ \max_{t\in[0,t_0]} \big|\mu_1(t) - \varphi(0)\big| + \bigg|\int_0^t \int_0^1 G(y,t,\eta,\tau)\bigg(\frac{(2M_5H_1 + M_4)\eta + M_6}{H_2} M_3 \\
&+ \max_{(\eta,\tau)\in\overline{Q}_{t_0}} \bigg|f\big(\eta h(\tau),\tau\big) - \mu_1'(\tau) - \eta\big(\mu_2'(\tau) - \mu_1'(\tau)\big) + \frac{h_0^2 a(\tau)\tau^\beta}{h^2(\tau)} \varphi''(\eta h_0) \\
&+ c\big(\eta h(\tau),\tau\big)\Big(\varphi(\eta h_0) - \varphi(0) + \mu_1(\tau) + \eta\big(\mu_2(\tau) - \mu_1(\tau) - \mu_2(0) + \mu_1(0)\big)\Big)\bigg)\bigg|\,d\eta d\tau\bigg| \\
\le{}& C_{12}t,
\end{aligned}$$

$$\begin{aligned}
\Big|v^*(y,t)\Big| \le{}& \max_{t\in[0,t_0]} \big|\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\big| \\
&+ \bigg|\int_0^t \int_0^1 G_y(y,t,\eta,\tau)\bigg(\frac{(2M_5H_1 + M_4)\eta + M_6}{H_2} M_3 \\
&+ \max_{(\eta,\tau)\in\overline{Q}_{t_0}} \bigg|f\big(\eta h(\tau),\tau\big) - \mu_1'(\tau) - \eta\big(\mu_2'(\tau) - \mu_1'(\tau)\big) + \frac{h_0^2 a(\tau)\tau^\beta}{h^2(\tau)} \varphi''(\eta h_0) \\
&+ c\big(\eta h(\tau),\tau\big)\Big(\varphi(\eta h_0) - \varphi(0) + \mu_1(\tau) + \eta\big(\mu_2(\tau) - \mu_1(\tau) - \mu_2(0) + \mu_1(0)\big)\Big)\bigg)\bigg|\,d\eta d\tau\bigg| \\
\le{}& C_{13}t + C_{14}t^{\frac{1-\beta}{2}}.
\end{aligned}$$

We fix the number T_0, $0 < T_0 \le t_0$, as follows

$$C_{12}T_0 \le \frac{1}{2}\min_{y\in[0,l]}\varphi(yh_0), \qquad C_{13}T_0 + C_{14}T^{\frac{1-\beta}{2}} \le \frac{h_0}{2}\max_{y\in[0,1]}\varphi'(yh_0).$$

As a result, we get

$$\frac{M_0}{2} \le P_1\omega \le M_1 + \frac{M_0}{2}, \quad M_2 \le P_2\omega \le M_3, \quad (y,t)\in\overline{Q}_{T_0}. \tag{44}$$

We consider the operator equation (39) on a convex closed set

$$N \equiv \Big\{ (w,v,h,p,b_1,b_2) \in \left(C\left(\overline{Q}_{T_0}\right)\right)^2 \times \left(C[0,T_0]\right)^4 : \frac{M_0}{2} \le w(y,t) \le M_1 + \frac{M_0}{2}, \\ M_2 \le v(y,t) \le M_3,\ H_2 \le h(t) \le 2H_1, \\ |p(t)| \le M_4,\ |b_1(t)| \le M_5,\ |b_2(t)| \le M_6 \Big\}$$

in the Banach space $\mathcal{B} \equiv \left(C(\overline{Q}_{T_0})\right)^2 \times \left(C[0,T_0]\right)^4$. The estimates (40)–(43), (44) guarantee that the operator P maps the set N into itself. The compactness of the operator P on the set N is based on the Arzela-Ascoli theorem and can be proved as in [21, p. 27]. Applying the Schauder fixed point theorem, we state that there exists the solution to the system of equations (22), (23), (27)–(30) in $\overline{Q}_{T_0}$ and therefore to inverse problem (7)–(12) in $\overline{Q}_{T_0}$.

3 Uniqueness of the solution to the problem (1)–(6)

To prove the uniqueness of the solution to inverse problem (7)–(12) we assume that the system of equations (22), (23), (27)–(30) has two solutions $(w_i, v_i, h_i, p_i, b_{1i}, b_{2i})$, $i = 1,2$, in $\overline{Q}_{T_0}$. Denote $w(x,t) = w_1(x,t) - w_2(x,t)$, $v(x,t) = v_1(x,t) - v_2(x,t)$, $h(t) = h_1(t) - h_2(t)$, $p(t) = p_1(t) - p_2(t)$, $b_1(t) = b_{11}(t) - b_{12}(t)$, $b_2(t) = b_{21}(t) - b_{22}(t)$. Using (22), (23), (27)–(30), we find

$$\begin{aligned} w(y,t) = \int_0^t\int_0^1 G(y,t,\eta,\tau)\Bigg(&\frac{\big(b_{11}(\tau)h_1(\tau)+p_1(\tau)\big)\eta + b_{21}(\tau)}{h_1(\tau)}v(\eta,\tau) \\ &-\Bigg(\frac{\big(\big(b_{11}(\tau)h_1(\tau)+p_1(\tau)\big)\eta + b_{21}(\tau)\big)h(\tau)}{h_1(\tau)h_2(\tau)} \\ &-\frac{\big(b_{11}(\tau)h(\tau)+b_1(\tau)h_2(\tau)+p(\tau)\big)\eta + b_2(\tau)}{h_2(\tau)}\Bigg)v_2(\eta,\tau)\Bigg)d\eta d\tau, \quad (y,t)\in\overline{Q}_{T_0}, \end{aligned} \tag{45}$$

$$\begin{aligned} v(y,t) = \int_0^t\int_0^1 G_y(y,t,\eta,\tau)\Bigg(&\frac{\big(b_{11}(\tau)h_1(\tau)+p_1(\tau)\big)\eta + b_{21}(\tau)}{h_1(\tau)}v(\eta,\tau) \\ &-\Bigg(\frac{\big(\big(b_{11}(\tau)h_1(\tau)+p_1(\tau)\big)\eta + b_{21}(\tau)\big)h(\tau)}{h_1(\tau)h_2(\tau)} \\ &-\frac{\big(b_{11}(\tau)h(\tau)+b_1(\tau)h_2(\tau)+p(\tau)\big)\eta + b_2(\tau)}{h_2(\tau)}\Bigg)v_2(\eta,\tau)\Bigg)d\eta d\tau, \quad (y,t)\in\overline{Q}_{T_0}, \end{aligned} \tag{46}$$

$$h(t) = -\frac{\mu_3(t)}{\int_0^1 w_1(y,t)dy\int_0^1 w_2(y,t)dy}\int_0^1 w(y,t)dy, \qquad t\in[0,T_0], \tag{47}$$

$$
\begin{aligned}
p(t) = &-\Bigg(F_{11}(t)\big(4h_1^2(t)\mu_2(t)\mu_4(t) - 4\mu_4^2(t) - 3h_1(t)\mu_2(t)\mu_5(t) \\
&\qquad - h_1^3(t)\mu_2(t)\mu_3(t) + 3\mu_3(t)\mu_5(t)\big) \\
&\quad + F_{21}(t)\big(h_1^3(t)\mu_1(t)\mu_2(t) + 3\mu_2(t)\mu_5(t) - 3\mu_1(t)\mu_5(t) \\
&\qquad - 2h_1(t)\mu_2(t)\mu_4(t) - h_1^2(t)\mu_2(t)\mu_3(t) + 2\mu_3(t)\mu_4(t)\big) \\
&\quad + F_{31}(t)\big(2h_1(t)\mu_2(t)\mu_3(t) - \mu_3^2(t) - 2\mu_2(t)\mu_4(t) \\
&\qquad - h_1^2(t)\mu_1(t)\mu_2(t) + 2\mu_1(t)\mu_4(t)\big)\Bigg)\frac{\Delta^*(t)}{\mu_2(t)\Delta_1(t)\Delta_2(t)} \\
&+\Bigg(\widetilde{F}_1(y,t)\big(4h_1^2(t)\mu_2(t)\mu_4(t) - 4\mu_4^2(t) - 3h_1(t)\mu_2(t)\mu_5(t) \\
&\qquad - h_1^3(t)\mu_2(t)\mu_3(t) + 3\mu_3(t)\mu_5(t)\big) \\
&\quad + F_{12}(y,t)\Big(4h^2(t)\mu_2(t)\mu_4(t) - 3h(t)\mu_2(t)\mu_5(t) - h^3(t)\mu_2(t)\mu_3(t)\Big) \\
&\quad + \widetilde{F}_2(y,t)\big(h_1^3(t)\mu_1(t)\mu_2(t) + 3\mu_2(t)\mu_5(t) - 3\mu_1(t)\mu_5(t) \\
&\qquad - 2h_1(t)\mu_2(t)\mu_4(t) - h_1^2(t)\mu_2(t)\mu_3(t) + 2\mu_3(t)\mu_4(t)\big) \\
&\quad + F_{22}(y,t)\Big(h^3(t)\mu_1(t)\mu_2(t) - 2h(t)\times\mu_2(t)\mu_4(t) - h^2(t)\mu_2(t)\mu_3(t)\Big) \\
&\quad + \widetilde{F}_3(y,t)\big(2h_1(t)\mu_2(t)\mu_3(t) - 2\mu_2(t)\mu_4(t) + 2\mu_1(t)\mu_4(t) \\
&\qquad - h_1^2(t)\mu_1(t)\mu_2(t) - \mu_3^2(t)\big) \\
&\quad + F_{32}(y,t)\Big(2h(t)\mu_2(t)\mu_3(t) - h^2(t)\mu_1(t)\mu_2(t)\Big)\Bigg)\frac{1}{\mu_2(t)\Delta_2(t)}, \qquad t\in[0,T_0],
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
b_1(t) = &-\Bigg(F_{11}(t)\big(h_1^2(t)\mu_3(t) - 2h_1(t)\mu_4(t)\big) + F_{21}(t)\big(2\mu_4(t) - h_1^2(t)\mu_1(t)\big) \\
&\quad + F_{31}(t)\big(h_1(t)\mu_1(t) - \mu_3(t)\big)\Bigg)\frac{\Delta^*(t)}{\Delta_1(t)\Delta_2(t)} \\
&+\Bigg(\widetilde{F}_1(y,t)\big(h_1^2(t)\mu_3(t) - 2h_1(t)\mu_4(t)\big) \\
&\quad + F_{12}(y,t)\big(h^2(t)\mu_3(t) - 2h(t)\mu_4(t)\big) \\
&\quad + \widetilde{F}_2(y,t)\big(2\mu_4(t) - h_1^2(t)\mu_1(t)\big) - F_{22}(y,t)h^2(t)\mu_1(t) \\
&\quad + \widetilde{F}_3(y,t)\big(h_1(t)\mu_1(t) - \mu_3(t)\big) + F_{32}(y,t)h(t)\mu_1(t)\Bigg)\frac{1}{\Delta_2(t)}, \qquad t\in[0,T_0],
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
b_2(t) = &-\Bigg(F_{11}(t)\big(3h_1(t)\mu_5(t) - 2h_1^2(t)\mu_4(t)\big) + F_{21}(t)\big(h_1^2(t)\mu_3(t) - 3\mu_5(t)\big) \\
&\quad + F_{31}(t)\big(2\mu_4(t) - h_1(t)\mu_3(t)\big)\Bigg)\frac{\Delta^*(t)}{\Delta_1(t)\Delta_2(t)} \\
&+\Bigg(\widetilde{F}_1(y,t)\big(3h_1(t)\mu_5(t) - 2h_1^2(t)\mu_4(t)\big) \\
&\quad + F_{12}(y,t)\big(3h(t)\mu_5(t) - 2h^2(t)\mu_4(t)\big) \\
&\quad + \widetilde{F}_2(y,t)\big(h_1^2(t)\mu_3(t) - 3\mu_5(t)\big) + F_{22}(y,t)h^2(t)\mu_3(t) \\
&\quad + \widetilde{F}_3(y,t)\big(2\mu_4(t) - h_1(t)\mu_3(t)\big) - F_{32}(y,t)h(t)\mu_3(t)\Bigg)\frac{1}{\Delta_2(t)}, \qquad t\in[0,T_0].
\end{aligned} \tag{50}
$$

We use the following notations in the formulas (48)–(50):

$$\Delta_i(t) \equiv \mu_3(t)\big(2h_i(t)\mu_4(t) - h_i^2(t)\mu_3(t)\big) + 2\mu_4(t)\big(h_i^2(t)\mu_1(t) - 2\mu_4(t)\big) + 3\mu_5(t)\big(\mu_3(t) - h_i(t)\mu_1(t)\big), \quad i = 1,2,$$

$$F_{1i}(t) \equiv \mu_3'(t) - \frac{a(t)t^\beta}{h_i(t)}\big(v_i(1,t) - v_i(0,t)\big) - h_i(t)\int_0^1 \Big(c\big(yh_i(t),t\big)w_i(y,t) + f\big(yh_i(t),t\big)\Big)dy, \quad i = 1,2,$$

$$F_{2i}(t) \equiv \mu_4'(t) - a(t)t^\beta\big(v_i(1,t) - \mu_2(t) + \mu_1(t)\big) - h_i^2(t)\int_0^1 y\Big(c\big(yh_i(t),t\big)w_i(y,t) + f\big(yh_i(t),t\big)\Big)dy, \quad i = 1,2,$$

$$F_{3i}(t) \equiv \mu_5'(t) - a(t)t^\beta h_i(t)\left(v_i(1,t) - 2\mu_2(t) + \frac{2\mu_3(t)}{h_i(t)}\right) - h_i^3(t)\int_0^1 y^2\Big(c\big(yh_i(t),t\big)w_i(y,t) + f\big(yh_i(t),t\big)\Big)dy, \quad i = 1,2,$$

$$\Delta^*(t) \equiv \mu_3(t)\Big(2h(t)\mu_4(t) - h^2(t)\mu_3(t)\Big) + 2h^2(t)\mu_4(t)\mu_1(t) - 3h(t)\mu_1(t)\mu_5(t),$$

$$\widetilde{F}_1(t) \equiv \frac{h(t)a(t)t^\beta}{h_1(t)h_2(t)}\big(v_1(1,t) - v_1(0,t)\big) - \frac{a(t)t^\beta}{h_2(t)}\big(v(1,t) - v(0,t)\big) - h(t)\int_0^1 \Big(f\big(yh_1(t),t\big) + c\big(yh_1(t),t\big)w_1(y,t)\Big)dy - h_2(t)\int_0^1 \Big(c\big(yh_1(t),t\big)w(y,t) + \Big(c\big(yh_1(t),t\big) - c\big(yh_2(t),t\big)\Big)w_2(y,t) + f\big(yh_1(t),t\big) - f\big(yh_2(t),t\big)\Big)dy,$$

$$\widetilde{F}_2(t) \equiv -a(t)t^\beta v(1,t) - \Big(h_1^2(t) - h_2^2(t)\Big)\int_0^1 y\Big(f\big(yh_1(t),t\big) + c\big(yh_1(t),t\big)w_1(y,t)\Big)dy - h_2^2(t)\int_0^1 y\Big(c\big(yh_1(t),t\big)w(y,t) + \Big(c\big(yh_1(t),t\big) - c\big(yh_2(t),t\big)\Big)w_2(y,t) + f\big(yh_1(t),t\big) - f\big(yh_2(t),t\big)\Big)dy,$$

$$\widetilde{F}_3(t) \equiv -a(t)t^\beta\,\big(h_1(t)v(1,t) + h(t)v_2(y,t)\big) - \Big(h_1^3(t) - h_2^3(t)\Big)\int_0^1 y^2\Big(f\big(yh_1(t),t\big) + c\big(yh_1(t),t\big)w_1(y,t)\Big)dy - h_2^3(t)\int_0^1 y^2\Big(c\big(yh_1(t),t\big)w(y,t) + \Big(c\big(yh_1(t),t\big) - c\big(yh_2(t),t\big)\Big)w_2(y,t) + f\big(yh_1(t),t\big) - f\big(yh_2(t),t\big)\Big)dy.$$

Applying the Lagrange mean value theorem and condition **(A4)** of the Theorem 1 we deduce

$$f\big(yh_1(y),t\big) - f\big(yh_2(t),t\big) = yh(t)\int_0^1 f_x\Big(y\big(h_2(t) + \sigma\big(h_1(t) - h_2(t)\big)\big),t\Big)d\sigma. \tag{51}$$

The analogous equality holds also for the functions $c\big(yh(t),t\big)$. Moreover, the formulas

$$h_1^2(t) - h_2^2(t) = h(t)\big(h_1(t) + h_2(t)\big), \tag{52}$$

$$h_1^3(t) - h_2^3(t) = h(t)\big(h_1^2(t) + h_1(t)h_2(t) + h_2^2(t)\big) \tag{53}$$

are fulfilled. Substituting (47)–(53) into (45), (46), we obtain the linear system of homogeneous integral Volterra equations of the second kind with respect to the functions $w = w(y,t)$, $v = v(y,t)$:

$$w(y,t) = \int_0^t \big(K_{11}(t,\tau)w(y,\tau) + K_{12}(t,\tau)v(y,\tau)\big)d\tau, \quad t \in [0,T_0],$$

$$v(y,t) = \int_0^t \big(K_{21}(t,\tau)w(y,\tau) + K_{22}(t,\tau)w(y,\tau)\big)d\tau, \quad t \in [0,T_0].$$

Taking into account (25)–(26) we conclude that the kernels of this system possess integrable singularities. It means that the system has only trivial solution

$$w(y,t) \equiv 0, \quad v(y,t) \equiv 0, \quad (y,t) \in \overline{Q}_{T_0}. \tag{54}$$

Substituting (54) into (47)–(50), we get $h(t) \equiv 0$, $p(t) \equiv 0$, $b_1(t) \equiv 0$, $b_2(t) \equiv 0$, $t \in [0,T_0]$. It completes the proof of the Theorem 1.

4 Conclusions

There is studied the coefficient inverse problem for degenerate parabolic equation in a free boundary domain. It is known that the minor coefficient of this equation is the first order polynomial with respect to space variable with two time-dependent functions in it. The Dirichlet boundary conditions and the values of heat moments as overdetermination conditions are given. The case of weak degeneration is investigated. Using the apparatus of Green's functions for the heat equation and Schauder fixed point theorem the existence of the local solution to the stated problem is established. The proof of the uniqueness of the local solution to this problem is based on the properties of the solutions to the homogeneous integral equations with integrable kernels.

Note that the system of equations (22), (23), (27)–(30), obtained in the paper, can serve the base for application of some numerical methods for construction the approximate solutions to the considered problem.

Results of this paper can be used in research of inverse problems of identification the younger coefficients in parabolic equation which depend on both space and time variables. They also can be used in investigation such problems for the multidimensional degenerate parabolic equations, for the case of strong degeneration or for the case, when two parts of boundary are unknown.

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Гузик Н.М., Бродяк О.Я., Пукач П.Я., Вовк М.І. *Обернена задача з вільною межею для параболічного рівняння з виродженням* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 230–245.

В області з вільною межею досліджується обернена задача для параболічного рівняння з виродженням. Виродження рівняння спричинене залежною від часу функцією при старшій похідній невідомої функції. Припускаємо, що коефіцієнт перед молодшою похідною в рівнянні є многочленом першого степеня за просторовою змінною з двома невідомими залежними від часу функціями. Умови існування та єдиності класичного розв'язку вказаної задачі встановлено для випадку слабкого виродження при заданих крайових умовах Діріхле та значеннях теплових моментів у якості умов перевизначення.

Ключові слова і фрази: коефіцієнтна обернена задача, задача з вільною межею, слабке степеневе виродження, параболічне рівняння, молодший коефіцієнт.