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Global solutions and their limit behavior for parabolic inclusions with an unbounded right-hand part

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We study global resolvability for parabolic inclusions with an upper semicontinuous multi-valued right-hand part of more than linear growth. Theorems about the existence of global mild solutions in different phase spaces are proved. Limit sets for the obtained global solutions in the corresponding phase spaces are investigated.

Key words and phrases: parabolic inclusion, mild solution, weak solution, semigroup, limit set.

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Introduction

Differential inclusions, i.e. differential equations with multi-valued right-hand parts, are proper mathematical models for describing processes with discontinuous interaction functions, control problems and variational inequalities [1, 19]. For systems with distributed parameters, inclusions with partial derivatives are naturally arisen [3, 6, 7, 13, 15, 16, 22]. In the parabolic case, the key point for proving global resolvability is no more than the linear growth of the multi-valued right-hand part [7,15,22]. At the same time, in the case of classical parabolic equations with locally Lipschitz nonlinear terms, the existence of mild solutions can be proved under much more general assumptions [9, 11, 17]. For parabolic problems with continuous nonlinearities, similar results have been recently proved in [8, 12, 20].

Since an upper semicontinuous multi-valued map may have no continuous selectors [2], adopting equations' techniques for inclusions is a challenging problem. Resolving this problem in the phase space of continuous functions as well as in L^2 space is the main task of the paper. Using approximation results [1] and comparison theorems [5, 14, 23], we have proved the global mild resolvability for parabolic inclusion with multi-valued right-hand parts without growth assumptions. Limit sets for the obtained global solutions in the corresponding phase spaces are also investigated.

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1 Setting of the problem

In a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, for an unknown function $u = u(t,x)$, $t > 0$, $x \in \Omega$, we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} - Au \in f(u) + h, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (1)$$

Here, $h = h(t,x)$ is a given function,

$$Au = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

and the following conditions are satisfied:

$$A \text{ is a strictly elliptic symmetric differential operator with bounded coefficients,} \quad (2)$$

$$f : \mathbb{R} \to \mathbb{C}_v(\mathbb{R}) \text{ is an upper semicontinuous map, } 0 \in f(0), \quad (3)$$

$$\exists\, B > 0 \quad \exists\, C > 0 \quad \forall\, s : |s| > B \quad \forall\, \xi \in f(s) \quad \xi \cdot s \leq C. \quad (4)$$

Here, $\mathbb{C}_v(\mathbb{R})$ is a set of all convex compact subsets of $\mathbb{R}$.

We will consider problem (1) in the phase space of continuous functions

$$X = \mathbb{C}_0(\Omega) = \left\{ v \in \mathbb{C}(\overline{\Omega}) \, : \, v|_{\partial\Omega} = 0 \right\}$$

with the norm

$$\|v\|_\infty = \sup_{x\in\Omega} |v(x)|,$$

and also in the phase space $X = L^2(\Omega)$ with the norm

$$\|v\|_{L^2} = \left(\int_\Omega |v(x)|^2 \, dx \right)^{1/2}.$$

For given $u_0 \in X$, $T > 0$, $h \in L^1(0,T;X)$, a solution of (1) will be understood in the sense of the following definition.

Definition 1. *A function $u \in \mathbb{C}([0,T];X)$ is called a solution of (1) on $[0,T]$ if $u(0) = u_0$ and there exists $l \in L^1(0,T;X)$ such that*

$$u(t) = \mathbb{T}(t)u_0 + \int_0^t \mathbb{T}(t-s)l(s)\,ds + \int_0^t \mathbb{T}(t-s)h(s)\,ds, \ \ \forall\, t \in [0,T],$$

$$l(s,x) \in f\big(u(s,x)\big) \ \text{ almost everywhere (a.e.) on } \ (0,T) \times \Omega,$$

where $\mathbb{T}(t)$ is the C_0-semigroup generated by A in X.

In the case of a single-valued continuous $f : \mathbb{R} \to \mathbb{R}$, this definition coincides with the classical definition of a mild solution [17]. In the paper, for locally Lipschitz multi-valued map f, we prove the global resolvability of (1) in the phase space $X = \mathbb{C}_0(\Omega)$ (Theorem 1). Using this result, some approximation procedure, and the connection between mild and weak solutions for parabolic equations [4], for upper semicontinuous f we prove global resolvability of (1) in the phase space $X = L^2(\Omega)$ (Theorem 2). We also investigate ω-limit attracting sets for the obtained global solutions (Theorem 3).

2 Preliminary notions and results

First, we introduce notions that are used in the paper.

A map $f : \mathbb{R} \to \mathbb{C}_v(\mathbb{R})$ is called upper semicontinuous if for every $s_0 \in \mathbb{R}$ we have

$$\operatorname{dist}_{\mathbb{R}}\big(f(s), f(s_0)\big) \to 0 \ \text{ as } \ s \to s_0,$$

where $\operatorname{dist}_{\mathbb{R}}$ is the Hasdorff semidistance in $\mathbb{R}$ (see [2]).

A map $f : \mathbb{R} \to \mathbb{C}_v(\mathbb{R})$ is called locally Lipschitz if for every $s_0 \in \mathbb{R}$ there exist $r = r(s_0) > 0$, $K = K(s_0) > 0$, such that

$$|s - s_0| < r \implies \max\big\{\operatorname{dist}_{\mathbb{R}}\big(f(s), f(s_0)\big), \operatorname{dist}_{\mathbb{R}}\big(f(s_0), f(s)\big)\big\} < K|s - s_0|.$$

If X is a Banach space, then $L^p(0,T;X)$, $1 \le p < \infty$, is the space of (classes of) L^p functions from $(0,T)$ into X, which is Banach with the norm

$$\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v(t)\|_X^p \, dt\right)^{1/p}.$$

For $p = \infty$, $L^\infty(0,T;X)$ is the space of (classes of) measurable functions from $(0,T)$ into X, which are essentially bounded. This space is Banach with respect to the norm

$$\|v\|_{L^\infty(0,T;X)} = \sup_{t\in(0,T)} \operatorname{ess} \|v(t)\|_X.$$

Similarly, we denote by $\mathbb{C}\big([0,T];X\big)$ the space of continuous functions from $[0,T]$ into X, which is Banach with the norm

$$\|v\|_{\mathbb{C}([0,T];X)} = \sup_{t\in[0,T]} \|v(t)\|_X.$$

It is known [17], that under condition (2), the operator A is the infinitesimal generator of an analytic compact semigroup $\mathbb{T}(t)$ both in $X = \mathbb{C}_0(\Omega)$ and in $X = L^2(\Omega)$. Moreover, there exist constants $M \ge 1$, $\lambda > 0$ (which depend on X) such that

$$\forall\, t \ge 0 \quad \|\mathbb{T}(t)\| \le Me^{-\lambda t}. \tag{5}$$

This allows us to claim that all results of the paper stay true if we replace assumption (4) with the following one:

$$\forall\, s \in \mathbb{R} \quad \forall\, \xi \in f(s) \quad \xi \cdot s \le \delta \cdot s^2 + C, \quad \text{where} \quad C > 0,\ \delta \in (0, \lambda).$$

Indeed, it is sufficient to consider the maps

$$\hat{A}u = Au + \hat{\lambda}u, \quad \hat{f}(u) = f(u) - \hat{\lambda}u, \quad \hat{\lambda} \in (\delta, \lambda),$$

which satisfy conditions (2)–(4).

It is known [2], that for any multi-valued map $f : \mathbb{R} \to \mathbb{C}_v(\mathbb{R})$, there exist single-valued maps $\underline{f} : \mathbb{R} \to \mathbb{R}$, $\overline{f} : \mathbb{R} \to \mathbb{R}$ such that

$$f(s) = \big[\, \underline{f}(s), \overline{f}(s) \,\big].$$

Moreover, f is locally Lipschitz (continuos) if and only if $\underline{f}, \overline{f}$ are locally Lipschitz (continuos); f is upper semicontinuous if and only if $\underline{f}$ is lower semicontinuous and $\overline{f}$ is upper semicontinuous. In particular, assumption (4) holds for $\underline{f}$ and $\overline{f}$.

Additionally, for an upper semicontinuous map $f : \mathbb{R} \to \mathbb{C}_v(\mathbb{R})$, the image of any compact set is a compact set. Therefore, the following values

$$\left|f(s)\right|_+ := \max_{\xi \in f(s)} |\xi|, \qquad \max_{|s| \le r} \left|f(s)\right|_+$$

are finite for every $s \in \mathbb{R}$ and $r > 0$, respectively.

Remark 1. *It is known* [2]*, that a locally Lipschitz (continuous) map $f : \mathbb{R} \to \mathbb{C}_v(\mathbb{R})$ has a locally Lipschitz (continuous) selector, i.e. a single-valued locally Lipschitz (continuous) map $g : \mathbb{R} \to \mathbb{R}$ such that $g(s) \in f(s)$ for all $s \in \mathbb{R}$. At the same time, an upper semicontinuous map $f : \mathbb{R} \to \mathbb{C}_v(\mathbb{R})$ may have no continuous selectors. Moreover, for such f, the formula $F(u)(x) = f\big(u(x)\big)$ does not define the map $F : X \to 2^X$ neither in $X = \mathbb{C}_0(\Omega)$, nor in $X = L^2(\Omega)$. Therefore, we cannot directly apply the operator approach* [7,13,15–17,22] *to the problem* (1).

In the sequel, we will use the following results from the theory of parabolic equations.

Lemma 1 ([17])**.** *Assume that we consider problem* (1) *in the phase space $X = \mathbb{C}_0(\Omega)$ with $h \in L^\infty_{loc}(\mathbb{R}_+; X)$, and $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz map with $f(0) = 0$. Then, for every $u_0 \in X$, there exists a unique mild solution of* (1)*, defined on some interval $[0, \tau]$, where $\tau = \tau(u_0)$. Such local existence (but without uniqueness) is also guaranteed in the case of continuous map $f : \mathbb{R} \to \mathbb{R}$ (see* [8,21]*).*

Lemma 2 ([5,14,23])**.** *Assume that we consider problem* (1) *in the phase space $X = \mathbb{C}_0(\Omega)$ with right-hand parts $f^1 + h^1$ and $f^2 + h^2$, and initial data u^1_0 and u^2_0. Let $u^1 = u^1(t,x)$ and $u^2 = u^2(t,x)$ be corresponding solutions of* (1) *on $[0,T]$. If $f^1, f^2 : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz, $h^1, h^2 \in L^\infty(0,T;X)$, and*

$$f^1(s) \le f^2(s),\ \forall s \in \mathbb{R},\ h^1(t,x) \le h^2(t,x) \text{ a.e., } u^1_0(x) \le u^2_0(x)\ \forall x \in \Omega,$$

then $u^1(t,x) \le u^2(t,x)$ for all $t \in [0,T]$ and $x \in \Omega$.

Lemma 3 ([4])**.** *Assume that we consider problem* (1) *in the phase space $X = L^2(\Omega)$, $f \equiv 0$, $h \in L^2(0,T;X)$, $u_0 \in X$. Then u is a mild solution of* (1) *on $[0,T]$, i.e. $u \in \mathbb{C}\big([0,T];X\big)$, $u(0) = u_0$, and*

$$u(t) = \mathbb{T}(t)u_0 + \int_0^t \mathbb{T}(t-s)h(s)\,ds \quad \forall\, t \in [0,T]$$

if and only if u is a weak solution of (1) *on $[0,T]$, i.e. $u \in L^2\left(0,T; H^1_0(\Omega)\right)$, $u(0) = u_0$, and for every $v \in H^1_0(\Omega)$, $\eta \in C^\infty_0(0,T)$ we have*

$$-\int_0^T (u(t),v)_{L^2}\,\eta_t\,dt + \int_0^T a(u(t),v)\eta\,dt = \int_0^T (h(t),v)_{L^2}\,\eta\,dt,$$

where $(u,v)_{L^2}$ is the scalar product in $L^2(\Omega)$, $a(u,v)$ is the bilinear continuous form generated by A in $H^1_0(\Omega)$.

3 Global resolvability in $X = \mathbb{C}_0(\Omega)$

Theorem 1. *Assume that conditions* (2)–(4) *are fulfilled,* $h \in L^\infty(\mathbb{R}_+; \mathbb{C}_0(\Omega))$*,* $f : \mathbb{R} \to 2^{\mathbb{R}}$ *is locally Lipschitz map. Then for every* $u_0 \in X = \mathbb{C}_0(\Omega)$*, and every* $T > 0$*, the problem* (1) *has a solution in the sense of Definition* 1*. Moreover,*

$$\forall t \geq 0 \quad \|u(t)\|_\infty \leq Me^{-\lambda t} \cdot \|u_0\|_\infty + \frac{M}{\lambda} \cdot \left(\frac{C}{B} + \max_{|s| \leq B} \left| f(s) \right|_+ + \sup_{t \geq 0} \|h(t)\|_\infty \right). \tag{6}$$

Proof. Let g be a locally Lipschitz selector of f defined as

$$g(s) = \alpha \overline{f}(s) + (1-\alpha)\underline{f}(s), \quad \alpha \in [0,1].$$

Then, $g(0) = 0$, and for all s such that $|s| > B$, we have $sg(s) \leq C$. Now we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} - Au = g(u) + h, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{7}$$

The existence and uniqueness of a local solution for (7) (and, therefore, for (1)) in the sense of Definition 1 is a consequence of Lemma 1. Let us derive a priori estimates to guarantee global existence.

Let $u = u(t,x)$, $t \in [0,\tau]$, $x \in \Omega$, be a solution of (7). We denote $\overline{z} = \overline{z}(t,x)$ as the solution of (7) with $h \equiv 0$ and $\overline{z}\,\big|_{t=0} = |u_0|$. Then, from Lemma 2, we have

$$\overline{z}(t,x) \geq 0 \quad \text{on} \quad [0,\tau] \times \Omega.$$

Let $v = v(t,x)$ be a solution of (7) with the right-hand part $g + |h|$ and initial data $v\big|_{t=0} = |u_0|$. Then, from Lemma 2, we deduce:

$$v(t,x) \geq u(t,x) \quad \text{on} \quad [0,\tau] \times \Omega, \tag{8}$$

$$v(t,x) \geq \overline{z}(t,x) \quad \text{on} \quad [0,\tau] \times \Omega.$$

Moreover, due to (4), we have

$$g\big(v(t,x)\big) \leq \max\left\{ \frac{C}{B}, \max_{s \in [0,B]} \left| g(s) \right| \right\} \leq \frac{C}{B} + \max_{s \in [0,B]} \left| f(s) \right|_+. \tag{9}$$

Using (9), we obtain

$$v(t) = \mathbb{T}(t)\,|u_0| + \int_0^t \mathbb{T}(t-s) g\big(v(s)\big) ds + \int_0^t \mathbb{T}(t-s) \big|h(s)\big| ds.$$

Thus, for $(t,x) \in [0,\tau] \times \Omega$, we have

$$0 \leq v(t,x) \leq Me^{-\lambda t} \cdot \|u_0\|_\infty + \frac{M}{\lambda} \cdot \left(\frac{C}{B} + \max_{s \in [0,\tau]} \left| f(s) \right|_+ + \sup_{t \geq 0} \|h(t)\|_\infty \right). \tag{10}$$

Now, let $\underline{z} = \underline{z}(t,x)$ be a solution of (7) with $h \equiv 0$ and $\underline{z}\,\big|_{t=0} = -\,|u_0|$. Then, from Lemma 2,

$$\underline{z}(t,x) \leq 0 \quad \text{on} \quad [0,\tau] \times \Omega.$$

Let $w = w(t,x)$ be a solution of (7) with the right-hand part $g - |h|$ and initial data $w\big|_{t=0} = -|u_0|$. Then, Lemma 2 implies

$$w(t,x) \leq u(t,x) \quad \text{on} \quad [0,\tau] \times \Omega, \tag{11}$$

$$w(t,x) \leq \underline{z}(t,x) \quad \text{on} \quad [0,\tau] \times \Omega.$$

Due to (4), we get

$$g\big(w(t,x)\big) \geq \max\left\{\frac{C}{B}, \max_{s\in[-B,0]} \big|g(s)\big|\right\} \geq -\frac{C}{B} + \max_{s\in[-B,0]} \big|f(s)\big|_{+}.$$

So, for $(t,x) \in [0,\tau] \times \Omega$ we obtain

$$0 \geq w(t,x) \geq -Me^{-\lambda t} \cdot \|u_0\|_\infty - \frac{M}{\lambda} \cdot \left(\frac{C}{B} + \max_{s\in[-B,0]} \big|f(s)\big|_{+} + \sup_{t\geq 0} \|h(t)\|_\infty\right). \tag{12}$$

Combining (8), (10), (11), and (12), for $u(t,x)$, we obtain (6). This estimate guarantees the existence of a global solution for (1). The theorem is proved. □

Remark 2. *Let us show that the set of solutions of* (1) *in the sense of Definition* 1 *is wider than the set of solutions of* (7)*, where g is a continuous selector of f.*

Indeed, let

$$\Omega = (0,\pi)\,, \quad Au = \Delta u = \frac{\partial^2 u}{\partial x^2}, \quad h \equiv 0,$$

$$\underline{f}(u) = \begin{cases} u + \frac{1}{4}u^3, & u < -2, \\ -u-2, & u \in [-2;-1), \\ -1, & u \in [-1;1), \\ u-2, & u \in [1;2), \\ -u + \frac{1}{4}u^3, & u \geq 2; \end{cases} \qquad \overline{f}(u) = \begin{cases} u + \frac{1}{4}u^3, & u < -2, \\ u+2, & u \in [-2;-1), \\ 1, & u \in [-1;1), \\ -u+2, & u \in [1;2), \\ -u + \frac{1}{4}u^3, & u \geq 2. \end{cases}$$

Then, conditions (2)–(4) *are fulfilled.*

The function

$$u(x) = \frac{1}{2}\sin x + \frac{1}{8}\sin 2x$$

satisfies (1)*, because*

$$u_t - \Delta u = \frac{1}{2}\sin x + \frac{1}{2}\sin 2x \in [-1,1].$$

But there is no $g \in \mathbb{C}(\mathbb{R})$, $g(u) \in [-1,1]$, such that $-\Delta u(x) = g\big(u(x)\big)$ for all $x \in (0,\pi)$. Indeed, suppose that such a function exists. As on $[0,\pi]$ the equation

$$\frac{1}{2}\sin x + \frac{1}{8}\sin 2x = \frac{1}{2}$$

has a solution $x = \frac{\pi}{2}$ and $x = x^* \neq \frac{\pi}{2}$, so for $x = \frac{\pi}{2}$ we have

$$g\left(\frac{1}{2}\right) = -u''\left(\frac{\pi}{2}\right) = \frac{1}{2},$$

and for $x = x^*$ we get

$$g\left(\frac{1}{2}\right) = \frac{1}{2}\sin x^* + \frac{1}{8}\sin 2x^* = \frac{1}{2} + \frac{3}{8}\sin 2x^* \neq \frac{1}{2}.$$

4 Global resolvability in $X = L^2(\Omega)$

Theorem 2. *Assume that conditions* (2)–(4) *are fulfilled, and* $h \in L^\infty(\mathbb{R}_+; L^\infty(\Omega))$. *Then for every* $u_0 \in L^\infty(\Omega)$ *and every* $T > 0$*, the problem* (1) *in the phase space* $X = L^2(\Omega)$ *has a solution in the sense of Definition* 1*. Moreover,*

$$\forall t \geq 0 \quad \|u(t)\|_\infty \leq M, \tag{13}$$

where the constant $M > 0$ *depends only on* $\|u_0\|_\infty$.

Proof. We can pass to the equivalent problem with

$$\widehat{A}u = Au + \widehat{\lambda}u, \quad \widehat{f}(u) = f(u) - \widehat{\lambda}u,$$

where $\widehat{A}$ satisfies (5), $\widehat{\lambda} \in (0, \lambda)$. Therefore, without loss of generality, we can assume that the right-hand part of (1) has the form $f(u) - \varepsilon u + h$, where $\varepsilon > 0$ is sufficiently small, and for some $\beta > 0$ we have

$$\forall s : |s| > \beta, \quad \forall \xi \in f(s) \quad \xi \cdot s \leq 0. \tag{14}$$

For sufficiently large $R > \beta$, we put

$$f_R(s) = \begin{cases} f(s), & |s| \leq R, \\ f\left(\frac{Rs}{|s|}\right), & |s| > R. \end{cases} \tag{15}$$

Then f is an upper semicontinuous map. Indeed, for compact-valued maps, it is sufficient to verify that *graph* (f) is closed [2]:

$$s_n \to s_0, \quad \xi_n \in f_R(s_n), \quad \xi_n \to \xi_0 \quad \Longrightarrow \quad \xi_0 \in f_R(s_0). \tag{16}$$

So, if $|s_n| \leq R$ then $|s_0| \leq R$. Therefore, $f_R(s_n) = f(s_n)$, $f_R(s_0) = f(s_0)$ and condition (16) is a consequence of the upper semicontinuity of f.

If $|s_n| > R$, $|s_0| > R$, then for $\xi_n \in f_R(s_n)$, we have

$$\xi_n \in f\left(\frac{Rs_n}{|s_n|}\right) \quad \Longrightarrow \quad \xi_n \to \xi_0 \in f\left(\frac{Rs_0}{|s_0|}\right) = f_R(s_0),$$

which implies (16).

Moreover, f_R satisfies (14). So, $f_R : \mathbb{R} \to C_v(\mathbb{R})$ is an upper semicontinuous map, satisfying (14), and

$$\forall s \in \mathbb{R} \quad |f_R(u) - \varepsilon u|_+ \leq C(R) + \varepsilon|u|, \tag{17}$$

where constant $C(R)$ depends on R.

The linear growth condition (17) allows us (see [7]) to assert, that for every $T > 0$ and $u_0 \in X = L^2(\Omega)$, the problem

$$\begin{cases} \frac{\partial u}{\partial t} - Au \in f_R(u) - \varepsilon u + h, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases}$$

has a solution on $[0, T]$ in the sense of Definition 1 for $X = L^2(\Omega)$. In other words, there exist $u^R \in \mathbb{C}([0,T]; L^2(\Omega))$, $l^R \in L^\infty(0,T; L^2(\Omega))$ such that $u^R(0) = u_0$ and

$$u^R(t) = \mathbb{T}(t)u_0 + \int_0^t \mathbb{T}(t-s)\big(l^R(s) + h(s)\big)ds, \ \ \forall\, t \in [0,T],$$
$$l^R(t,x) \in f_R\big(u^R(t,x)\big) - \varepsilon u^R(t,x) \ \text{ a.e. on } (0,T)\times\Omega.$$

Then, due to Lemma 3, we deduce that $u^R \in L^2\left(0,T; H_0^1(\Omega)\right)$ is a weak solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t} - Au = l^R + h, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{18}$$

Since $l^R + h \in L^\infty\left(0,T; L^2(\Omega)\right)$, due to [21], for $u_0 \in H_0^1(\Omega)$, the weak solution of (18) satisfies

$$u^R \in \mathbb{C}\left([0,T]; H_0^1(\Omega)\right) \cap L^2\left(0,T; H^2(\Omega)\cap H_0^1(\Omega)\right), \quad \frac{\partial u^R}{\partial t} \in L^2\left(0,T; L^2(\Omega)\right). \tag{19}$$

Taking into account that $u^R(\tau) \in H_0^1(\Omega)$ for any arbitrary small $\tau > 0$, we infer that on $[\tau, T]$ solution of (18) satisfies (19). For a fixed function v let us denote

$$v_+ = \begin{cases} v, & v \geq 0, \\ 0, & v < 0, \end{cases} \quad v_- = \begin{cases} -v, & v \leq 0, \\ 0, & v > 0. \end{cases}$$

We put

$$M = \max\left\{B, \frac{2\|h\|_\infty}{\varepsilon}, \|u_0\|_\infty\right\}.$$

Due to (19), we can multiply equation (18) by $\left(u^R - M\right)_+$ in $L^2(\Omega)$ and obtain for almost all (a.a.) $t \in (\tau, T)$ the following

$$\begin{aligned} &\frac{1}{2}\cdot\frac{d}{dt}\left\|\left(u^R - M\right)_+\right\|_{L^2}^2 + a\left(\left(u^R - M\right)_+, \left(u^R - M\right)_+\right) \\ &\quad = \int_\Omega l^R(t,x)\left(u^R(t,x) - M\right)_+ dx + \int_\Omega h(t,x)\left(u^R(t,x) - M\right)_+ dx \\ &\quad = \int_\Omega \left(l^R(t,x) + \frac{\varepsilon}{2}u^R(t,x)\right)\left(u^R(t,x) - M\right)_+ dx \\ &\qquad + \int_\Omega \left(h(t,x) - \frac{\varepsilon}{2}u^R(t,x)\right)\left(u^R(t,x) - M\right)_+ dx. \end{aligned} \tag{20}$$

Let us estimate the right-hand part of (20). According to (14) and the choice of M, the inequality $u^R(t,x) > M$ yields

$$l^R(t,x) + \frac{\varepsilon}{2}u^R(t,x) \leq -\frac{\varepsilon}{2}u^R(t,x) \leq 0,$$
$$h(t,x) - \frac{\varepsilon}{2}u^R(t,x) \leq \|h\|_\infty - \frac{\varepsilon}{2}u^R(t,x) \leq 0.$$

So, from (20) and ellipticity of A, we deduce that there exists $\nu > 0$ such that for a.a. $t \in (\tau, T)$ we have

$$\frac{1}{2}\cdot\frac{d}{dt}\left\|\left(u^R - M\right)_+\right\|_{L^2}^2 + \nu\cdot\left\|\left(u^R - M\right)_+\right\|_{L^2}^2 \leq 0.$$

Therefore, for all $t \in [\tau, T]$, we have

$$\int_\Omega \left(u^R(t,x) - M\right)_+^2 dx \le e^{-2\nu(t-\tau)} \cdot \int_\Omega \left(u^R(\tau,x) - M\right)_+^2 dx.$$

Since $u^R : [0,T] \to L^2(\Omega)$ is a continuous function, we can pass to the limit as $\tau \to 0+$ and derive that

$$\int_\Omega \left(u^R(t,x) - M\right)_+^2 dx \le e^{-2\nu t} \cdot \int_\Omega \left(u_0(x) - M\right)_+^2 dx \tag{21}$$

for all $t \in [0,T]$. Inequality (21) implies that for all $t \in [0,T]$ and for a.a. $x \in \Omega$ we have

$$u^R(t,x) \le M.$$

Repeating previous arguments for $\left(u^R + M\right)_-$, we get that for all $t \in [0,T]$ and for a.a. $x \in \Omega$

$$u^R(t,x) \ge -M.$$

Combining these inequalities, we have that

$$\left|u^R(t,x)\right| \le M \tag{22}$$

for all $t \ge 0$ and for a.a. $x \in \Omega$, where M does not depend on R.

Thus, for every $u_0 \in L^\infty(\Omega)$, we choose in (15)

$$R > \max\left\{B, \frac{2\|h\|_\infty}{\varepsilon}, \|u_0\|_\infty\right\},$$

and obtain from (22) that

$$\left|u^R(t,x)\right| \le R$$

for all $t \ge 0$ and a.a. $x \in \Omega$. So, u^R is a solution of the problem (1) in the sense of Definition 1. The theorem is proved. □

Remark 3. *In* [7, 12, 21] *theorems about global resolvability in the phase space $X = L^2(\Omega)$ were proved under one of the following growth assumptions:*

$$\exists\, C_1, C_2 \ge 0\ \forall\, s \in \mathbb{R}\ \left|f(s)\right|_+ \le C_1|s| + C_2;$$

$\exists\, p \ge 2,\ \alpha_1, \alpha_2, k > 0\ \forall\, s \in \mathbb{R}\ \forall\, \xi \in f(s)$

$$-k - \alpha_1 |s|^p \le \xi s \le k - \alpha_2 |s|^p.$$

It is easy to verify that the function $f(u) = u \exp(-u)$ does not meet these assumptions but satisfies condition (4).

5 Limit behaviour of global solutions

Assume that $h(t,x) \equiv h(x)$, and X is either $\mathbb{C}_0(\Omega)$ or $L^2(\Omega)$. Our previous results guarantee that for all $u_0 \in X$ there exists at least one solution of (1) on $[0,+\infty)$ with $u(0) = u_0$ that satisfies Definition 1 for every $[0,T]$.

We consider the set

$$\omega(u) = \bigcap_{T>0} cl_X \left(\bigcup_{t \ge T} u(t)\right). \tag{23}$$

Theorem 3. *For every global solution* $u : [0, +\infty) \mapsto X$ *of* (1) *obtained in Theorem* 1 *(in the phase space* $X = \mathbb{C}_0(\Omega)$*) or in Theorem* 2 *(in the phase space* $X = L^2(\Omega)$*), we have that the* ω*-limit set* (23) *is a non-empty compact connected subset of* X*, and*

$$\operatorname{dist}_X \big(u(t), \omega(u)\big) \to 0 \ \ \textit{as} \ \ t \to \infty.$$

Proof. Due to estimates (6) and (13), we have that

$$\sup_{t \geq 0} \|u(t)\|_\infty \leq K \tag{24}$$

for some $K > 0$. It means that the set $\{u(t) : t \geq 0\}$ is bounded in X, and for proving the theorem, it is sufficient to verify that for every $t_n \to \infty$

$$\text{the sequence } \{u(t_n)\} \text{ is precompact in } X. \tag{25}$$

First, let us analyze the case $X = \mathbb{C}_0(\Omega)$. Let us consider functions $u_s(\tau) = u(s + \tau)$ and $l_s(\tau) = l(s + \tau)$ for $\tau \geq 0$. Then, for every $T > 0$, we have

$$u_s \in \mathbb{C}\big([0, T]; X\big),\ l_s \in L^1(0, T; X) \text{ and } u_s(0) = u(s).$$

Additionally,

$$l_s(\tau, x) = l(s + \tau, x) \in f\big(u(s + \tau, x)\big) \quad \forall\, x \in \Omega, \text{ for a.a. } \tau > 0.$$

Then, for every $t \geq 0$, we get

$$\begin{aligned}
u_s(t) = u(t+s) &= \mathbb{T}(t+s)u_0 + \int_0^{t+s} \mathbb{T}(t+s-\tau) l(\tau) d\tau + \int_0^{t+s} \mathbb{T}(t+s-\tau) h d\tau \\
&= \mathbb{T}(t) \cdot \mathbb{T}(s) u_0 + \int_0^{s} \mathbb{T}(t+s-\tau) l(\tau) d\tau + \int_0^{s} \mathbb{T}(t+s-\tau) h d\tau \\
&\quad + \int_s^{s+t} \mathbb{T}(t+s-\tau) l(\tau) d\tau + \int_s^{s+t} \mathbb{T}(t+s-\tau) h d\tau \\
&= \mathbb{T}(t) \left[\mathbb{T}(s) u_0 + \int_0^{s} \mathbb{T}(s-\tau) l(\tau) d\tau + \int_0^{s} \mathbb{T}(s-\tau) h d\tau \right] \\
&\quad + \int_0^{t} \mathbb{T}(t-\tau) l(\tau+s) d\tau + \int_0^{t} \mathbb{T}(t-\tau) h d\tau \\
&= \mathbb{T}(t) u_s(0) + \int_0^{t} \mathbb{T}(t-\tau) l_s(\tau) d\tau + \int_0^{t} \mathbb{T}(t-\tau) h d\tau.
\end{aligned}$$

In particular,

$$u(t_n) = \mathbb{T}(1) u(t_n - 1) + \int_0^1 \mathbb{T}(1-\tau) l(t_n - 1 + \tau) d\tau + \int_0^1 \mathbb{T}(1-\tau) h d\tau, \tag{26}$$

where $l(t_n - 1 + \tau, x) \in f\big(u(t_n - 1 + \tau, x)\big)\ \forall\, x \in \Omega,$ for a.a. $\tau > 0$.

Upper semicontinuity of f and (24) imply

$$\big\|l(t_n - 1 + \tau)\big\|_\infty \leq K_1, \tag{27}$$

where K_1 does not depend on τ, n. After that, we will use the following estimate for semigroup $\mathbb{T}(t)$ in the phase space $X = \mathbb{C}_0(\Omega)$ [10]:

$\exists\, C > 0,\ \alpha \in (0,1),\ \delta \in (1/2, 1)$ such that

$$\forall\, u_0 \in \mathbb{C}_0(\Omega)\ \ \forall\, t \in (0,1]\ \ \|\mathbb{T}(t)u_0\|_{C^{1+\alpha}} \le \frac{C}{t^\delta}\|u_0\|_\infty.$$

Therefore, from (26), we get

$$\begin{aligned}\big\|u(t_n)\big\|_{C^{1+\alpha}} &\le C\big\|u(t_n-1)\big\|_\infty + \int_0^1 Cs^{-\delta}\big\|l(t_n-s)\big\|_\infty\,ds + \int_0^1 Cs^{-\delta}\|h\|_\infty\,ds \\ &\le CK + \frac{C}{1-\delta}\big(K_1 + \|h\|_\infty\big).\end{aligned}$$

This inequality guarantees the required compactness property (25).

Let us analyze the case $X = L^2(\Omega)$. From (24), we deduce the precompactness of $\{u(t_n)\}$ in the weak topology of $L^2(\Omega)$. So, it remains to prove strong convergence of a subsequence. We put $u_0^n := u(t_n-1)$, $l_n(t) := l(t_n-1+t)$ and consider the problem

$$\begin{cases} \frac{\partial v}{\partial t} - Av = l_n + h, \\ v|_{\partial\Omega} = 0, \\ v|_{t=0} = u_0^n. \end{cases} \tag{28}$$

This problem on $(0,T)$, $T>1$, has the unique weak solution $v_n(t) = u(t_n-1+t)$. Multiplying (28) by v_n in $L^2(\Omega)$, we get that

$$\frac{1}{2}\frac{d}{dt}\big\|v_n(t)\big\|^2_{L^2} + a\big(v_n(t), v_n(t)\big) = \big(l_n(t), v_n(t)\big)_{L^2} + \big(h, v_n(t)\big)_{L^2} \tag{29}$$

for a.a. $t \in (0,T)$. This energy equality, coercivity of a, continuous embedding $H_0^1(\Omega) \subset L^2(\Omega)$ and (27) yield that, for some positive constants γ, L, which do not depend on n, we have

$$\forall\, 0 \le s \le t \le T\ \ \big\|v_n(t)\big\|^2_{L^2} + \gamma\int_s^t \big\|v_n(\tau)\big\|^2_{H_0^1}\,d\tau \le \big\|v_n(s)\big\|^2_{L^2} + L(t-s). \tag{30}$$

From (24), (28) and (29), we deduce that the sequence $\{v_n\}$ is bounded in $W(0,T)$, where

$$W(0,T) = \left\{v \in L^2\left(0,T; H_0^1(\Omega)\right)\ :\ \frac{\partial v}{\partial t} \in L^2\left(0,T; H^{-1}(\Omega)\right)\right\}.$$

Due to the compact embedding $W(0,T) \subset L^2\left(0,T; L^2(\Omega)\right)$ [18], we conclude that for some $v \in W(0,T)$ up to subsequence, the following convergences hold:

$$\begin{aligned} v_n &\to v\ \text{ weakly in } W(0,T), \\ v_n &\to v\ \text{ in } L^2\big(0,T; L^2(\Omega)\big), \\ v_n(t) &\to v(t)\ \text{ weakly in } L^2(\Omega) \text{ for all } t \in (0,T), \\ v_n(t) &\to v(t)\ \text{ in } L^2(\Omega) \text{ for a.a. } t \in (0,T). \end{aligned} \tag{31}$$

At the same time, we may assume that for some bounded $\bar{l}$ and $\bar{u}$ up to a subsequence

$$\begin{aligned} l_n &\to \bar{l}\ \text{ weakly in } L^2\big(0,T; L^2(\Omega)\big), \\ u_0^n &\to \bar{u}\ \text{ weakly in } L^2(\Omega). \end{aligned}$$

Therefore, v is a solution of (28) with the right-hand part $\bar{l} + h$ and initial data $\bar{u}$. Consequently, equality (29) and inequality (30) hold for v. Let us consider functions

$$J_n(t) = \left\|v_n(t)\right\|_{L^2}^2 - Lt, \quad J(t) = \left\|v(t)\right\|_{L^2}^2 - Lt.$$

Due to the continuous embedding $W(0,T) \subset \mathbb{C}\left([0,T]; L^2(\Omega)\right)$ and (30), we conclude that J_n and J are continuous and monotonically decreasing. Moreover, using (31), we have $J_n(t) \to J(t)$ for a.a. $t \in (0,T)$. Therefore, by applying a variant of Dini's Lemma [24], we obtain that

$$J_n(t) \to J(t) \ \text{ for all } t \in (0,T).$$

In particular, for $t = 1$, we deduce that $\left\|v_n(1)\right\|_{L^2} \to \left\|v(1)\right\|_{L^2}$. Combining this fact with weak convergence, we get strong convergence of $\left\{v_n(1) = u(t_n)\right\}$ in $L^2(\Omega)$. The theorem is proved. $\square$

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Капустян О., Безущак Д., Станжицький О., Король І. *Глобальні розв'язки та їх гранична поведінка для параболічних включень з необмеженою правою частиною* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 414–426.

Досліджено глобальну розв'язність для параболічних включень з напівнеперервною зверху багатозначною правою частиною більш ніж лінійного росту. Доведено теореми про існування глобальних м'яких розв'язків у різних фазових просторах. Досліджено граничні множини для отриманих глобальних розв'язків у відповідних фазових просторах.

Ключові слова і фрази: параболічне включення, м'який розв'язок, слабкий розв'язок, напівгрупа, гранична множина.