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A truncation error bound for branched continued fractions of the special form on subsets of angular domains

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Truncation error bounds for branched continued fractions of the special form are established. These fractions can be obtained by fixing the values of variables in branched continued fractions with independent variables, which is an effective tool for approximating complex functions of two variables. The main result is a two-dimensional analog of the theorem considered in [SCIAM J. Numer. Anal. 1983, **20** (3), 1187–1197] for van Vleck's continued fractions. For its proving, the $\mathcal{C}$-figure convergence and estimates of the difference between approximants of fractions in an angular domain are significantly used. In comparison with the previously established results, the elements of a branched continued fraction of the special form can tend to zero at a certain rate. An example of the effectiveness of using a two-dimensional analog of van Vleck's theorem is considered.

Key words and phrases: branched continued fraction with independent variables, branched continued fraction of the special form, truncation error bound, approximation.

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Introduction

Different types of functional continued fractions are used to approximate analytical functions. Often, these fractions converge in wider regions than the corresponding power series, and their approximants give better approximations than partial sums of series [26,29,33].

Multidimensional generalizations of continued fractions in terms of a function of many variables are branched continued fractions (BCFs), and their approximants give multidimensional rational approximations. One such generalization is BCFs with independent variables. In particular, the tools of BCFs with independent variables is effective in the approximation of complex functions of two variables. Numerical experiments confirm the effectiveness of the existing expansion algorithms [8,22,23]. However, the question of strict proof of their convergence remains open. At fixed values of the variables, they are called BCFs of the special form. So, the results obtained for BCFs of the special form can be used to study functional BCFs.

Let

$$b_0+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{b_{i(k)}}=b_0+\sum_{i_1=1}^{N}\cfrac{a_{i(1)}}{b_{i(1)}+\sum\limits_{i_2=1}^{i_1}\cfrac{a_{i(2)}}{b_{i(2)}+\sum\limits_{i_3=1}^{i_2}\cfrac{a_{i(3)}}{b_{i(3)}+\ddots}}}$$

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be a BCF of the special form, where b_0, $a_{i(k)}$, $b_{i(k)} \in \mathbb{C}$, $i(k) \in \mathcal{I}$,

$$\mathcal{I} = \{i(k) = (i_1, i_2, \ldots, i_k) : \ 1 \leq i_k \leq i_{k-1} \leq \ldots \leq i_0; \ k \geq 1; \ i_0 = N\}$$

is the set of multiindices, N is a fixed positive integer, it is the dimension of the BCF when its elements are functions of N variables.

Analyzing the one-dimensional results for continued fractions, we can see that many effective theorems on the convergence of continued fractions are formulated using the element and value regions when the belonging of the fraction elements in a certain region guarantees their convergence. In particular, it is considered angular, parabolic, and circular sets of convergence. This paper is concerned with the study of convergence on angular sets.

J. Jensen and E.B. van Vleck first studied the angular regions of convergence of continued fractions. Various proofs of the main result (van Vleck's theorem) are known [26,29,32,33]. Not all proof methods gave an estimate of the rate of convergence, for example, the one proposed in [33]. Quite common in the study of convergence of numerical and functional fractions is the use of the Stieltjes-Vitali theorem on the convergence of a sequence of analytic functions, which does not give an estimate of the rate of convergence of these fractions. In particular, such a proof was proposed in the book by W.J. Thron and W.B. Jones [26]. However, Jensen's work proposes a proof method that uses an estimate of the convergence rate, but this estimate is not stated in the formulation of the theorem. In the paper by D.D. Warner and W.B. Gragg [24], it is proved the van Vlack-Jensen theorem in the following formulation.

Theorem 1 (van Vleck-Jensen, 1901, 1909)**.** *Let the elements of the continued fraction*

$$\mathop{\mathrm{D}}_{n=1}^{\infty} \frac{1}{b_n}$$

satisfy the conditions

$$b_k \neq 0, \quad |\arg b_k| < \theta, \quad \theta < \pi/2, \quad k = 1, 2, \ldots .$$

Then

1) *there exist the finite limits of even and odd approximants;*

2) *the sequence of approximants $\{f_n\}$ converges iff the series $\sum\limits_{n=1}^{\infty} |b_n|$ diverges;*

3) *the bound*
$$|f_m - f_{n-1}| \leq \frac{1}{d_n}, \quad m \geq n,$$
holds with
$$d_n \geq \frac{\mathrm{Re}(b_1)}{2 + \mathrm{Re}(b_1)} \cos\theta \ln\left(1 + (\mathrm{Re}(b_1))^2 \min\left\{1, \frac{1}{|b_1|^2}\right\} \cos\theta \sum_{k=1}^{n} |b_k|\right), \quad n \geq 1.$$

The authors of this formulation refer to the monograph of O. Perron [29], who adapted the proof proposed by J. Jensen [25].

Theorem 1, where only items 1) and 2) are considered, is often called van Vleck's theorem [26,33].

1 Multidimensional analogs of van Vleck's theorem

Convergence sets have been studied for different types and various structures of BCFs (BCFs with N branches of branching [15, 20, 30], two-dimensional continued fractions [27, 28], BCFs of the special form [3, 6, 10, 14]). The obtained results can be used to study different types and various structures of functional BCFs with independent variables [4,5,7–9,13,18,19,21–23].

The angular sets of convergence were studied in [2, 11, 12, 15, 17, 31]. The proof of the analog of van Vleck's theorem for different types of BCFs was carried out only with the help of the Stieltjes-Vitali theorem. Therefore, the problem of estimating the rate of convergence of BCFs in angular regions remains relevant. In the works [11, 12], estimates of the rate of convergence of BCFs of the special form on some subsets of angular domains were established. Additionally, the conditions were imposed that the elements of BCFs are located at a positive distance from zero [12]. Also, they were allowed to converge to zero with a certain speed [11].

Using a multidimensional generalization of the Seidel criterion [16], a multidimensional analog of the van Vleck theorem was established [12]. Let us show the effectiveness of this result for BCFs of the special form with $N = 2$. We apply the formulation of this result to BCFs of the special form with $N = 2$ when studying the convergence of a certain expansion of a function of two variables into a two-dimensional J-fraction with independent variables.

In the following, we will use the abbreviated multiindex notation

$$m[k] = (\underbrace{m, m, \ldots, m}_{k}).$$

Theorem 2. *Let the partial denominators of the two-dimensional BCF of the special form*

$$\sum_{i_1=1}^{2} \frac{1}{b_{i(1)}} + \mathop{\mathrm{D}}_{k=2}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}} \tag{1}$$

belong to the domain

$$G(\varepsilon) = \left\{ z \in \mathbb{C} : \; z \neq 0, \; |\arg z| < \frac{\pi}{2} - \varepsilon \right\}, \tag{2}$$

where ε is an arbitrary positive number, $0 < \varepsilon < \pi/2$.

Then

1) *every nth approximation f_n of the two-dimensional BCF* (1) *belongs to the domain* (2);

2) *there exist finite limits of even and odd approximants;*

3) *the BCF* (1) *converges if the series $\sum_{m=1}^{\infty} |b_{1[m]}|$ and $\sum_{m=1}^{\infty} |b_{2[m]}|$ diverge, as well as for each r, $r \in \mathbb{N}$, the series $\sum_{m=1}^{\infty} |b_{2[r],1[m]}|$ diverge.*

In the paper [22], R. Dmytryshyn used the asymptotic expansion of the function

$$\Psi_1(z_1, z_2) = \int_0^{\infty} \frac{te^{-tz_1}}{1 - e^{-t}} dt + \int_0^{\infty} \frac{s}{1 - e^{-s}} \exp\left\{ -sz_2 - s \int_0^{\infty} \frac{te^{-tz_1}}{1 - e^{-t}} dt \right\} ds$$

in a formal double Laurent series [1] to construct the corresponding two-dimensional J-fraction with independent variables

$$\sum_{i_1=1}^{2} \frac{p_{e_{i(1)}}}{q_{e_{i(1)}} + z_{i_1} + \sum_{i_2=1}^{i_1} \frac{p_{e_{i(2)}}}{q_{e_{i(2)}} + z_{i_2} + \sum_{i_3=1}^{i_2} \frac{p_{e_{i(3)}}}{q_{e_{i(3)}} + z_{i_3} + \ldots}}}, \tag{3}$$

where $e_{i(k)} = e_{i_1} + e_{i_2} + \cdots + e_{i_k}$, $k \geq 1$, $e_r = (\delta_{r,1}, \delta_{r,2})$, $r = 1, 2$, $\delta_{i,j}$ is a Kronecker symbol,

$$p_{e_1+re_2} = p_{e_2} = 1, \quad r \geq 0,$$
$$p_{me_1+re_2} = p_{me_2} = \frac{(m-1)^4}{4(2m-3)(2m-1)}, \quad m \geq 2,\ r \geq 0,$$
$$q_{me_1+re_2} = q_{me_2} = -\frac{1}{2}, \quad m \geq 1,\ r \geq 0.$$

Using a two-dimensional generalization of van Vleck's theorem (Theorem 2), we will prove the convergence of this expansion. To do this, we rewrite the fraction (3) using the notation of element indices mentioned in Theorem 2. That is, the BCF (3) obtain the form

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{p_{i(k)}}{q_{i(k)} + z_{i_k}},\ i_0 = 2, \tag{4}$$

where

$$p_1 = p_{2[r],1} = p_2 = 1, \quad r \geq 0,$$
$$p_{2[r],1[m]} = p_{2[m]} = \frac{(m-1)^4}{4(2m-3)(2m-1)}, \quad m \geq 2,\ r \geq 0,$$
$$q_{2[r],1[m]} = q_{2[m]} = -\frac{1}{2}, \quad m \geq 1,\ r \geq 0.$$

After making the equivalent transformations of the fraction (4) into a fraction with partial denominators equal to 1, we get the fraction

$$\mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}(z_1, z_2)}, \quad i_0 = 2, \tag{5}$$

where

$$b_{i(k)}(z_1, z_2) = (q_{i(k)} + z_{i_k}) \prod_{r=1}^{k} p_{i(r)}^{(-1)^{k+r-1}} = \left(z_{i_k} - \frac{1}{2}\right) \prod_{r=1}^{k} p_{i(r)}^{(-1)^{k+r-1}}.$$

For the elements $b_{i(k)}(z_1, z_2)$, $i(k) \in \mathcal{I}$, the estimates are true

$$|b_1(z_1, z_2)| = |b_{2,1}(z_1, z_2)| = \left|z_1 - \frac{1}{2}\right|, \quad |b_2(z_1, z_2)| = \left|z_2 - \frac{1}{2}\right|,$$
$$|b_{1[m]}(z_1, z_2)| \geq \frac{1}{8e^2}\left|z_1 - \frac{1}{2}\right| \frac{2m-1}{(m-1)^2}, \quad |b_{2,1[m]}(z_1, z_2)| \geq \frac{1}{64e^4}\left|z_1 - \frac{1}{2}\right| \frac{2m-1}{(m-1)^2}, \quad m \geq 2,$$
$$|b_{2[r]}(z_1, z_2)| \geq \frac{1}{8e^2}\left|z_2 - \frac{1}{2}\right| \frac{2r-1}{(r-1)^2}, \quad |b_{2[r],1}(z_1, z_2)| \geq \frac{1}{8e^2}\left|z_1 - \frac{1}{2}\right| \frac{2r-1}{(r-1)^2}, \quad r \geq 2,$$
$$|b_{2[r],1[m]}(z_1, z_2)| \geq \frac{1}{64e^4}\left|z_1 - \frac{1}{2}\right| \frac{2r-3}{(r-1)^2} \frac{2m-1}{(m-1)^2}, \quad m \geq 2,\ r \geq 2.$$

Therefore, in the domain

$$G = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \; \left| \arg \left(z_m - \frac{1}{2} \right) \right| < \frac{\pi}{2} - \varepsilon, \; m = 1, 2 \right\}$$

the series $\sum\limits_{m=1}^{\infty} |b_{1[m]}(z_1, z_2)|$ and $\sum\limits_{m=1}^{\infty} |b_{2[m]}(z_1, z_2)|$ diverge, as well as for each r, $r \in \mathbb{N}$, the series $\sum\limits_{m=1}^{\infty} |b_{2[r],1[m]}(z_1, z_2)|$ diverge. As a result, the conditions of Theorem 2 are fulfilled, and therefore, the BCF (5) converges, and the equivalent BCF (4) converges as well.

2 Estimation of the rate of convergence of a BCF of the special form

In this section, using the estimates of the rate of convergence of continued fractions in the angular domain, we establish an estimate of the rate of convergence of the BCFs of the special form under certain restrictions on the rate of tending to zero of fractions elements.

Theorem 3. *Let the elements of the BCF of the special form*

$$b_0 + \sum_{i_1=1}^{2} \frac{1}{b_{i(1)}} + \mathop{\mathrm{D}}_{k=2}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{1}{b_{i(k)}} \tag{6}$$

satisfy the conditions

$$\mathrm{Re}\,(b_{i(k)}) \geq \frac{\delta}{k^{\beta}}, \quad |\arg b_{i(k)}| \leq \theta, \quad \theta < \frac{\pi}{4}, \quad 0 < \beta \leq \frac{1}{2}, \quad \delta^2 > \frac{1+\beta}{2\beta \cos 2\theta}, \quad i(k) \in \mathcal{I}.$$

Then BCF (6) *converges to a value f, and the following truncation error bound holds*

$$|f - f_{s(n)}| < \frac{M}{\ln \left(1 + \frac{\alpha}{1-\beta} ((n+1)^{1-\beta} - 1) \right)},$$

where $s(n) = (n+1) \sqrt[1-\beta]{(n+1)^{2\beta} + 1}$, *if* $(n+1) \sqrt[1-\beta]{(n+1)^{2\beta} + 1}$ *is integer, otherwise* $s(n) = [(n+1) \sqrt[1-\beta]{(n+1)^{2\beta} + 1}] + 1$, *$M$ and α are some positive constants independent of n and $s(n)$.*

Remark. *In the case $\beta = 0$, the statement of the theorem holds without additional restrictions on δ, and $s(n) = 2n$ (see [12]).*

Proof. The BCF (6) converges because its elements satisfy the conditions of Theorem 2. Let us estimate from above the modulus of the difference between the approximations of the BCF (6). To do this, we write them in the form

$$f_r = b_0^{(1,r)} + \mathop{\mathrm{D}}_{k=1}^{r} \frac{1}{b_{2[k]}^{(1,r-k)}}, \quad r \geq 1,$$

where

$$b_0^{(1,r)} = b_0 + \mathop{\mathrm{D}}_{l=1}^{r} \frac{1}{b_{1[l]}}, \quad b_{2[r]}^{(1,0)} = b_{2[r]}, \quad b_{2[k]}^{(1,r-k)} = b_{2[k]} + \mathop{\mathrm{D}}_{l=1}^{r-k} \frac{1}{b_{2[k],1[l]}}, \quad k = \overline{1, r-1}.$$

Let us consider the $\mathcal{C}$-figured approximant of BCF (6) [14]

$$\widetilde{f}_r = b_0^{(1)} + \mathop{\mathrm{D}}_{k=1}^{r} \frac{1}{b_{2[k]}^{(1)}}, \quad r \geq 1,$$

where $b_0^{(1)}$, $b_{2[k]}^{(1)}$ are values of infinite continued fractions

$$b_0^{(1)} = b_0 + \mathop{\mathrm{D}}_{l=1}^{\infty} \frac{1}{b_{1[l]}}, \quad b_{2[k]}^{(1)} = b_{2[k]} + \mathop{\mathrm{D}}_{l=1}^{\infty} \frac{1}{b_{2[k],1[l]}}, \quad k = \overline{1,r},$$

which converge according to Theorem 1.

There is an obvious inequality

$$|f_m - f_r| \leq |f_m - \widetilde{f}_r| + |\widetilde{f}_r - f_r|, \quad m \geq r.$$

Let us establish an estimate for each of the terms on the right hand side of the above inequality. To do this, use the following triangle inequality

$$|f_p - \widetilde{f}_r| \leq |f_p - g_{p,n}| + |\widetilde{f}_r - g_{p,n}|, \quad p \geq r, \tag{7}$$

where

$$g_{p,n} = b_0^{(1)} + \frac{1}{b_{2[1]}^{(1)}} + \cdots + \frac{1}{b_{2[n]}^{(1)}} + \frac{1}{b_{2[n+1]}^{(1,p-n-1)}} + \cdots + \frac{1}{b_{2[p]}^{(1,0)}}, \quad n \leq p-1.$$

Using the method of proving the formula for the difference between approximants through the tails of fractions approximants, it is easy to show that

$$|f_p - g_{p,n}| \leq |b_0^{(1,p)} - b_0^{(1)}| + \sum_{k=1}^{n} \frac{|b_{2[k]}^{(1,p-k)} - b_{2[k]}^{(1)}|}{\prod\limits_{s=1}^{k} |\widetilde{Q}_{2[s]}^{(p)} Q_{2[s]}^{(p)}|}, \tag{8}$$

where $\widetilde{Q}_{2[s]}^{(p)}$, $Q_{2[s]}^{(p)}$ are sth tails of the continued fractions $g_{p,n}$ and f_p, respectively, $1 \leq s \leq n$,

$$\begin{gathered} Q_{2[s]}^{(p)} = b_{2[s]}^{(1,p-s)} + \frac{1}{Q_{2[s+1]}^{(p)}}, \quad s = 1,2,\ldots,p-1, \quad Q_{2[p]}^{(p)} = b_{2[p]}^{(1,0)}, \\ \widetilde{Q}_{2[s]}^{(p)} = \begin{cases} b_{2[s]}^{(1)} + \dfrac{1}{\widetilde{Q}_{2[s+1]}^{(p)}}, & s = 1,2,\ldots,n, \\ Q_{2[s]}^{(p)}, & s = n+1, n+2, \ldots, p. \end{cases} \end{gathered} \tag{9}$$

Let us apply Theorem 1 to evaluate the numerators of the right-hand side of the inequality (8) since these expressions are the modules of the difference between the values of the continued fractions $b_0^{(1)}$, $b_{2[k]}^{(1)}$ and their approximants. Thus, taking into account the conditions of the theorem, we have

$$|b_{2[k]}^{(1,p-k)} - b_{2[k]}^{(1)}| \le \frac{2+\mathrm{Re}\,(b_{2[k],1})}{\mathrm{Re}\,(b_{2[k],1})\cos\theta} \frac{1}{\ln\left(1+(\mathrm{Re}\,(b_{2[k],1}))^2\min\left\{1,\dfrac{1}{|b_{2[k],1}|^2}\right\}\cos\theta\sum\limits_{s=1}^{p-k}|b_{2[k],1[s]}|\right)}$$
$$\le \frac{2+\dfrac{\delta}{(k+1)^\beta}}{\dfrac{\delta\cos\theta}{(k+1)^\beta}} \frac{1}{\ln\left(1+\min\left\{\dfrac{\delta^2}{(k+1)^{2\beta}},\cos^2\theta\right\}\cos\theta\sum\limits_{s=1}^{p-k}\dfrac{\delta}{(k+s)^\beta}\right)}$$
$$\le \frac{2(k+1)^\beta+\delta}{\delta\cos\theta}\frac{1}{\ln\left(1+\alpha_{k+1}\sum\limits_{s=1}^{p-k}\dfrac{1}{(k+s)^\beta}\right)},$$

where $\alpha_{k+1} = \min\{(\delta^3\cos\theta)/(k+1)^{2\beta}, \delta\cos^3\theta\}$. Taking into account the estimation

$$\sum_{s=1}^{p-k}\frac{1}{(k+s)^\beta} = \sum_{s=k+1}^{p}\frac{1}{s^\beta} \ge \int_{k+1}^{p+1}\frac{dx}{x^\beta} = \frac{(p+1)^{1-\beta}-(k+1)^{1-\beta}}{1-\beta} \ge \frac{(p+1)^{1-\beta}-(n+1)^{1-\beta}}{1-\beta},$$

with $p \ge (n+1)\sqrt[1-\beta]{(n+1)^{2\beta}+1}$ and $1 \le k \le n$, we obtain

$$|b_{2[k]}^{(1,p-k)} - b_{2[k]}^{(1)}| \le \frac{2(k+1)^\beta+\delta}{\delta\cos\theta}\frac{1}{\ln\left(1+\alpha_{n+1}\dfrac{(p+1)^{1-\beta}-(n+1)^{1-\beta}}{1-\beta}\right)}$$
$$\le \frac{2(k+1)^\beta+\delta}{\delta\cos\theta}\frac{1}{\ln\left(1+\dfrac{\alpha_{n+1}}{1-\beta}(n+1)^{1+\beta}\right)} \tag{10}$$
$$< \frac{2(k+1)^\beta+\delta}{\delta\cos\theta}\frac{1}{\ln\left(1+\dfrac{\alpha}{1-\beta}((n+1)^{1-\beta}-1)\right)},$$

since $\alpha_{n+1}(n+1)^{1+\beta} = \min\{\delta^3\cos\theta, (n+1)^{2\beta}\delta\cos^3\theta\}(n+1)^{1-\beta} \ge \min\{\delta^3\cos\theta, \delta\cos^3\theta\}\cdot(n+1)^{1-\beta} = \alpha(n+1)^{1-\beta}$. Similarly, for the first term of the right-hand side of the inequality (8), the following estimate holds

$$|b_0^{(1,p)} - b_0^{(1)}| < \frac{2+\delta}{\delta\cos\theta}\frac{1}{\ln\left(1+\dfrac{\alpha}{1-\beta}((n+1)^{1-\beta}-1)\right)}, \tag{11}$$

with $p \ge (n+1)\sqrt[1-\beta]{(n+1)^{2\beta}+1}$.

Consider the products in the denominators of the right-hand side of the inequality (8). If $k = 2l$, $l \ge 1$, then

$$\prod_{s=1}^{2l}|\widetilde{Q}_{2[s]}^{(p)}Q_{2[s]}^{(p)}| = \prod_{s=1}^{l}(|\widetilde{Q}_{2[2s-1]}^{(p)}\widetilde{Q}_{2[2s]}^{(p)}||Q_{2[2s-1]}^{(p)}Q_{2[2s]}^{(p)}|).$$

Let us estimate separately each factor of the product, taking into account the conditions of the theorem, then for each s, $1 \leq s \leq l$, we obtain

$$\begin{aligned}|\widetilde{Q}^{(p)}_{2[2s-1]}\widetilde{Q}^{(p)}_{2[2s]}| &= \left|\widetilde{Q}^{(p)}_{2[2s]}\left(b^{(1)}_{2[2s-1]}+\frac{1}{\widetilde{Q}^{(p)}_{2[2s]}}\right)\right| \geq \mathrm{Re}(b^{(1)}_{2[2s-1]}\widetilde{Q}^{(p)}_{2[2s]}+1)\\ &= \mathrm{Re}\left(b^{(1)}_{2[2s-1]}\left(b^{(1)}_{2[2s]}+\frac{1}{\widetilde{Q}^{(p)}_{2[2s+1]}}\right)\right)+1 \geq \mathrm{Re}(b_{2[2s-1]}b_{2[2s]})+1\\ &\geq \frac{A}{(2s(2s-1))^{\beta}}+1,\end{aligned}$$

where $A=\delta^2\cos 2\theta$.

Similarly, it can be proved that $|Q^{(p)}_{2[2s-1]}Q^{(p)}_{2[2s]}| \geq A/(2s(2s-1))^{\beta}+1$, $s=\overline{1,l}$.

Therefore,

$$\prod_{s=1}^{2l}|\widetilde{Q}^{(p)}_{2[s]}Q^{(p)}_{2[s]}| \geq \prod_{s=1}^{l}\Big(\frac{A}{(2s(2s-1))^{\beta}}+1\Big)^2.$$

Let k be an odd number. Taking into account the conditions of the theorem and the equation (9), we obtain that if $k=1$, then $|\widetilde{Q}^{(p)}_2 Q^{(p)}_2| \geq \delta^2$.

Let $k=2l+1$, $l \geq 1$, then

$$\begin{aligned}\prod_{s=1}^{2l+1}|\widetilde{Q}^{(p)}_{2[s]}Q^{(p)}_{2[s]}| &= |\widetilde{Q}^{(p)}_2 Q^{(p)}_2|\prod_{s=1}^{l}(|\widetilde{Q}^{(p)}_{2[2s]}\widetilde{Q}^{(p)}_{2[2s+1]}||Q^{(p)}_{2[2s]}Q^{(p)}_{2[2s+1]}|)\\ &\geq \delta^2\prod_{s=1}^{l}(\mathrm{Re}(b_{2[2s]}b_{2[2s+1]})+1)^2 \geq \delta^2\prod_{s=1}^{l}\Big(\frac{A}{(2s(2s+1))^{\beta}}+1\Big)^2.\end{aligned}$$

Hence,

$$\prod_{s=1}^{k}|\widetilde{Q}^{(p)}_{2[s]}Q^{(p)}_{2[s]}| \geq \delta^{1-(-1)^k}\prod_{s=1}^{[k/2]}\Big(\frac{A}{(2s(2s+(-1)^{k+1}))^{\beta}}+1\Big)^2. \tag{12}$$

Thus, continuing the estimation of (8), (10), and (11) with the inequalities (12), we have

$$\begin{aligned}|f_p-g_{p,n}| \leq{}& \frac{1}{\delta\cos\theta\ln\Big(1+\frac{\alpha}{1-\beta}((n+1)^{1-\beta}-1)\Big)}\\ &\times\left(\frac{2+\delta}{1}+\frac{2\times 2^{\beta}+\delta}{\delta^2}+\frac{2\cdot 3^{\beta}+\delta}{\Big(\frac{A}{(2\cdot 1)^{\beta}}+1\Big)^2}+\frac{2\cdot 4^{\beta}+\delta}{\delta^2\Big(\frac{A}{(2\cdot 3)^{\beta}}+1\Big)^2}\right.\\ &\left.+\dots+\frac{2(2l+1)^{\beta}+\delta}{\prod\limits_{q=1}^{l}\Big(\frac{A}{(2q(2q-1))^{\beta}}+1\Big)^2}+\frac{2(2l+2)^{\beta}+\delta}{\delta^2\prod\limits_{q=1}^{l}\Big(\frac{A}{(2q(2q+1))^{\beta}}+1\Big)^2}+\dots\right).\end{aligned}$$

Let us investigate the convergence of the series

$$\sum_{l=1}^{\infty}\frac{2(2l+1)^{\beta}+\delta}{\prod\limits_{q=1}^{l}\Big(\frac{A}{(2q(2q-1))^{\beta}}+1\Big)^2},\qquad \sum_{l=1}^{\infty}\frac{2(2l+2)^{\beta}+\delta}{\delta^2\prod\limits_{q=1}^{l}\Big(\frac{A}{(2q(2q+1))^{\beta}}+1\Big)^2}. \tag{13}$$

Using the basic comparison test and the limit comparison test for series with non-negative terms, the last series will converge if the following series converges

$$\sum_{l=1}^{\infty} \frac{2(2l)^{\beta}+\delta}{\delta^2 \prod\limits_{q=1}^{l} \Big(\dfrac{A}{(2q)^{2\beta}}+1\Big)^2}. \tag{14}$$

For the convergence of this series, it is necessary the product $\prod_{q=1}^{l}(A/(2q)^{2\beta}+1)^2$ tends to infinity, that is $\sum_{q=1}^{l}(A/(2q)^{2\beta}) = \infty$. For $\beta \le 1/2$ this condition is satisfied. Let us estimate the denominator of the term of this series

$$\prod_{q=1}^{l} \Big(\frac{A}{(2q)^{2\beta}}+1\Big)^2 = \exp\Big(2\sum_{q=1}^{l} \ln\Big(\frac{A}{(2q)^{2\beta}}+1\Big)\Big). \tag{15}$$

Since the function $f(x) = \ln(A/(2x)^{2\beta}+1)$ is monotonically decreasing at $x \in [1;\infty)$, then the following estimate is true

$$\sum_{q=1}^{l} \ln\Big(\frac{A}{(2q)^{2\beta}}+1\Big) \ge \int_1^{l+1} \ln\Big(\frac{A}{(2q)^{2\beta}}+1\Big)dq. \tag{16}$$

To simplify the calculation of the last integral, let us estimate the subintegral function from below. The following inequality holds

$$\ln\Big(\frac{A}{(2q)^{2\beta}}+1\Big) \ge \ln\Big(\frac{A}{2q}+1\Big)^{2\beta}, \quad 0<\beta\le 1/2, \quad q \ge 1. \tag{17}$$

It is easy to see that the equality holds for $\beta = 1/2$. If $0<\beta<1/2$, $q \ge 1$, then we consider the difference

$$\frac{A}{(2q)^{2\beta}}+1-\Big(\frac{A}{2q}+1\Big)^{2\beta} = \frac{A+(2q)^{2\beta}-(A+2q)^{2\beta}}{(2q)^{2\beta}}.$$

Let us investigate the function $g(x) = A + x^{2\beta} - (A+x)^{2\beta}$ with $x \in [1/2;+\infty]$. Since $g(x)$ is monotonically increasing on the interval, and the inequality $g(1/2) > 0$ holds at $x = 1/2$, the inequality $g(1/2) > 0$ holds for all points in the considered interval. Therefore, we have proved the inequality (17).

Let us apply it to the continuation of the evaluation of (16). Calculating the resulting integral, we get

$$\begin{aligned}\int_1^{l+1} \ln\Big(\frac{A}{(2q)^{2\beta}}+1\Big)dq &\ge \int_1^{l} \ln\Big(\frac{A}{2q}+1\Big)^{2\beta}dq\\ &= 2\beta\Big(\ln\Big(\frac{2l+A}{2l}\Big)^{l}+\frac{A}{2}\ln(2l+A)+\ln\frac{2}{(2+A)^{1+A/2}}\Big)\\ &\ge \ln\Big((2l+A)^{A\beta}\Big(\frac{4}{(2+A)^{2+A}}\Big)^{\beta}\Big).\end{aligned}$$

Using the result, we estimate the value of (15). As a result, we obtain

$$\prod_{q=1}^{l} \Big(\frac{A}{(2q)^{2\beta}}+1\Big)^2 \ge (2l+A)^{2A\beta}C,$$

where $C = (4/(2+A)^{2+A})^{2\beta}$, $0 < \beta \leq 1/2$, $l \geq 1$. Thus, we can estimate from above the general term of the series (14) as follows

$$\frac{2(2l)^{\beta}+\delta}{\prod\limits_{q=1}^{l}\left(\frac{A}{(2q)^{2\beta}}+1\right)^{2}} \leq \frac{2(2l)^{\beta}+\delta}{(2l+A)^{2A\beta}C}.$$

Using the basic comparison test and the limit comparison test for series with non-negative terms, we study the series with the terms $(2(2l)^{\beta}+\delta)/((2l+A)^{2A\beta}C)$ by comparing it with the generalized harmonic series $\sum_{n=1}^{\infty}(1/n^{2A\beta-\beta})$. The last one converges, since $\beta > 1/2A-1$ by the condition of the theorem.

So, to estimate the first summand of the inequality (7), we have the result

$$|f_p - g_{p,n}| \leq \frac{K}{\delta^3 \cos\theta \ln\left(1+\frac{\alpha}{1-\beta}((n+1)^{1-\beta}-1)\right)} \sum_{l=1}^{\infty} \frac{2(2l)^{\beta}+\delta}{(2l+A)^{2A\beta}C}, \tag{18}$$

where K is the sum of the series (13).

Let us evaluate the second term on the right-hand side of inequality (7). Let us use Theorem 1 for the continued fraction obtained by replacing all the continued fractions in the two-dimensional fraction (6) with their values. It is not difficult to show that the conditions of Theorem 1 are fulfilled for them. We get $|g_{p,n} - \widetilde{f}_r| \leq \frac{1}{\nu_n}$, $n \geq 1$, $p \geq r$, where

$$\nu_n = \frac{\mathrm{Re}\,(b_2^{(1)})}{2+\mathrm{Re}\,(b_2^{(1)})} \cos\theta \ln\left(1+(\mathrm{Re}\,(b_2^{(1)}))^2 \min\left\{1, \frac{1}{|b_2^{(1)}|^2}\right\} \cos\theta \sum_{s=1}^{n} |b_{2[s]}^{(1)}|\right).$$

Since $\mathrm{Re}\,(b_{2[k]}^{(1)}) \geq \delta/k^{\beta}$, $k = 1, 2, \ldots$, then, following the same reasoning as in the proof of the inequality (10), we obtain

$$|g_{p,n} - \widetilde{f}_r| < \frac{2+\delta}{\delta\cos\theta} \frac{1}{\ln\left(1+\frac{\alpha}{1-\beta}((n+1)^{1-\beta}-1)\right)}, \quad p \geq r. \tag{19}$$

Taking into account the estimates of the summands of the right-hand side of inequality (7) (namely (18) and (19)), we obtain

$$|f_p - \widetilde{f}_r| < \frac{2+\delta}{\delta\cos\theta} \frac{2K+1}{\ln\left(1+\frac{\alpha}{1-\beta}((n+1)^{1-\beta}-1)\right)}, \quad p \geq (n+1)\sqrt[1-\beta]{(n+1)^{2\beta}+1}, \quad p \geq r.$$

This leads to the estimate formulated in the statement of Theorem 3. □

3 Conclusions and prospects

The tools of two-dimensional BCFs is effective in the approximation of complex functions of two variables. Numerical experiments confirm the effectiveness of the existing expansion algorithms [8,22,23]. However, the question of a strict justification of their convergence remains open. The obtained results can be used to study existing different types and various structures of BCF.

In the future, it is possible to obtain more general results for fractions whose partial numerators can be arbitrary complex numbers that do not equal one to avoid equivalent transformations during the study. It may also be expedient to weaken the conditions set on the elements or prove similar results for multidimensional generalizations of the continued fractions.

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Боднар Д.І., Боднар О.С., Біланик І.Б. *Оцінка швидкості збіжності гіллястих ланцюгових дробів спеціального вигляду на підмножинах кутових областей* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 437–448.

Встановлено оцінку похибки наближення гіллястих ланцюгових дробів спеціального вигляду. Ці дроби отримують при фіксуванні значень змінних у гіллястих ланцюгових дробах з нерівнозначними змінними, які є ефективним апаратом для наближення функції двох комплексних змінних. Основним результатом є двовимірний аналог теореми, розглянутої в [SCIAM J. Numer. Anal. 1983, **20** (3), 1187–1197] для неперервних дробів Ван Флека. При його доведенні значно використовуються збіжність $\mathcal{C}$-фігур та оцінки різниці апроксимантів дробів у кутовій області. У порівнянні з раніше встановленими результатами елементи гіллястого ланцюгового дробу спеціального вигляду можуть прямувати до нуля з певною швидкістю. Розглянуто приклад ефективності використання двовимірного аналогу теореми Ван Флека.

Ключові слова і фрази: гіллястий ланцюговий дріб спеціального вигляду, гіллястий ланцюговий дріб з нерівнозначними змінними, оцінка похибки наближення, наближення.