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Approximation of classes of periodic functions of several variables with given majorant of mixed moduli of continuity

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We obtain the exact-order estimates of approximation of the Nikol'skii-Besov-type classes $B^{\Omega}_{\infty,\theta}$ of periodic functions of several variables with a given function $\Omega(t)$ of a special form by using linear operators satisfying certain conditions. The approximation error is estimated in the metric of the space L_∞. The obtained estimates of the considered approximation characteristic, in addition to independent interest, can be used to establish the lower bounds of the orthowidths of the corresponding functional classes.

Key words and phrases: mixed modulus of continuity, Bari-Stechkin condition, Nikol'skii-Besov-type class, linear operator, Vallée Poussin kernel, Fejér kernel.

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Introduction

The purpose of the work is to obtain the estimates of one approximation characteristic of the Nikol'skii-Besov-type classes $B^{\Omega}_{\infty,\theta}$ of periodic functions of several variables with a given function $\Omega(t)$ of a special form. Approximation of functions from corresponding classes is carried out using linear operators that are subject to certain conditions, and at the same time the approximation error is estimated in the space L_∞. The obtained estimates complement and generalize some of the results of the work [17], related to approximation of the Nikol'skii-type classes H^{Ω}_{∞}.

The mentioned functional classes and the approximation characteristic are defined below. First, let us present some necessary notation and definitions.

Let $\mathbb{R}^d, d \geq 1$, be a d-dimensional real space with elements $x = (x_1, \dots, x_d)$ and $(x,y) = x_1 y_1 + \dots + x_d y_d$ be the scalar product of the elements $x, y \in \mathbb{R}^d$. Let $L_p\left(\mathbb{T}^d\right)$, where $\mathbb{T}^d = \prod\limits_{j=1}^{d} [0, 2\pi)$, denote a space of functions f that are 2π-periodic in each of their variable and satisfying

$$\|f\|_p := \|f\|_{L_p(\mathbb{T}^d)} = \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{T}^d)} = \operatorname*{ess\,sup}_{x \in \mathbb{T}^d} |f(x)| < \infty.$$

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Further, we assume that for functions $f \in L_p\left(\mathbb{T}^d\right)$ the following additional condition

$$\int_0^{2\pi} f(x)dx_j = 0\,, \;\; j = 1, \dots, d,$$

holds. By $L_p^0\left(\mathbb{T}^d\right)$ we denote the set of such functions.

For $f \in L_p^0\left(\mathbb{T}^d\right), 1 \le p \le \infty$, and $t = (t_1, \dots, t_d)$, $t_j \ge 0$, $j = 1, \dots, d$, we consider the mixed modulus of continuity of the order l, namely

$$\Omega_l(f,t)_p = \sup_{\substack{|h_j| \le t_j \\ j=1,\dots,d}} \left\| \Delta_h^l f(\cdot) \right\|_p,$$

where $l \in \mathbb{N}$, $\Delta_h^l f(x) = \Delta_{h_1}^l \dots \Delta_{h_d}^l f(x) = \Delta_{h_d}^l \left(\dots \left(\Delta_{h_1}^l f(x) \right) \right)$ is a mixed difference of the order l with a vector step $h = (h_1, \dots, h_d)$. Here the difference of the lth order with a step h_j in the variable x_j is defined as follows

$$\Delta_{h_j}^l f(x) = \sum_{n=0}^{l} (-1)^{l-n} C_l^n f(x_1, \dots, x_{j-1}, x_j + nh_j, x_{j+1}, \dots, x_d).$$

Let $\Omega(t) = \Omega(t_1, \dots, t_d)$ be a given function of the type of a mixed modulus of continuity of the order l, which satisfies the following conditions:

1) $\Omega(t) > 0$ for $t_j > 0$, $j = 1, \dots, d$, and $\Omega(t) = 0$ for $\prod\limits_{j=1}^{d} t_j = 0$;

2) $\Omega(t)$ is nondecreasing in each variable;

3) $\Omega\left(m_1 t_1, \dots, m_d t_d\right) \le \left(\prod_{j=1}^d m_j \right)^l \Omega(t)$, $m_j \in \mathbb{N}$, $j = 1, \dots, d$;

4) $\Omega(t)$ is continuous for $t_j \ge 0$, $j = 1, \dots, d$.

We also assume that $\Omega(t)$ satisfies the conditions (S) and (S_l), which are called the Bari-Stechkin conditions (see [27]). This means the following.

A function of one variable $\varphi(\tau) \ge 0$ is said to satisfy the condition (S) if $\varphi(\tau)/\tau^\alpha$ almost increases for some $\alpha > 0$, i.e. there exists a constant $C_1 > 0$ independent of τ_1 and τ_2 such that

$$\frac{\varphi(\tau_1)}{\tau_1^\alpha} \le C_1 \frac{\varphi(\tau_2)}{\tau_2^\alpha}, \quad 0 < \tau_1 \le \tau_2 \le 1.$$

A function $\varphi(\tau) \ge 0$ is said to satisfy the condition (S_l) if $\varphi(\tau)/\tau^\gamma$ almost decreases for some $0 < \gamma < l$, i.e. there exists a constant $C_2 > 0$ independent of τ_1 and τ_2 such that

$$\frac{\varphi(\tau_1)}{\tau_1^\gamma} \ge C_2 \frac{\varphi(\tau_2)}{\tau_2^\gamma}, \quad 0 < \tau_1 \le \tau_2 \le 1.$$

In the case of $d > 1$ we say that $\Omega(t)$ satisfies the conditions (S) and (S_l) if $\Omega(t)$ satisfies these conditions in each variable t_j for fixed t_i, $i \ne j$.

Let $1 \le p \le \infty$, $1 \le \theta \le \infty$, and let $\Omega(t)$ be a given function of the type of a mixed modulus of continuity of the order l. Then the classes $B^{\Omega}_{p,\theta}$ are defined in the following way (see [29]):

$$B^{\Omega}_{p,\theta} = \left\{ f \in L^0_p(\mathbb{T}^d) : \|f\|_{B^{\Omega}_{p,\theta}} \le 1 \right\},$$

where

$$\|f\|_{B^{\Omega}_{p,\theta}} = \left\{ \int_{\mathbb{T}^d} \left(\frac{\Omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \prod_{j=1}^{d} \frac{dt_j}{t_j} \right\}^{1/\theta}, \quad 1 \le \theta < \infty,$$

$$\|f\|_{B^{\Omega}_{p,\infty}} = \sup_{t>0} \frac{\Omega_l(f,t)_p}{\Omega(t)}.$$

Here and subsequently, the expression $t > 0$ for $t = (t_1, \dots, t_d)$ is equivalent to $t_j > 0$ for all $j = 1, \dots, d$.

We note that for $\theta = \infty$ the classes $B^{\Omega}_{p,\theta}$ coincide with the classes H^{Ω}_p, which were considered by N.N. Pustovoitov in [16].

In what follows, it will be convenient for us to use the equivalent (to within absolute constants) definition of the classes $B^{\Omega}_{p,\theta}$. For this purpose, we need more appropriate notations.

Let $V_n(t)$ be the Vallée Poussin kernel of the order $2n-1$, i.e.

$$V_n(t) = 1 + 2\sum_{k=1}^{n} \cos kt + 2\sum_{k=n+1}^{2n-1} \left(1 - \frac{k-n}{n}\right) \cos kt.$$

To every vector $s = (s_1, \dots, s_d)$, $s_j \in \mathbb{N}$, $j = 1, \dots, d$, we assign the polynomial

$$A_s(x) = \prod_{j=1}^{d} \left(V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j) \right)$$

in correspondence. For $f \in L_p(\mathbb{T}^d)$, $1 \le p \le \infty$, by $A_s(f)$ we denote the convolution

$$A_s(f) := A_s(f,x) = (f * A_s)(x).$$

Let $1 < p < \infty$, $1 \le \theta \le \infty$ and let $\Omega(t)$ be a given function of the type of a mixed modulus of continuity of the order l that satisfies the conditions 1) – 4), (S) and (S_l). Then, to within absolute constants, the classes $B^{\Omega}_{p,\theta}$ can be defined as follows:

$$B^{\Omega}_{p,\theta} = \left\{ f \in L^0_p(\mathbb{T}^d) : \|f\|_{B^{\Omega}_{p,\theta}} = \left(\sum_{s} \Omega^{-\theta}(2^{-s}) \big\| A_s(f) \big\|_p^{\theta} \right)^{1/\theta} \le 1 \right\}, \tag{1}$$

for $1 \le \theta < \infty$ and

$$B^{\Omega}_{p,\infty} = \left\{ f \in L^0_p(\mathbb{T}^d) : \|f\|_{B^{\Omega}_{p,\infty}} = \sup_{s} \frac{\big\| A_s(f) \big\|_p}{\Omega(2^{-s})} \le 1 \right\}. \tag{2}$$

Throughout the paper, $\Omega\,(2^{-s}) = \Omega\,(2^{-s_1}, \dots, 2^{-s_d})$, $s_j \in \mathbb{N}$, $j = 1, \dots, d$.

We note that relations (1) and (2) were obtained in works [24] and [16], respectively.

We note also that for $\Omega(t) = \prod_{j=1}^{d} t_j^{r_j}$, $0 < r_j < l$, the classes $B^{\Omega}_{p,\theta}$ coincide with the well-known Besov $B^{r}_{p,\theta}$, $1 \le \theta < \infty$, and Nikol'skii $B^{r}_{p,\infty} = H^{r}_p$ classes (see, e.g., [11]).

In what follows, we study the classes $B^{\Omega}_{p,\theta}$ that are defined by the function

$$\Omega(t)=\Omega\left(t_1,\dots,t_d\right)=\begin{cases}\prod\limits_{j=1}^{d}\dfrac{t_j^r}{\left(\log 1/t_j\right)_+^{b_j}}, & \text{if } t_j>0, j=1,\dots,d,\\ 0, & \text{if } \prod\limits_{j=1}^{d} t_j=0.\end{cases} \tag{3}$$

Here and subsequently, we consider the logarithms to base 2, and

$$\left(\log(1/t_j)\right)_+=\max\left\{1,\log(1/t_j)\right\}.$$

In addition, we assume that $b_j\in\mathbb{R}$, $j=1,\dots,d$, and $0<r<l$. Hence, properties 1) – 4) and the conditions (S) and (S_l) are satisfied for the function $\Omega(t)$ of the form (3).

Let us define the approximation characteristic, introduced in [26].

For the functional class $F\subset L_q\left(\mathbb{T}^d\right)$ we denote

$$d^B_M(F,L_q)=\inf_{G\in L_M(B)_q}\ \sup_{f\in F\cap D(G)}\|f-Gf\|_q, \tag{4}$$

where $L_M(B)_q$ is the set of linear operators satisfying the following conditions:

a) the domain $D(G)$ of these operators contains all trigonometric polynomials, and their range of values is contained in an M-dimensional subspace of the space $L_q\left(\mathbb{T}^d\right)$;

b) there exists a number $B\geq 1$ such that for all vectors $k=(k_1,\dots,k_d)$, $k_j\in\mathbb{Z}$, $j=1,\dots,d$, the inequality $\left\|Ge^{i(k,\cdot)}\right\|_2\leq B$ holds.

We note that $L_M(1)_2$ contains the operators of orthogonal projection onto the spaces of dimension M and the operators that are set on an orthonormal system of functions with the help of the multiplier defined by a sequence $\{\lambda_m\}$ such that $|\lambda_m|\leq 1$ for all m.

There are a lot of papers where the quantity $d^B_M\left(F,L_q\right)$ was studied for various classes of functions. We mention works [1, 8, 9, 17–23, 25, 28], where the quantity (4) and similar to it approximation characteristics were considered for the classes of functions of many variables $W^r_{p,\alpha}$, H^r_p, $B^r_{p,\theta}$, and H^{Ω}_p (see also numerous references therein).

In what follows, we study the quantities $d^B_M\left(B^{\Omega}_{p,\theta},L_q\right)$ for $p=q=\infty$. Note that the obtained estimates complement the results established in the works [2–7, 10].

1 Auxiliary assertions

Now we give several known assertions, which are used in the subsequent considerations. To every vector $s=(s_1,\dots,s_d)$, $s_j\in\mathbb{N}$, $j=1,\dots,d$, we assign the set

$$\rho(s)=\Big\{k=(k_1,\dots,k_d): 2^{s_j-1}\leq|k_j|<2^{s_j},\ k_j\in\mathbb{Z},\ j=1,\dots,d\Big\}$$

in correspondence. For natural N, we set

$$\chi(N)=\Big\{s=(s_1,\dots,s_d):\ s_j\in\mathbb{N},\ j=1,\dots,d,\ \Omega(2^{-s})\geq\frac{1}{N}\Big\},\quad Q(N)=\bigcup_{s\in\chi(N)}\rho(s).$$

The sets $Q(N)$ are generated by the level surfaces of the function $\Omega(t)$. They are analogs of step hyperbolic crosses, which were used in the study of approximation characteristics of the Nikol'skii H^r_p and Besov $B^r_{p,\theta}$ classes.

Using definition (3), we can define the set $\chi(N)$ as follows

$$\chi(N) = \Big\{ s = (s_1, \dots, s_d): \ s_j \in \mathbb{N}, \, j = 1, \dots, d, \, \prod_{j=1}^{d} 2^{rs_j} s_j^{b_j} \le N \Big\}.$$

Further, let

$$\chi^{\perp}(N) = \mathbb{N}^d \setminus \chi(N), \quad \Theta(N) = \chi(2^l N) \setminus \chi(N).$$

We note that, according to (3), the definition of the set $\Theta(N)$ takes the form

$$\Theta(N) = \Big\{ s = (s_1, \dots, s_d): \ s_j \in \mathbb{N}, \, j = 1, \dots, d, \, N < \prod_{j=1}^{d} 2^{rs_j} s_j^{b_j} \le 2^l N \Big\}.$$

Let $\mathfrak{M}$ be certain finite set. By $|\mathfrak{M}|$, we denote the number of its elements. The following proposition is true.

Lemma 1 ([17]). *For the number of elements of the set $Q(N)$, the following ordinal equalities hold:*

$$\begin{aligned} \big|Q(N)\big| &\asymp N^{1/r} (\log N)^{-b_1/r - \dots - b_d/r + d - 1}, \quad && \text{if} \quad b_1 \le \ldots \le b_d < r; \\ \big|Q(N)\big| &\asymp N^{\frac{1}{r}} (\log N)^{-b_1/r - \dots - b_\nu/r + \nu - 1}, \quad && \text{if} \quad b_1 \le \ldots \le b_\nu < r < b_{\nu+1} \le \ldots \le b_d; \\ \big|Q(N)\big| &\asymp N^{\frac{1}{r}} (\log N)^{-b_1/r}, \quad && \text{if} \quad r \le b_1 \le \ldots \le b_d, \ b_2 > r. \end{aligned}$$

Here and below, the notation $\mu_1 \ll \mu_2$ for positive functions $\mu_1(N)$ and $\mu_2(N)$ means that there exists a constant $C > 0$ such that the inequality $\mu_1(N) \le C\mu_2(N)$ holds for all $N \in \mathbb{N}$. The relation $\mu_1 \asymp \mu_2$ holds if $\mu_1 \ll \mu_2$ and $\mu_1 \gg \mu_2$. We note also that all constants C_i, $i = 1, 2, \dots$, which are used in what follows, can depend only on parameters that are contained in the definitions of a class of functions of a dimension d.

Lemma 2 ([13]). *The number of elements of the set $\Theta(N)$ satisfies the relation*

$$\big|\Theta(N)\big| \asymp (\log N)^{d-1}.$$

Lemma 3 ([13]). *For a function $\Omega(t)$ defined by (3) with $r > 0$, the following relation*

$$\sum_{s \in \chi^{\perp}(N)} \big(\Omega(2^{-s})\big)^p \ll \sum_{s \in \Theta(N)} \big(\Omega(2^{-s})\big)^p$$

holds for $0 < p < \infty$.

Lemma 4 ([17]). *If $\gamma_1 \le \ldots \le \gamma_\nu < 1 < \gamma_{\nu+1} \le \ldots \le \gamma_d$, then*

$$\sum_{s \in \Theta(N)} \prod_{j=1}^{d} s_j^{-\gamma_j} \asymp (\log N)^{-\gamma_1 - \dots - \gamma_\nu + \nu - 1}.$$

Next, for natural N, we set

$$\chi'(N) = \Big\{ s = (s_1, \dots, s_d) : \ s_j \in \mathbb{N}, \, j = 1, \dots, d, \, \prod_{j=1}^{d} 2^{rs_j} s_j^{b'_j} \le N \Big\}, \quad \Theta'(N) = \chi'(2^l N) \backslash \chi'(N),$$

where $b'_j = b_j < r$, $j = 1, \dots, \nu$, and $r < b'_j < b_j$, $j = \nu + 1, \dots, d$. In the case of

$$r < b_1 = \dots = b_\nu < b_{\nu+1} \le \dots \le b_d,$$

we put $b'_j = b_1$, $j = 1, \dots, \nu$, and $b_1 < b'_j < b_j$, $j = \nu + 1, \dots, d$.

The sets

$$Q'(N) = \bigcup_{s \in \chi'(N)} \rho(s)$$

are analogs of the so-called "improper" step hyperbolic crosses (see, e.g., [17]).

In [14], it was established that the number of elements of the set $Q'(N)$ satisfies the order relation

$$\big|Q'(N)\big| \asymp \big|Q(N)\big|.$$

2 Main results

Passing to the statement of the propositions and their proof, we assume that $M = \big|Q(N)\big|$. First, we consider case $b_1 \le \dots \le b_d < r$. Then, according to Lemma 1, we have

$$M \asymp N^{1/r} (\log N)^{-b_1/r - \dots - b_d/r + d - 1}, \quad \log M \asymp \log N, \quad N \asymp M^r (\log M)^{b_1 + \dots + b_d - (d-1)r}.$$

Theorem 1. *Let $1 \le \theta < \infty$ and $\Omega(t)$ be a function of the form* (3)*. Then for $b_1 \le \dots \le b_d < r$, $0 < r < l$, the relation*

$$d_M^B (B_{\infty,\theta}^\Omega, L_\infty) \asymp M^{-r} (\log M)^{-b_1 - \dots - b_d + (d-1)(r + 1 - 1/\theta)} \tag{5}$$

holds.

Proof. First, we establish the upper bound in (5). For this purpose, we consider an approximation of the functions $f \in B_{\infty,\theta}^\Omega$ by polynomials of the form

$$V_{Q(N)}(x) = \sum_{s \in \chi(N)} A_s(f, x),$$

where the number N is chosen for a given M from the relation $M = \big|Q(N)\big|$. The operator G that associates the function f with the polynomial of the form $V_{Q(N)}(x)$ belongs to $L_M(1)_2$. Taking into account that $f(x) = \sum\limits_s A_s(f, x)$ (see, e.g., [12, p. 304]), and using Minkowski's inequality, we obtain

$$\begin{aligned} \|f - V_{Q(N)}\|_\infty &= \Big\| \sum_{s \in \chi^\perp(N)} A_s(f) \Big\|_\infty \le \sum_{s \in \chi^\perp(N)} \|A_s(f)\|_\infty \\ &= \sum_{s \in \chi^\perp(N)} \Omega^{-1}(2^{-s}) \|A_s(f)\|_\infty \Omega(2^{-s}) = I_1. \end{aligned}$$

To estimate I_1, we consider two cases.

First, let $1 < \theta < \infty$. Applying Hölder's inequality with exponent θ to I_1 and using Lemmas 2 and 3, we obtain

$$
\begin{aligned}
I_1 &\leq \Big(\sum_{s \in \chi^{\perp}(N)} \Omega^{-\theta}(2^{-s}) \big\| A_s(f) \big\|_\infty^\theta \Big)^{1/\theta} \Big(\sum_{s \in \chi^{\perp}(N)} \big(\Omega(2^{-s})\big)^{\theta/(\theta-1)} \Big)^{1-1/\theta} \\
&\ll \|f\|_{B^{\Omega}_{\infty,\theta}} \Big(\sum_{s \in \chi^{\perp}(N)} \big(\Omega(2^{-s})\big)^{\theta/(\theta-1)} \Big)^{1-1/\theta} \ll \Big(\sum_{s \in \Theta(N)} \big(\Omega(2^{-s})\big)^{\theta/(\theta-1)} \Big)^{1-1/\theta} \\
&\ll N^{-1} \Big(\sum_{s \in \Theta(N)} 1 \Big)^{1-1/\theta} = N^{-1} \big|\Theta(N)\big|^{1-1/\theta} \asymp N^{-1} (\log N)^{(d-1)(1-1/\theta)} \\
&\asymp M^{-r} (\log M)^{-b_1 - \cdots - b_d + (d-1)r} (\log M)^{(d-1)(1-1/\theta)} = M^{-r} (\log M)^{-b_1 - \cdots - b_d + (d-1)(r+1-1/\theta)}.
\end{aligned}
$$

In the case $\theta = 1$ we have

$$
I_1 \ll N^{-1} \sum_{s \in \chi^{\perp}(N)} \Omega^{-1}(2^{-s}) \big\| A_s(f) \big\|_\infty \ll N^{-1} \, \|f\|_{B^{\Omega}_{\infty,1}} \asymp M^{-r} (\log M)^{-b_1 - \cdots - b_d + (d-1)r}.
$$

Thus, the upper bound in (5) is established. Let us find the corresponding lower bound. Let N be a sufficiently large number. With the help of the reasoning analogous to that in [15], we can prove the existence of a set $\Theta_1(N) \subset \Theta(N)$ such that the following relations $s_j \asymp \log N$, $j = 1, \ldots, d$, and $\big|\Theta_1(N)\big| \asymp (\log N)^{d-1}$ are satisfied for $s = (s_1, \ldots, s_d) \in \Theta_1(N)$.

Let K_n be the Fejér kernel of the order n, i.e.

$$
K_n(t) = \sum_{|k| \leq n} \left(1 - \frac{|k|}{n+1} \right) e^{ikx}.
$$

We denote by k^s the vector $k^s = \big(k_1^{s_1}, \ldots, k_d^{s_d}\big)$, where

$$
k_j^{s_j} = \begin{cases} 2^{s_j - 1} + 2^{s_j - 2}, & s_j \geq 2; \\ 1, & s_j = 1, j = 1, \ldots, d. \end{cases}
$$

Consider the function

$$
g_1(x) = \sum_{s \in \Theta_1(N)} \mathcal{K}_s(x),
$$

where

$$
\mathcal{K}_s(x) = e^{i(k^s, x)} \prod_{j=1}^{d} K_{2^{s_j - 2}}(x_j).
$$

Suppose that the operator G belongs to $L_M(B)_\infty$. In [10], it was established that there exists a vector $y^* = \big(y_1^*, \ldots, y_d^*\big)$ such that the following relation

$$
\big\| g_1\left(x - y^*\right) - G g_1\left(x - y^*\right) \big\|_\infty \gg M \tag{6}
$$

holds.

Consider the function

$$
g_2(x) = C_3 N^{-1} \left(N^{1/r} (\log N)^{-b_1/r - \cdots - b_d/r} \right)^{-1} (\log N)^{-(d-1)/\theta} g_1(x), \quad C_3 > 0.
$$

Now, we show that at the corresponding choice of the constant C_3 this function belongs to the class $B^{\Omega}_{\infty,\theta}, 1 \le \theta < \infty$.

Since $\|K_n\|_\infty \asymp n$ for the Fejér kernel, we have $\|\mathcal{K}_s\|_\infty \asymp 2^{\|s\|_1}$, where $\|s\|_1 = s_1 + \cdots + s_d$, $s_j \in \mathbb{N}$. Using the relation $\|A_s(g_1)\|_\infty \ll \|\mathcal{K}_s\|_\infty \asymp 2^{\|s\|_1}$, $s \in \Theta_1(N)$, we can write

$$\|g_2\|_{B^{\Omega}_{\infty,\theta}} = \left(\sum_s \Omega^{-\theta}(2^{-s}) \|A_s(g_2)\|^\theta_\infty \right)^{1/\theta}$$
$$\ll N^{-1} \left(N^{1/r}(\log N)^{-b_1/r-\cdots-b_d/r}\right)^{-1} (\log N)^{-\frac{d-1}{\theta}} \left(\sum_{s\in\Theta_1(N)} \Omega^{-\theta}(2^{-s}) \|A_s(g_1)\|^\theta_\infty \right)^{1/\theta}$$
$$\ll \left(N^{1/r}(\log N)^{-b_1/r-\cdots-b_d/r}\right)^{-1} (\log N)^{-\frac{d-1}{\theta}} \left(\sum_{s\in\Theta_1(N)} 2^{\|s\|_1\theta} \right)^{1/\theta} = I_2.$$

Since the following relations

$$2^{\|s\|_1} \asymp N^{1/r} \prod_{j=1}^d s_j^{-b_j/r} \qquad \text{and} \qquad s_j \asymp \log N, \quad j = 1, \ldots, d,$$

hold for $s \in \Theta_1(N) \subset \Theta(N)$, we get

$$I_2 \asymp \left(N^{1/r}(\log N)^{-b_1/r-\cdots-b_d/r}\right)^{-1} (\log N)^{-(d-1)/\theta} N^{1/r}(\log N)^{-b_1/r-\cdots-b_d/r} \left|\Theta_1(N)\right|^{1/\theta}$$
$$\asymp (\log N)^{-(d-1)/\theta}(\log N)^{(d-1)/\theta} = 1.$$

So, we conclude that $g_2 \in B^{\Omega}_{p,\theta}$ with the corresponding constant $C_3 > 0$.

Now, using the relation (6), we obtain

$$\left\|g_2\left(x-y^*\right) - Gg_2\left(x-y^*\right)\right\|_\infty$$
$$\gg N^{-1} \left(N^{1/r}(\log N)^{-b_1/r-\cdots-b_d/r+d-1}\right)^{-1} (\log N)^{(d-1)(1-1/\theta)} \left\|g_1(x-y^*) - Gg_1(x-y^*)\right\|_\infty$$
$$\gg M^{-r}(\log M)^{-b_1-\cdots-b_d+(d-1)r} M^{-1} (\log M)^{(d-1)(1-1/\theta)} M$$
$$= M^{-r}(\log M)^{-b_1-\cdots-b_d+(d-1)(r+1-1/\theta)}.$$

The lower bound in (5) is established. □

Further, we consider the case $b_1 \le \ldots \le b_\nu < r < b_{\nu+1} \le \ldots \le b_d$. Then, according to Lemma 1, we have

$$M \asymp N^{\frac{1}{r}} \left(\log N\right)^{-b_1/r-\cdots-b_\nu/r+\nu-1}, \quad \log M \asymp \log N, \quad N \asymp M^r \left(\log M\right)^{b_1+\cdots+b_\nu-(\nu-1)r}.$$

The following statement is valid.

Theorem 2. *Let* $1 \le \theta < \infty$ *and* $\Omega(t)$ *be a function of the form* (3). *Then for* $0 < r < l$, $b_j > r+1$, $j = \nu+1, \ldots, d$, *the relation*

$$d^B_M \left(B^{\Omega}_{\infty,\theta}, L_\infty\right) \asymp M^{-r} \left(\log M\right)^{-b_1-\cdots-b_\nu+(\nu-1)(r+1-1/\theta)} \tag{7}$$

holds.

Proof. First, we obtain the upper bound in (7). We approximate the functions $f \in B^{\Omega}_{\infty,\theta}$ by polynomials of the form

$$V_{Q'(N)}(x) = \sum_{s \in \chi'(N)} A_s(f, x),$$

which corresponds to the sets $Q'(N)$, where $1 < b_{\nu+1} - b'_{\nu+1} \leq \ldots \leq b_d - b'_d$.

Using Minkowski's inequality, we have

$$\begin{aligned} \|f - V_{Q'(N)}\|_\infty &= \Big\| \sum_{s \in \mathbb{N}^d \setminus \chi'(N)} A_s(f) \Big\|_\infty \leq \sum_{s \in \mathbb{N}^d \setminus \chi'(N)} \|A_s(f)\|_\infty \\ &= \sum_{s \in \mathbb{N}^d \setminus \chi'(N)} \Omega^{-1}(2^{-s}) \|A_s(f)\|_\infty \Omega(2^{-s}) = I_3. \end{aligned}$$

To estimate I_3, we consider two cases.

Let $1 < \theta < \infty$. Applying Hölder's inequality with exponent θ to I_3 and using Lemma 3, we get

$$\begin{aligned} I_3 &\leq \Bigg(\sum_{s \in \mathbb{N}^d \setminus \chi'(N)} \Omega^{-\theta}\left(2^{-s}\right) \|A_s(f)\|^{\theta}_\infty \Bigg)^{1/\theta} \Bigg(\sum_{s \in \mathbb{N}^d \setminus \chi'(N)} \left(\Omega(2^{-s})\right)^{\theta/(\theta-1)} \Bigg)^{1-1/\theta} \\ &\ll \|f\|_{B^{\Omega}_{\infty,\theta}} \Bigg(\sum_{s \in \mathbb{N}^d \setminus \chi'(N)} \left(\Omega(2^{-s})\right)^{\theta/(\theta-1)} \Bigg)^{1-1/\theta} \ll \Bigg(\sum_{s \in \Theta'(N)} \left(\Omega(2^{-s})\right)^{\theta/(\theta-1)} \Bigg)^{1-1/\theta} = I_4. \end{aligned}$$

Taking into account that

$$\prod_{j=1}^{d} 2^{rs_j} s_j^{b'_j} \asymp N$$

for $s \in \Theta'(N)$, we obtain

$$\Omega(2^{-s}) \asymp N^{-1} \prod_{j=\nu+1}^{d} s_j^{-(b_j - b'_j)}.$$

We use Lemma 4, which is valid for the set $\Theta'(N)$. Putting in it $\gamma_1 = \ldots = \gamma_\nu = 0$, we write

$$\begin{aligned} I_4 &\asymp N^{-1} \Bigg(\sum_{s \in \Theta'(N)} \prod_{j=\nu+1}^{d} s_j^{-\theta(b_j - b'_j)/(\theta-1)} \Bigg)^{1-1/\theta} \\ &\asymp N^{-1} (\log N)^{(\nu-1)(1-1/\theta)} \asymp M^{-r} (\log M)^{-b_1 - \cdots - b_\nu + (\nu-1)r} (\log M)^{(\nu-1)(1-1/\theta)} \\ &= M^{-r} (\log M)^{-b_1 - \cdots - b_\nu + (\nu-1)(r+1-1/\theta)}. \end{aligned}$$

Let $\theta = 1$. Then we have

$$I_3 \ll N^{-1} \sum_{s \in \mathbb{N}^d \setminus \chi'(N)} \Omega^{-1}(2^{-s}) \|A_s(f)\|_\infty \ll N^{-1} \|f\|_{B^{\Omega}_{\infty,1}} \asymp M^{-r} \big(\log M\big)^{-b_1 - \cdots - b_\nu + (\nu-1)r}.$$

Thus, the upper bound in (7) is established.

Let us find the lower bound. Similarly, as in the proof of Theorem 1, we can assert that there exists a set

$$\Theta_1^{(\nu)}(N) = \{s \in \Theta(N) : s_j \asymp \log N,\ j = 1,\ldots,\nu,\ s_j = 1,\ j = \nu+1,\ldots,d\}$$

such that

$$\left|\Theta_1^{(\nu)}(N)\right| \asymp \left(\log N\right)^{\nu-1}.$$

We set

$$g_3(x) = \sum_{s\in\Theta_1^{(\nu)}(N)} \mathcal{K}_s^{(\nu)}(x) \prod_{j=\nu+1}^{d} e^{ix_j},$$

where

$$\mathcal{K}_s^{(\nu)}(x) = \prod_{j=1}^{\nu} e^{ik_j^{s_j} x_j} K_{2^{s_j-2}}(x_j),$$

$$k_j^{s_j} = \begin{cases} 2^{s_j-1} + 2^{s_j-2}, & s_j \ge 2; \\ 1, & s_j = 1,\ j = 1,\ldots,\nu. \end{cases}$$

Consider the function

$$g_4(x) = C_4 N^{-1} \left(N^{1/r}(\log N)^{-b_1/r-\cdots-b_\nu/r}\right)^{-1} (\log N)^{-(\nu-1)/\theta} g_3(x), \quad C_4 > 0.$$

Now, we show that, at a certain choice of the constant C_4, this function belongs to the class $B^{\Omega}_{\infty,\theta}$. Indeed, since

$$\left\|A_s(g_3)\right\|_\infty \ll \left\|K_s^{(\nu)}\right\|_\infty \asymp 2^{\|s\|_1}, \quad s \in \Theta_1^{(\nu)}(N),$$

we have

$$\begin{aligned}
\|g_4\|_{B^{\Omega}_{p,\theta}} &= \left(\sum_s \Omega^{-\theta}(2^{-s})\left\|A_s(g_4)\right\|_p^\theta\right)^{1/\theta} \\
&\ll N^{-1}\left(N^{1/r}(\log N)^{-b_1/r-\cdots-b_\nu/r}\right)^{-1}(\log N)^{-\frac{\nu-1}{\theta}}\left(\sum_{s\in\Theta_1^{(\nu)}(N)} \Omega^{-\theta}(2^{-s})\left\|A_s(g_3)\right\|_p^\theta\right)^{1/\theta} \\
&\ll \left(N^{1/r}(\log N)^{-b_1/r-\cdots-b_\nu/r}\right)^{-1}(\log N)^{-\frac{\nu-1}{\theta}}\left(\sum_{s\in\Theta_1^{(\nu)}(N)} 2^{\|s\|_1\theta}\right)^{1/\theta} = I_5.
\end{aligned}$$

Taking into account that for $s \in \Theta_1^{(\nu)}(N) \subset \Theta(N)$ the following relations

$$2^{\|s\|_1} \asymp N^{1/r}\prod_{j=1}^{\nu} s_j^{-b_j/r}, \qquad s_j \asymp \log N,\ j = 1,\ldots,\nu,\ s_j = 1,\ j = \nu+1,\ldots d,$$

$$\left|\Theta_1^{(\nu)}(N)\right| \asymp (\log N)^{\nu-1},$$

hold, we get

$$\begin{aligned}
I_5 &\asymp \left(N^{1/r}(\log N)^{-b_1/r-\cdots-b_\nu/r}\right)^{-1}(\log N)^{-(\nu-1)/\theta} N^{1/r}(\log N)^{-b_1/r-\cdots-b_\nu/r}\left|\Theta_1^{(\nu)}(N)\right|^{1/\theta} \\
&\asymp (\log N)^{-(\nu-1)/\theta}(\log N)^{(\nu-1)/\theta} = 1.
\end{aligned}$$

Therefore, $g_4 \in B^{\Omega}_{p,\theta}$ with some constant $C_4 > 0$.

In [17], it was established that there exists a vector $y^* = \big(y_1^*, \dots, y_d^*\big)$ such that the following relation

$$\big\|g_3(x-y^*) - Gg_3(x-y^*)\big\|_\infty \gg M \tag{8}$$

holds for $G \in L_M(B)_\infty$. Thus, using estimate (8), we obtain

$$\begin{aligned}
&\big\|g_4(x-y^*) - Gg_4(x-y^*)\big\|_\infty \\
&\quad\gg N^{-1}\big(N^{1/r}(\log N)^{-b_1/r-\dots-b_\nu/r+\nu-1}\big)^{-1}(\log N)^{(\nu-1)(1-1/\theta)}\big\|g_1(x-y^*) - Gg_1(x-y^*)\big\|_\infty \\
&\quad\gg M^{-r}\big(\log M\big)^{-b_1-\dots-b_\nu+(\nu-1)r}M^{-1}(\log M)^{(\nu-1)(1-1/\theta)}M \\
&\quad= M^{-r}(\log M)^{-b_1-\dots-b_\nu+(\nu-1)(r+1-1/\theta)}.
\end{aligned}$$

The lower bound in (7) is established. □

In the following proposition, we consider other relations for the numbers $r, b_1, \dots, b_d$. Let $r \le b_1 \le \ldots \le b_d$, $b_2 > r$. In this case, by Lemma 1, we obtain

$$M \asymp N^{1/r}\big(\log N\big)^{-b_1/r}, \quad \log M \asymp \log N, \quad N \asymp M^r\big(\log M\big)^{b_1}.$$

Theorem 3. *Let $1 \le \theta < \infty$ and $\Omega(t)$ be a function of the form (3). Then for $0 < r < l$, $b_2 > b_1 + 1$ the following relation*

$$d_M^B\left(B^{\Omega}_{\infty,\theta}, L_\infty\right) \asymp M^{-r}\big(\log M\big)^{-b_1} \tag{9}$$

holds.

Proof. For $1 \le \theta < \infty$, the embedding $B^{\Omega}_{\infty,\theta} \subset H^{\Omega}_\infty$ is valid. Therefore, the upper bound in (9) follows from the corresponding estimate of $d_M^B\left(H^{\Omega}_\infty, L_\infty\right)$, proved in [17].

Let us prove the corresponding lower bound. We choose a vector $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_d) \in \Theta(N)$ such that

$$\tilde{s}_1 \asymp \log N, \quad \tilde{s}_2 = \ldots = \tilde{s}_d = 1,$$

and set

$$g_5(x) = \mathcal{K}_{\tilde{s}}(x) = e^{i\left(k^{\tilde{s}},x\right)}K_{2^{\tilde{s}_1-2}}(x_1),$$

where $k^{\tilde{s}} = \big(2^{\tilde{s}_1-1} + 2^{\tilde{s}_1-2}, 1, \dots, 1\big)$.

Consider the function

$$g_6(x) = C_5N^{-1}2^{-\|\tilde{s}\|_1}g_5(x), \quad C_5 > 0.$$

It is easy to see that, at the corresponding choice of the constant $C_5 > 0$, the function g_6 belongs to the class $B^{\Omega}_{\infty,\theta}$.

Indeed, by the property

$$\big\|A_{\tilde{s}}(g_5)\big\|_\infty \ll \|K_{\tilde{s}}\|_\infty \asymp 2^{\|\tilde{s}\|_1}$$

of the Fejér kernel, we have

$$\begin{aligned}
\|g_6\|_{B^{\Omega}_{\infty,\theta}} &= \left(\sum_s \Omega^{-\theta}(2^{-s})\big\|A_s(g_6)\big\|_\infty^\theta\right)^{1/\theta} \\
&\ll N^{-1}2^{-\|\tilde{s}\|_1}\left(\Omega^{-\theta}(2^{-\tilde{s}})\big\|A_{\tilde{s}}(g_5)\big\|_\infty^\theta\right)^{1/\theta} \ll 2^{-\|\tilde{s}\|_1}\big\|A_{\tilde{s}}(g_5)\big\|_\infty \ll 2^{-\|\tilde{s}\|_1}2^{\|\tilde{s}\|_1} = 1.
\end{aligned}$$

Hence, g_6 belongs to $B^{\Omega}_{\infty,\theta}$ with the corresponding constant $C_5 > 0$.

In [17], it was shown that there exists a vector $y^* = \left(y_1^*, \dots, y_d^*\right)$ such that the following relation

$$\left\|g_5(x-y^*) - Gg_5(x-y^*)\right\|_\infty \gg M \tag{10}$$

holds for $G \in L_M(B)_\infty$.

Taking into account that

$$2^{\|\tilde{s}\|_1} \asymp N^{\frac{1}{r}} (\log N)^{-b_1/r},$$

and using estimate (10), we get

$$\begin{aligned}\left\|g_6(x-y^*) - Gg_6(x-y^*)\right\|_\infty &\gg N^{-1} 2^{-\|\tilde{s}\|_1} \left\|g_5(x-y^*) - Gg_5(x-y^*)\right\|_\infty \\ &\gg M^{-r}\left(\log M\right)^{-b_1} M^{-1} M = M^{-r}\left(\log M\right)^{-b_1}.\end{aligned}$$

The lower bound in (9) is established. □

At the end of the work, we make two important remarks.

Remark 1. *In Theorem 1, the order of $d^B_M\left(B^{\Omega}_{\infty,\theta}, L_\infty\right)$ is realized by using trigonometric polynomials with "number" harmonics from $Q(N)$, while in Theorems 2 and 3, from $Q'(N)$.*

Remark 2. *The analogues of Theorems 1 – 3 for the classes H^{Ω}_∞ are obtained in [17]. Moreover, if the conditions of Theorem 3 are fulfilled, the following relation*

$$d^B_M\left(B^{\Omega}_{\infty,\theta}, L_\infty\right) \asymp d^B_M\left(H^{\Omega}_\infty, L_\infty\right)$$

holds. In other words, estimate of the quantity $d^B_M\left(B^{\Omega}_{\infty,\theta}, L_\infty\right)$ does not depend on parameter θ.

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Встановлено точні за порядком оцінки наближення класів типу Нікольського-Бєсова $B^{\Omega}_{\infty,\theta}$ періодичних функцій багатьох змінних із заданою функцією $\Omega(t)$ спеціального вигляду за допомогою лінійних операторів, які підпорядковані певним умовам. Похибка наближення оцінюється в метриці простору L_∞. Одержані оцінки розглянутої апроксимаційної характеристики, окрім самостійного інтересу, можуть бути використані для встановлення оцінок знизу ортопоперечників відповідних функціональних класів.

Ключові слова і фрази: мішаний модуль неперервності, умова Барі-Стєчкіна, клас типу Нікольського-Бєсова, лінійний оператор, ядро Валле Пуссена, ядро Фейєра.