ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2023, **15** (2), 559–562
doi:10.15330/cmp.15.2.559-562

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2023, Т.15, №2, С.559–562

Derivations of Mackey algebras

Bezushchak O.

We describe derivations of finitary Mackey algebras over fields of characteristics not equal to 2. We prove that an arbitrary derivation of an associative finitary Mackey algebra or one of the Lie algebras $\mathfrak{sl}_\infty(V|W)$, $\mathfrak{o}_\infty(f)$ is an adjoint operator of an element in the corresponding Mackey algebra. It provides a description of the derivations of all algebras in the Baranov-Strade classification of finitary simple Lie algebras. The proof is based on N. Jacobson's result on derivations of associative algebras of linear transformations of an infinite-dimensional vector space and the results on Herstein's conjectures.

Key words and phrases: derivation, Mackey algebra.

Taras Shevchenko National University of Kyiv, Volodymyrska 60, Kyiv 01033, Ukraine
E-mail: bezushchak@knu.ua

Introduction

Let V be an infinite-dimensional vector space over a field $\mathbb{F}$, and let V^* denote the vector space of all linear functionals on V. For an element $v \in V$ and a linear functional $w \in V^*$, we denote $w(v) = (v|w)$. A subspace $W \subset V^*$ is called *total* if for $v \in V$ the equality $(v|W) = 0$ implies $v = 0$.

Define $\mathrm{End}_{\mathbb{F}}(V)$ as the associative algebra of all linear transformations $V \to V$, and let

$$\mathrm{End}_{fin}(V) = \{\varphi \in \mathrm{End}_{\mathbb{F}}(V) : \dim_{\mathbb{F}} \varphi(V) < \infty\}.$$

The space V^* has a natural structure of a right $\mathrm{End}_{\mathbb{F}}(V)$-module. Given a linear functional $\chi : V \to \mathbb{F}$ and a linear transformation $\varphi : V \to V$, we define $(\chi\varphi)(v) = \chi(\varphi(v))$, $v \in V$.

The subalgebra

$$A\,(V|W) = \{\varphi \in \mathrm{End}_{\mathbb{F}}(V) : W\varphi \subseteq W\}$$

and the subalgebra

$$A_{fin}\,(V|W) = A\,(V|W) \cap \mathrm{End}_{fin}(V)$$

are called the *Mackey algebra* and the *finitary Mackey algebra*, respectively.

Clearly, $A_{fin}\,(V|W)$ is an ideal of the algebra $A\,(V|W)$. For more information about Mackey algebras, see [11, 12, 14, 15].

The algebra $A_{fin}\,(V|W)$ can be identified with the tensor product $V \otimes_{\mathbb{F}} W$ by

$$(v_1 \otimes w_1)\,(v_2 \otimes w_2) = (v_2|w_1)\,v_1 \otimes w_2.$$

An element $\sum_i v_i \otimes w_i$ we treat as the linear transformation

$$v \to \sum_i (v|w_i)v_i, \quad v \in V.$$

УДК 512.554, 512.554.37, 512.552.16
2020 *Mathematics Subject Classification:* 17A36, 16W25.

The algebra $A_{fin}(V|W)$ of a total subspace $W \subset V^*$ is a nonunital locally matrix algebra (see [6–9, 13]).

The algebras $A(V|W)$, $A_{fin}(V|W)$ also give rise to Lie algebras

$$\mathfrak{gl}(V|W) = \big(A(V|W), \quad [a,b] = ab - ba\big)$$

and

$$\mathfrak{gl}_\infty(V|W) = \big(A_{fin}(V|W), \quad [a,b] = ab - ba\big).$$

Let

$$\mathfrak{sl}_\infty(V|W) = \big[\mathfrak{gl}_\infty(V|W), \mathfrak{gl}_\infty(V|W)\big].$$

The structure of these Lie algebras was studied in [5].

A bijective linear transformation (respectively additive map) φ of an associative algebra (respectively ring) A is called an *anti-automorphism* if $\varphi(ab) = \varphi(b)\varphi(a)$ for all elements $a, b \in A$. An anti-automorphism $*$ is called an *involution* if $(a^*)^* = a$ for all elements $a \in A$.

N. Jacobson [12] showed that a finitary Mackey algebra $A_{fin}(V|W)$ has an involution if and only if the vector space V is equipped with a weakly Hermitian (see [12]) non-degenerate bilinear form $f(x,y)$ such that W is the space of linear functionals $\widehat{x} : v \to f(v,x)$, $v \in A$, where x runs over V. In this case, we call the pair (V,W) *dual*, and the anti-automorphism is the transpose

$$\varphi \to \varphi^t, \quad f\big(\varphi(v_1), v_2\big) = f\big(v_1, \varphi^t(v_2)\big).$$

Moreover, a linear transformation $\varphi \in \mathrm{End}_{\mathbb{F}}(V)$ has a transpose if and only if it lies in $A(V|W)$. The anti-automorphism t is an involution if and only if the form f is symmetric or skew-symmetric.

Let the characteristic of the field $\mathbb{F}$ be different from 2, and let

$$\mathfrak{o}(f) = \big\{\, a \in A(V|W) : a^t = -a \big\}$$

be the Lie algebra of skew-symmetric linear transformations. Let

$$\mathfrak{o}_\infty(f) = \mathfrak{o}(f) \cap \mathrm{End}_{fin}(V).$$

A. Baranov and H. Strade [1] obtained the classification of infinite-dimensional simple finitary Lie algebras over an algebraically closed field of characteristic not equal to 2 nor 3. An algebra belongs to this class if and only if it is isomorphic to one of the Mackey algebras $\mathfrak{sl}_\infty(V|W)$, $\mathfrak{o}_\infty(f)$, where f is a non-degenerate symmetric or skew-symmetric form. The proof used the classification of simple modular Lie algebras (see [16]), which explains the restriction on characteristics.

Recall that a linear map $d : A \to A$ is called a *derivation* if

$$d(xy) = d(x)y + xd(y)$$

for arbitrary elements x and y from A.

For an element $a \in A$, the adjoint operator

$$\mathrm{ad}(a) : A \to A, \quad x \mapsto [a,x],$$

is an *inner derivation* of the algebra A.

The following theorem describes derivations of finitary Mackey algebras. In particular, for characteristics not equal to 2 nor 3, it describes derivations of all infinite-dimensional simple finitary Lie algebras.

Theorem 1. (a) *An arbitrary derivation of the associative algebra* $A_{fin}(V|W)$ *is an adjoint operator* $\mathrm{ad}(a)$*, where* $a \in A\,(V|W)$.

(b) *Let* $\mathrm{char}\,\mathbb{F} \neq 2$*. Then an arbitrary derivation of the Lie algebra* $\mathfrak{sl}_\infty\,(V|W)$ *is an adjoint operator* $\mathrm{ad}(a)$*, where* $a \in A\,(V|W)$.

(c) *Let* $\mathrm{char}\,\mathbb{F} \neq 2$*. Then an arbitrary derivation of the Lie algebra* $\mathfrak{o}_\infty(f)$ *is an adjoint operator* $\mathrm{ad}(a)$*, where* $a \in A\,(V|W)$ *and* $a^t = -a$.

Remark. *In the special case when both spaces* V *and* W *are countable-dimensional, the algebras* $A\,(V|W)$ *and* $A_{fin}\,(V|W)$ *are isomorphic to the algebra* $M_{rcf}(\mathbb{F})$ *of countable matrices having finitely many nonzero entries in each row and each column, and to the algebra* $M_\infty(\mathbb{F})$ *of countable matrices with finitely many nonzero entries, respectively. Derivations of these algebras were described in [8].*

Proof. N. Jacobson showed [11] that if A is a subalgebra of the algebra $\mathrm{End}_{\mathbb{F}}(V)$ such that A acts irreducibly on V, and A contains a nonzero linear transformation of finite range, then for an arbitrary derivation d of the algebra A there exists an element $\varphi \in \mathrm{End}_{\mathbb{F}}(V)$ such that $d(a) = [\varphi, a]$ for every $a \in A$.

Let us show that the algebra $A_{fin}\,(V|W)$ acts on V irreducibly. Choose elements $v_1, v_2 \in V$, $v_1 \neq 0$. Since the subspace $W \subset V^*$ is total, there exists a linear functional $w \in W$ such that $(v_1|w) \neq 0$. The linear transformation $v_2 \otimes w \in A_{fin}\,(V|W)$ maps the element v_1 to $(v_1|w)\,v_2$.

All elements from $A_{fin}\,(V|W)$ have finite ranges. Hence, by N. Jacobson's theorem, there exists a linear transformation $\varphi \in \mathrm{End}_{\mathbb{F}}(V)$ such that $[\varphi, a] = d(a)$ for every element $a \in A_{fin}\,(V|W)$. Let us show that $\varphi \in A\,(V|W)$. For arbitrary elements $v \in V$, $w \in W$, we have $[v \otimes w, \varphi] = v \otimes w\,\varphi - \varphi(v) \otimes w \in V \otimes W$, which implies that $w\varphi \in W$, $\varphi \in A\,(V|W)$. This completes the proof of the assertion (a).

The assertion (b) immediately follows from the proof of Herstein's conjectures by K.I. Beidar et al. (see [2–4]).

Now, let d be a derivation of a Lie algebra $\mathfrak{o}_\infty(f)$. Again, by the result of K.I. Beidar et al. [2–4], the derivation d extends to a derivation of $A_{fin}\,(V|W)$. Hence, by (a), there exists an element $\varphi \in A\,(V|W)$ such that $d(k) = [\varphi, k]$ for an arbitrary element k from $\mathfrak{o}_\infty(f)$. We have $[\varphi, k] \in \mathfrak{o}_\infty(f)$. Hence, $[\varphi, k]^t = [k^t, \varphi^t] = [\varphi^t, k]$, and, on the other hand, $[\varphi, k]^t = -[\varphi, k]$. This implies $[\varphi^t + \varphi, k] = 0$.

The subspace $\mathfrak{o}_\infty(f)$ generates $A_{fin}\,(V|W)$ as an associative algebra (see [10]). Hence,

$$[\varphi^t + \varphi, A_{fin}\,(V|W)] = (0).$$

By Schur's lemma (see [11]), it implies that $\varphi^t + \varphi$ is a scalar multiplication, namely

$$\varphi^t + \varphi = \alpha \cdot \mathrm{Id}_V, \quad \alpha \in \mathbb{F},$$

where Id_V is the identity transformation of V.

Now,

$$\varphi = \frac{1}{2}\,(\varphi - \varphi^t) + \frac{1}{2}\,(\varphi + \varphi^t) = \frac{1}{2}\,(\varphi - \varphi^t) + \frac{1}{2}\,\alpha \cdot \mathrm{Id}_V.$$

The element

$$a = \frac{1}{2}\,(\varphi - \varphi^t)$$

lies in $A\,(V|W)$, $a^t = -a$, and the derivation d is the restriction of $\mathrm{ad}(a)$ to $\mathfrak{o}_\infty(f)$.

This completes the proof of the theorem. □

References

[1] Baranov A.A., Strade H. *Finitary Lie algebras*. J. Algebra 2002, **254** (1), 173–211. doi:10.1016/S0021-8693(02)00079-0

[2] Beidar K.I., Brešar M., Chebotar M.A., Martindale W.S. *On Herstein's Lie map conjectures, I*. Trans. Amer. Math. Soc. 2001, **353** (10), 4235–4260. doi:10.1090/S0002-9947-01-02731-3

[3] Beidar K.I., Brešar M., Chebotar M.A., Martindale W.S. *On Herstein's Lie map conjectures, II*. J. Algebra 2001, **238** (1), 239–264. doi:10.1006/jabr.2000.8628

[4] Beidar K.I., Brešar M., Chebotar M.A., Martindale W.S. *On Herstein's Lie map conjectures, III*. J. Algebra 2002, **249** (1), 59–94. doi:10.1006/jabr.2001.9076

[5] Bezushchak O. *On the Lie structure of locally matrix algebras*. Carpathian Math. Publ. 2020, **12** (2), 311–316. doi:10.15330/cmp.12.2.311-316

[6] Bezushchak O. *Derivations and automorphisms of locally matrix algebras*. J. Algebra 2021, **576**, 1–26. doi: 10.1016/j.jalgebra.2021.02.013

[7] Bezushchak O. *Spectra of locally matrix algebras*. Algebra Discrete Math. 2021, **31** (1), 17–36. doi: 10.12958/adm1734

[8] Bezushchak O. *Automorphisms and derivations of algebras of infinite matrices*. Linear Algebra Appl. 2022, **650** (1), 42–59. doi:10.1016/j.laa.2022.05.020

[9] Bezushchak O., Oliynyk B. *Primary decompositions of unital locally matrix algebras*. Bull. Math. Sci. 2020, **10** (1), 2050006. doi:10.1142/S166436072050006X

[10] Herstein I.N. Rings with involution. Univ. of Chicago Press, Chicago, 1976.

[11] Jacobson N. Structure of rings. Am. Math. Soc. Colloq. Publ. **37**, 1956.

[12] Jacobson N. Lectures in abstract algebra. Volume 2. Linear algebra, Springer-Verlag, 1975.

[13] Kurosh A. *Direct decompositions of simple rings*. Rec. Math. 1942, **11**, 245–264.

[14] Mackey G. *On infinite dimensional linear spaces*. Proc. Natl. Acad. Sci. USA 1943, **29** (7), 216–221. doi: 10.1073/pnas.29.7.216

[15] Penkov I., Serganova V. *Tensor representations of Mackey Lie algebras and their dense subalgebras*. Dev. Math. 2014, **38**, 291–330.

[16] Strade H. *Locally finite dimensional Lie algebras and their derivation algebras*. Abh. Math. Semin. Univ. Hambg. 1999, **69** (1), 373–391. doi:10.1007/BF02940886

Received 25.08.2023

Безущак О. *Диференціювання алгебр Макі* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 559–562.

Ми описуємо диференціювання фінітарних алгебр Макі над полями характеристики, яка не дорівнює 2. Доводимо, що довільне диференціювання асоціативної фінітарної алгебри Макі або однієї з алгебр Лі $\mathfrak{sl}_\infty(V|W)$, $\mathfrak{o}_\infty(f)$ є приєднаним оператором за допомогою деякого елемента з відповідної алгебри Макі. Тим самим, отримуємо опис диференціювань усіх алгебр з класифікації фінітарних простих алгебр Лі за Барановим-Штраде. Доведення базується на результатах Джекобсона про диференціювання асоціативних алгебр лінійних перетворень нескінченно вимірного векторного простору і результатів щодо гіпотез Херстейна.

Ключові слова і фрази: диференціювання, алгебра Макі.