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Weakly symmetric functions on spaces of Lebesgue integrable functions

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In this work, we present the notion of a weakly symmetric function. We show that the subset of all weakly symmetric elements of an arbitrary vector space of functions is a vector space. Moreover, the subset of all weakly symmetric elements of some algebra of functions is an algebra. Also we consider weakly symmetric functions on the complex Banach space $L_p[0,1]$ of all Lebesgue measurable complex-valued functions on $[0,1]$ for which the pth power of the absolute value is Lebesgue integrable. We show that every continuous linear functional on $L_p[0,1]$, where $p \in (1,+\infty)$, can be approximated by weakly symmetric continuous linear functionals.

Key words and phrases: symmetric function, weakly symmetric function, holomorphic function on an infinite dimensional space, spaces of Lebesgue integrable functions.

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Introduction

Symmetric polynomials on spaces ℓ_p and $L_p[0,1]$, where $p \in [1,+\infty)$, firstly were studied in [14]. Some of the results of [14] were generalized to sequence spaces with symmetric bases and to rearrangement invariant function spaces in [10]. Algebras of symmetric polynomials and symmetric holomorphic functions on sequence spaces were studied in [1,3–6,8,11–13,20]. Algebras of symmetric polynomials and symmetric holomorphic functions on function spaces were investigated in [7–9, 15–19, 21–23]. The most general approach to the investigation of symmetric functions was proposed in [2].

The theory of symmetric holomorphic functions has many advantages over the general theory of holomorphic functions on infinite dimensional spaces. Unfortunately, most of the methods of this theory cannot be applied for investigations of nonsymmetic holomorphic functions. In most cases, nonsymmetric holomorphic functions cannot be approximated by symmetric functions. For example, most of continuous linear functionals on $L_p[0,1]$ are not symmetric and cannot be approximated by symmetric continuous linear functionals. More precisely, only continuous linear functionals of the form $x \in L_p[0,1] \mapsto c \int_{[0,1]} x(t)\,dt$, where c is a constant, are symmetric on $L_p[0,1]$, where $p \in [1,+\infty)$. In this work we present the idea of approximation of nonsymmetric holomorphic functions by the so-called weakly symmetric functions. It particular, we show that every continuous linear functional on the complex space $L_p[0,1]$, where $p \in (1,+\infty)$, can be approximated by weakly symmetric continuous linear function-

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als on $L_p[0,1]$. As a consequence, the Fréchet algebra of all holomorphic functions which can be approximated by weakly symmetric holomorphic functions of bounded type on $L_p[0,1]$, where $p \in [1,+\infty)$, contains the Fréchet algebra generated by all continuous linear functionals on $L_p[0,1]$.

1 Preliminaries

1.1 Symmetric and weakly symmetric functions

Let X and Y be nonempty sets. Let F be a set of mappings which act from X to itself. A function $f : X \to Y$ is called F-symmetric if $f(a(x)) = f(x)$ for every $a \in F$ and $x \in X$.

Let

$$\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\} \tag{1}$$

be a family of sets F_α of mappings which act from X to itself, indexed by elements of some index set Λ, such that for every $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $F_\gamma \subset F_\alpha \cap F_\beta$. A function $f : X \to Y$ is called $\mathcal{F}$-weakly symmetric if there exists $\alpha \in \Lambda$ such that f is F_α-symmetric.

1.2 Weakly symmetric functions on $L_p[0,1]$

Let $L_p[0,1]$, where $p \in [1,+\infty)$, be the complex Banach space of all Lebesgue measurable functions $x : [0,1] \to \mathbb{C}$ for which the pth power of the absolute value is Lebesgue integrable with norm

$$\|x\|_p = \Big(\int_{[0,1]} |x(t)|^p\, dt\Big)^{1/p}.$$

Let $n \in \mathbb{N}$. Let $\Xi^{(n)}_{[0,1]}$ be the set of all bijections $\sigma : [0,1] \to [0,1]$ such that

$$\sigma(t + 1/n) = \sigma(t) + 1/n$$

for every $t \in [0, 1 - 1/n]$ and, for every Lebesgue measurable set $E \subset [0,1]$, both sets $\sigma(E)$ and $\sigma^{-1}(E)$ are Lebesgue measurable and

$$\mu\big(\sigma(E)\big) = \mu\big(\sigma^{-1}(E)\big) = \mu(E),$$

where μ is the Lebesgue measure.

Let $p \in [1,+\infty)$. For $\sigma \in \Xi^{(n)}_{[0,1]}$, let the operator $s_{\sigma,p}$ be defined by

$$s_{\sigma,p} : x \in L_p[0,1] \mapsto x \circ \sigma \in L_p[0,1].$$

Let

$$S_{n,p} = \big\{ s_{\sigma,p} : \ \sigma \in \Xi^{(n)}_{[0,1]} \big\}$$

and

$$\mathcal{S}_p = \big\{ S_{2^n,p} : \ n \in \mathbb{N} \big\}.$$

We shall consider $\mathcal{S}_p$-weakly symmetric functions on $L_p[0,1]$.

2 The main result

Theorem 1. *Let X be an arbitrary nonempty set. Let Y be a vector space over some field $\mathbb{K}$. Let V be a vector space over $\mathbb{K}$ of some mappings which act from X to Y with respect to pointwise operations of addition and multiplication by elements of $\mathbb{K}$. Let a family $\mathcal{F}$ be of the form* (1). *Let $V_{\mathcal{F}}$ be the set of all $\mathcal{F}$-weakly symmetric elements of V. Then $V_{\mathcal{F}}$ is a vector subspace of V. Moreover, if Y is an algebra and V is an algebra with respect to pointwise operations, then $V_{\mathcal{F}}$ is a subalgebra of V.*

Proof. Let $f \in V_{\mathcal{F}}$ and $\lambda \in \mathbb{K}$. Let us show that $\lambda f \in V_{\mathcal{F}}$. Since $V_{\mathcal{F}} \subset V$, it follows that $f \in V$. Consequently, since V is a vector space, $\lambda f \in V$. Since $f \in V_{\mathcal{F}}$, it follows that f is $\mathcal{F}$-weakly symmetric, i.e. there exists $\alpha \in \Lambda$ such that f is F_α-symmetric. Therefore λf is F_α-symmetric. Consequently, λf is $\mathcal{F}$-weakly symmetric. Thus, $\lambda f \in V_{\mathcal{F}}$.

Let $f, g \in V_{\mathcal{F}}$. Let us show that $f + g \in V_{\mathcal{F}}$. Since $V_{\mathcal{F}} \subset V$, it follows that $f, g \in V$. Consequently, since V is a vector space, $f + g \in V$. Let us show that $f + g$ is $\mathcal{F}$-weakly symmetric. Since f and g are $\mathcal{F}$-weakly symmetric, it follows that there exist $\alpha, \beta \in \Lambda$ such that f is F_α-symmetric and g is F_β-symmetric. By the definition of $\mathcal{F}$, there exists $\gamma \in \Lambda$ such that $F_\gamma \subset F_\alpha \cap F_\beta$. Consequently, both f and g are F_γ-symmetric. Therefore $f + g$ is F_γ-symmetric. Thus, $f + g$ is $\mathcal{F}$-weakly symmetric. Consequently, $f + g \in V_{\mathcal{F}}$. Hence, $V_{\mathcal{F}}$ is a vector subspace of V.

Suppose Y is an algebra and V is an algebra with respect to pointwise operations. As we have shown, $V_{\mathcal{F}}$ is a vector subspace of V. Note that the proof of the fact that $fg \in V_{\mathcal{F}}$ for every $f, g \in V_{\mathcal{F}}$ is analogical to the proof of the fact that $f + g \in V_{\mathcal{F}}$ for every $f, g \in V_{\mathcal{F}}$. Thus, $V_{\mathcal{F}}$ is a subalgebra of V. ☐

Let $H_b\left(L_p[0,1]\right)$ be the Fréchet algebra of all entire functions $f : L_p[0,1] \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets. Let $H^{(0)}_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ be the set of all $\mathcal{S}_p$-weakly symmetric elements of $H_b\left(L_p[0,1]\right)$. By Theorem 1, $H^{(0)}_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ is a subalgebra of $H_b\left(L_p[0,1]\right)$. Let $H_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ be the closure of $H^{(0)}_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ in $H_b\left(L_p[0,1]\right)$. Note that $H_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ is a Fréchet algebra.

Theorem 2. *For every $p \in (1, +\infty)$, the algebra $H_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ contains all continuous linear functionals on $L_p[0,1]$.*

Proof. Let f be a continuous linear functional on $L_p[0,1]$, where $p \in (1, +\infty)$. Let us show that $f \in H_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$. It is well known that the dual space to $L_p[0,1]$ is isometrically isomorphic to $L_q[0,1]$, where $q = p/(p-1)$. Let $g \in L_q[0,1]$ be the image of f under this isomorphism. Then

$$f(x) = \int_{[0,1]} x(t)g(t)\,dt$$

for every $x \in L_p[0,1]$. Note that the subspace

$$\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{2^n} a_j \mathbf{1}_{\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]} : a_1, \ldots, a_{2^n} \in \mathbb{C} \right\},$$

where $\mathbf{1}_{[a,b]}$ is the characteristic function of $[a,b]$, is dense in $L_q[0,1]$. Let $a_{n,j} \in \mathbb{C}$, where $n \in \mathbb{N}$ and $j \in \{1,\ldots,2^n\}$, be such that the sequence of functions

$$g_n = \sum_{j=1}^{2^n} a_{n,j}\mathbf{1}_{\left[\frac{j-1}{2^n},\frac{j}{2^n}\right]}$$

converges to g as $n \to \infty$ in $L_q[0,1]$. It can be verified that, for every $n \in \mathbb{N}$, the continuous linear functional $f_n : L_p[0,1] \to \mathbb{C}$, defined by

$$f_n : x \in L_p[0,1] \mapsto \int_{[0,1]} x(t)g_n(t)\,dt,$$

is $S_{2^n,p}$-symmetric and, consequently, it is $\mathcal{S}_p$-weakly symmetric. Thus, $f_n \in H^{(0)}_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} g_n = g$ in $L_q[0,1]$, it follows that $\lim_{n\to\infty} f_n = f$ in the strong topology of the dual space $\left(L_p[0,1]\right)'$. Since the strong topology of $\left(L_p[0,1]\right)'$ is the restriction of the topology of $H_b\left(L_p[0,1]\right)$, it follows that $\lim_{n\to\infty} f_n = f$ in $H_b\left(L_p[0,1]\right)$. Consequently, taking into account that $f_n \in H^{(0)}_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ for every $n \in \mathbb{N}$ and $H_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ is the closure of $H^{(0)}_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$, we have that $f \in H_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$. This completes the proof. □

Let $A_1(L_p[0,1])$ be the closure in $H_b\left(L_p[0,1]\right)$ of the algebra of all polynomials of finite type, i.e. polynomials which can be represented as linear combinations of products of continuous linear functionals on $L_p[0,1]$. Theorem 2 implies the following corollary.

Corollary 1. *For every $p \in (1,+\infty)$, the algebra $H_{b,\mathcal{S}_p-w.s.}\left(L_p[0,1]\right)$ contains the algebra $A_1(L_p[0,1])$.*

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Василишин Т.В., Загороднюк В.А. *Слабко симетричні функції на просторах інтегровних за Лебеґом функцій* // Карпатські матем. публ. — 2022. — Т.14, №2. — C. 437–441.

В даній роботі представлено поняття слабко симетричної функції. Показано, що підмножина всіх слабко симетричних елементів довільного векторного простору функцій сама є векторним простором. Більше того, підмножина всіх слабко симетричних елементів деякої алгебри функцій є алгеброю. Також розглянуто слабко симетричні функції на комплексному банаховому просторі $L_p[0,1]$ всіх вимірних за Лебеґом комплекснозначних функцій на відрізку $[0,1]$, для яких p-тий степінь абсолютного значення є інтегровним за Лебеґом. Показано, що кожен неперервний лінійний функціонал на просторі $L_p[0,1]$, де $p \in (1,+\infty)$, можна наблизити слабко симетричними неперервними лінійними функціоналами.

Ключові слова і фрази: симетрична функція, слабко симетрична функція, аналітична функція на нескінченновимірному просторі, простори інтегровних за Лебеґом функцій.