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Generalized selfadjointness of operators generated by Jacobi Hermitian matrices

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We investigate selfadjointness in sense of Hilbert space rigging and related questions. We proved that this generalized selfadjointness of some operator, which acts from positive into negative space, is equivalent to ordinary selfadjointness of some modification of this operator in basic ("zero") space.

Also we consider operators generated by classical and generalized Jacobi Hermitian matrices, their selfadjointness and generalized selfadjointness in sense of weight Hilbert space rigging. Some sufficient conditions of generalized selfadjointness of these operators are proved. Using obtained results we explain possibility of construction of example of gereralized selfadjoint opearator which is not selfadjoint in classical sence.

Key words and phrases: Hilbert space rigging, Hermitian operator, selfadjoint operator, block three-diagonal matrix, Jacobi matrix.

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1 Introduction

This article appears from one problem, which was described years ago by Yu.M. Berezansky. Investigating this problem we obtained results which are connected to notion of Hilbert space rigging. We consider operators acting in this chain and their selfadjointness in the sense of this construction, which we call generalized selfadjointness. Therefore, the main goal of this article is to explain more clearly the notion of generalized selfadjointness.

In the Section 2, we consider connection between generalized and ordinary selfadjointness. There we prove that instead of investigation of generalized selfadjointness of some operator it is sufficient to investigate ordinary selfadjointness of some new operator, which is a modification of given one.

In the Section 3, we consider operators generated by classical Jacobi matrix in space ℓ_2 of squared summable sequences and in weight Hilbert rigging of ℓ_2. Here we transfer results of Section 2 on this partial case. Also we obtain a few interesting results about: sufficient conditions of nonselfadjointness of operator generated by classical Jacobi matrix in ℓ_2, sufficient conditions of generalized selfadjointness of operator generated by classical Jacobi matrix in weight Hilbert rigging of ℓ_2 and connection between generalized and ordinary selfadjointness of these operators.

The Section 4 is dedicated to an example, which was considered in article [3]. It was the example of operator which is selfadjoint in ordinary sense but in the same time it is not gener-

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alized selfadjoint. As it was later noticed by Yu. M. Berezanskiy that example was constructed with a mistake. In this section, we show that such example can not be constructed or, in other words, such situation is not possible.

In the Section 5, we consider operator generated by generalized Jacobi matrix, i.e. three-diagonal block Jacobi type matrix. As it provided to be, in this case similar situation to described one in Section 3 takes place.

2 Connection between ordinary and generalized selfadjointness

Let us consider complex Hilbert space rigging

$$\mathcal{H}_- \supset \mathcal{H}_0 \supset \mathcal{H}_+. \tag{1}$$

The construction of this space was described in [2]. We will not repeat all the procedure of construction. But we just admit that $\mathcal{H}_+$ is dense in $\mathcal{H}_0$ as subspace and $\|u\|_{H_0} \le \|u\|_{H_+}$, $u \in H_+$.

Let $\mathbf{J}, J, \mathbf{I}$ be some isometric operators, which are constructed in specific way with respect to (1). In particular, $\mathbf{I} \upharpoonright \mathcal{H}_0$ is the adjoint operator to the operator of embedding of $\mathcal{H}_+ \to \mathcal{H}_0$. For these operators following equalities take place:

$$\begin{aligned}
&\mathbf{J} : \mathcal{H}_- \to \mathcal{H}_0, \quad D(\mathbf{J}) = \mathcal{H}_-, R(\mathbf{J}) = \mathcal{H}_0;\\
&J : \mathcal{H}_0 \to \mathcal{H}_+, \quad D(J) = \mathcal{H}_0, R(J) = \mathcal{H}_+;\\
&\mathbf{I} : \mathcal{H}_- \to \mathcal{H}_+, \quad D(\mathbf{I}) = \mathcal{H}_-, R(\mathbf{I}) = \mathcal{H}_+;\\
&\mathbf{I} = J\mathbf{J}.
\end{aligned} \tag{2}$$

Let us consider an operator $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ with a dense domain $D(\mathbf{A})$.

For $\mathbf{A}$ it is easy to define an adjoint operator $\mathbf{A}^+ : \mathcal{H}_+ \to \mathcal{H}_-$ [3]. Let $\psi \in \mathcal{H}_+$ be such that the functional $\varphi \to (\mathbf{A}\varphi, \psi)_{\mathcal{H}_0} \in \mathbb{C}$, which is defined on $D(\mathbf{A})$, is continuous and, therefore, has a representation $(\mathbf{A}\varphi, \psi)_{\mathcal{H}_0} = (\varphi, \psi^+)_{\mathcal{H}_0}, \psi^+ \in \mathcal{H}_-$. Then such ψ is an element of the domain $D(\mathbf{A}^+)$ of an operator $\mathbf{A}^+$ and $\mathbf{A}^+\psi := \psi^+$. If $\mathcal{H}_+ = \mathcal{H}_0$, then this is classical definition of adjoint operator.

Definition 1. *Operator* $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ *is called generalized Hermitian, i.e. Hermitian in sense of rigging* (1)*, if* $(\mathbf{A}u, v)_{\mathcal{H}_0} = (u, \mathbf{A}v)_{\mathcal{H}_0}$, $u, v \in D(\mathbf{A})$.

Definition 2. *Operator* $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ *is called generalized selfadjoint, i.e. selfadjoint in sense of rigging* (1)*, if* $\mathbf{A} = \mathbf{A}^+$.

In paper [3] it was proved the following result.

Proposition 1. *Operator* $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ *is generalized selfadjoint if and only if the operator* $\mathbf{IA} : \mathcal{H}_+ \to \mathcal{H}_+$ *is selfadjoint, i.e. if* $\mathbf{IA}$ *is selfadjoint in a classical sense as an operator in* $\mathcal{H}_+$.

But it is more convenient to check a selfadjointness in space $\mathcal{H}_0$. So, we prove the following theorem.

Theorem 1. *Operator* $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ *is generalized selfadjoint if and only if the operator* $\mathbf{J}\mathbf{A}J : \mathcal{H}_0 \to \mathcal{H}_0$ *is selfadjoint.*

Proof. Necessity. Let $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ be a generalized selfadjoint operator. Let us prove that the operator $\mathbf{J}\mathbf{A}J$ is Hermitian. Since $\mathbf{J}^+ = J$ (see [2]), we get

$$(\mathbf{J}\mathbf{A}Jf, g)_{\mathcal{H}_0} = (\mathbf{A}Jf, Jg)_{\mathcal{H}_0} = (Jf, \mathbf{A}Jg)_{\mathcal{H}_0} = \overline{(\mathbf{A}Jg, Jf)}_{\mathcal{H}_0}$$
$$= \overline{(\mathbf{J}\mathbf{A}Jg, f)}_{\mathcal{H}_0} = (f, \mathbf{J}\mathbf{A}Jg)_{\mathcal{H}_0}, \qquad f, g \in D(\mathbf{J}\mathbf{A}J).$$

Thus, for selfadjointness of $\mathbf{J}\mathbf{A}J$ it is sufficient to show that defect numbers of this operator are $(0,0)$, or, what is the same, a defect subspace is equal to $\{0\}$.

Since $\mathbf{A}$ is generalized selfadjoint operator and Proposition 1 holds true, for any fixed $z \in \mathbb{C} \setminus \mathbb{R}$ and for all $u \in D(\mathbf{A})$ from $\big((\mathbf{IA} - z\mathbf{1})u, v\big)_{\mathcal{H}_+} = 0$ it follows that $v = 0$, i.e. defect subspace of $\mathbf{IA}$ consist of $\{0\}$, where $v \in \mathcal{H}_+$ and $\mathbf{1}$ is identity operator.

For some $z \in \mathbb{C} \setminus \mathbb{R}$, $g \in \mathcal{H}_0$ and for all $f \in D(\mathbf{J}\mathbf{A}J)$ we have

$$0 = \big((\mathbf{J}\mathbf{A}J - z\mathbf{1})f, g\big)_{\mathcal{H}_0} = \Big(J^{-1}J(\mathbf{J}\mathbf{A}J - z\mathbf{1})f, J^{-1}Jg\Big)_{\mathcal{H}_0}$$
$$= \big(J(\mathbf{J}\mathbf{A}J - z\mathbf{1})f, Jg\big)_{\mathcal{H}_+} = \big((\mathbf{IA} - z\mathbf{1})Jf, Jg\big)_{\mathcal{H}_+}.$$

Since $Jf \in D(\mathbf{A})$ and $Jg \in \mathcal{H}_+$, from above mentioned and the last equality it follows that $Jg = 0$. Since operator J is isometric, then $g = 0$. So, the defect subspace of operator $\mathbf{J}\mathbf{A}J$ consists only of element 0 and, therefore, $\mathbf{J}\mathbf{A}J$ is selfadjoint.

Sufficiency. Let $\mathbf{J}\mathbf{A}J : \mathcal{H}_0 \to \mathcal{H}_0$ is selfadjoint. Let us show that $\mathbf{IA} : \mathcal{H}_+ \to \mathcal{H}_+$ is selfadjoint.

$$(\mathbf{IA}u, v)_{\mathcal{H}_+} = \Big(J^{-1}\mathbf{IA}u, J^{-1}v\Big)_{\mathcal{H}_0} = \Big(J^{-1}J\mathbf{J}\mathbf{A}JJ^{-1}u, J^{-1}v\Big)_{\mathcal{H}_0}$$
$$= \Big(J^{-1}u, \mathbf{J}AJJ^{-1}v\Big)_{\mathcal{H}_0} = \Big(J^{-1}u, J^{-1}J\mathbf{J}Av\Big)_{\mathcal{H}_0} = (u, \mathbf{IA}v)_{\mathcal{H}_+}, \qquad u, v \in D(\mathbf{A}).$$

Since $\mathbf{IA}$ is Hermitian, it is sufficient to show that its defect space is equal to $\{0\}$. Let us do it in the same way as for necessity.

Since $\mathbf{J}\mathbf{A}J$ is selfadjoint, from $\big((\mathbf{J}\mathbf{A}J - z\mathbf{1})f, g\big)_{\mathcal{H}_0} = 0$ it follows that $g = 0$, where $z \in \mathbb{C} \setminus \mathbb{R}, f \in D(\mathbf{J}\mathbf{A}J), g \in \mathcal{H}_0$. For $z \in \mathbb{C} \setminus \mathbb{R}, u \in D(\mathbf{A})$ and $v \in \mathcal{H}_+$ the following equality takes place

$$0 = \big((\mathbf{IA} - z\mathbf{1})u, v\big)_{\mathcal{H}_+} = \Big(J^{-1}(\mathbf{IA} - z\mathbf{1})u, J^{-1}v\Big)_{\mathcal{H}_0} = \Big((\mathbf{J}\mathbf{A}J - z\mathbf{1})J^{-1}u, J^{-1}v\Big)_{\mathcal{H}_0}.$$

Since $J^{-1}u \in D(\mathbf{J}\mathbf{A}J)$ and $J^{-1}v \in \mathcal{H}_0$, we have that $J^{-1}v = 0$. From the equality

$$0 = \Big(J^{-1}v, J^{-1}v\Big)_{\mathcal{H}_0} = (v, v)_{\mathcal{H}_+} = \|v\|^2_{\mathcal{H}_+}$$

it follows that $v = 0$. Thus, $\mathbf{IA} : \mathcal{H}_+ \to \mathcal{H}_+$ is selfadjoint. Therefore from Proposition 1 it follows that the operator $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ is generalized selfadjoint. $\square$

3 Ordinary and generalized selfadjointness of operators generated by classical Jacobi matrix

Let us consider the Hilbert space rigging

$$\ell_2(p^{-1}) \supset \ell_2 \supset \ell_2(p), \tag{3}$$

where $\ell_2(p)$ is the space of complex sequences $u = (u_0, u_1, \ldots)$ with scalar product

$$(u, v)_{\ell_2(p)} = \sum_{n=0}^{\infty} u_n \bar{v}_n p_n, \quad u, v \in \ell_2(p)$$

and weight $p = (p_n)_{n=0}^{\infty}$, $p_n \geq 1$, $n \in \mathbb{N}_0 := \{0, 1, \ldots\}$. In this case the operator J acts in the following way

$$\ell_2 \ni u \mapsto Ju \in \ell_2(p) : \ (Ju)_n = p_n^{-1/2} u_n.$$

The operator $\mathbf{J} : \ell_2\left(p^{-1}\right) \to \ell_2$ acts in the same way on elements of space $\ell_2\left(p^{-1}\right)$.

Let us consider classical Hermitian Jacobi matrix of view

$$A = \begin{pmatrix} b_0 & c_0 & 0 & 0 & 0 & \ldots \\ a_0 & b_1 & c_1 & 0 & 0 & \ldots \\ 0 & a_1 & b_2 & c_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \quad b_n \in \mathbb{R}, a_n = c_n > 0, n \in \mathbb{N}_0. \tag{4}$$

Consider an operator A', which acts on finite sequences $f \in \ell_{\text{fin}}$ as follows

$$\left(A'f\right)_n = (Af)_n = a_{n-1} f_{n-1} + b_n f_n + a_n f_{n+1}, \quad f_{-1} = 0, \quad \forall\, n \in \mathbb{N}_0.$$

Operator $A' : \ell_{\text{fin}} \to \ell_{\text{fin}}$ is Hermitian. Let us denote by $\mathcal{A} : \ell_2 \to \ell_2$ the closure of operator A' in ℓ_2. Also we can define an operator $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ as closure of operator $A' : \ell_2(p) \to \ell_2\left(p^{-1}\right)$, which we understand as an operator from $\ell_2(p)$ to $\ell_2\left(p^{-1}\right)$.

In what follows we will investigate generalized selfadjointness of operator $\mathbf{A}$ and its connection with selfadjointness of operator $\mathcal{A}$. Since the theory of classical Jacobi matrices is well known, then at first we will briefly characterize selfadjointness of $\mathcal{A} : \ell_2 \to \ell_2$.

Let us consider for some complex number z the following recurrence relation

$$\left(AP(z)\right)_n = a_{n-1} P_{n-1}(z) + b_n P_n(z) + a_n P_{n+1}(z) = z P_n(z), \quad n \in \mathbb{N}_0,$$

where $P(z) = \left(P_0(z), P_1(z), \ldots\right)$ is such a sequence of polynomials that $P_{-1}(z) = 0$, $P_0(z) = 1$. The following theorem gives necessary and sufficient conditions of selfadjointness of $\mathcal{A}$.

Proposition 2. *Operator $\mathcal{A} : \ell_2 \to \ell_2$ is selfadjoint if and only if for all $z \in \mathbb{C} \setminus \mathbb{R}$ the following equality*

$$\sum_{n=0}^{\infty} \left|P_n(z)\right|^2 = \infty$$

holds.

In the paper [6] it was formulated a few sufficient conditions of selfadjointness of operator $\mathcal{A}$ (see, also, [1]). We consider some of them.

Proposition 3. *The operator $\mathcal{A} : \ell_2 \to \ell_2$ is selfadjoint if any one of the following conditions holds:*

a) $\sum\limits_{n=0}^{\infty} \frac{1}{a_n} = \infty$;

b) $\sum\limits_{n=0}^{\infty} \left| \frac{b_{n+1}}{a_n a_{n+1}} \right| = \infty$;

c) $\liminf\limits_{n\to\infty} \max\{a_0, a_1, \ldots, a_{n-1}\} n^{-1} < \infty$;

d) $a_{n-1} + b_n + a_n \leq C < \infty$ *for all* $n \in \mathbb{N}_0$, *where* C *is some constant and* $a_{-1} = 0$.

In [1], it was proved the following sufficient conditions of nonselfadjointness of an operator $\mathcal{A} : \ell_2 \to \ell_2$.

Proposition 4. *Let* $|b_n| \leq c = const, n \in \mathbb{N}_0$. *Let the inequality* $a_{n-1}a_{n+1} \leq a_n^2$ *holds, starting from some* $n \in \mathbb{N}_0$ *and let* $\sum\limits_{n=0}^{\infty} \frac{1}{a_n} < \infty$. *Then the operator* $\mathcal{A} : \ell_2 \to \ell_2$ *is not selfadjoint.*

Let us prove some generalization of this theorem.

Theorem 2. *Let the inequality* $a_{n-1}a_{n+1} \leq a_n^2$ *holds, starting from some* $n \in \mathbb{N}_0$ *and let* $\sum\limits_{n=0}^{\infty} \frac{1+|b_n|}{a_n} < \infty$. *Then the operator* $\mathcal{A} : \ell_2 \to \ell_2$ *is not selfadjoint.*

Proof. From Proposition 4 it follows that it is sufficient to show that $\mathcal{A} : \ell_2 \to \ell_2$ is not selfadjoint if starting from some n we have $a_{n-1}a_{n+1} \leq a_n^2$, $|b_n| \to \infty, n \to \infty$ and $\sum\limits_{n=0}^{\infty} \frac{|b_n|}{a_n} < \infty$.

Let $n_0 \in \mathbb{N}$ is such that $a_{n-1}a_{n+1} \leq a_n^2$ for all $n \geq n_0$. The operator $\mathcal{A} : \ell_2 \to \ell_2$ is not selfadjoint if for some $z \in \mathbb{C} \setminus \mathbb{R}$ we have $\sum\limits_{n=0}^{\infty} |P_n(z)|^2 < \infty$. From the theorem conditions it follow that it is sufficient to show that $|P_n(z)| \leq \frac{C}{\sqrt{a_n}}$ for all $n \geq n_1$, where $n_1 \in \mathbb{N}$ and $C > 0$ is some constant.

Let us suppose that for all n such that $n_1 \leq n \leq m$ the inequality $\sqrt{a_n}|P_n(z)| \leq C_m$ holds, where $C_m > 0$ is some sequence of constants and $m \geq \max\{n_0, n_1\} + 1 =: m_0$. Now we will select C_{m+1}. Since $P_{m+1}(z) = \frac{1}{a_m}(z - b_m)P_m(z) - \frac{a_{m-1}}{a_m}P_{m-1}(z)$, for any fixed $z \in \mathbb{C} \setminus \mathbb{R}$ the following inequality

$$\begin{aligned}
\sqrt{a_{m+1}}|P_{m+1}(z)| &\leq \frac{\sqrt{a_{m+1}}}{a_m}|z - b_m||P_m(z)| + \frac{\sqrt{a_{m+1}}a_{m-1}}{a_m}|P_{m-1}(z)| \\
&\leq \left(k + |b_m|\right)\frac{\sqrt{a_{m+1}a_{m-1}}}{a_m}\frac{1}{\sqrt{a_{m-1}a_m}}\sqrt{a_m}|P_m(z)| \\
&\quad + \frac{\sqrt{a_{m+1}a_{m-1}}}{a_m}\sqrt{a_{m-1}}|P_{m-1}(z)| \\
&\leq C_m\left(1 + \frac{k + |b_m|}{\sqrt{a_{m-1}a_m}}\right)
\end{aligned}$$

takes place, where $k > 0$ is some constant. Consider C_m of view $C_{m+1} := C_m\left(1 + \frac{k+|b_m|}{\sqrt{a_{m-1}a_m}}\right)$, $m \in \mathbb{N}_0$. Therefore, for all $m > m_0$ we get

$$C_m = C_{m_0}\prod_{n=m_0}^{m-1}\left(1 + \frac{k + |b_n|}{\sqrt{a_{n-1}a_n}}\right) \leq C_{m_0}\prod_{n=1}^{\infty}\left(1 + \frac{k + |b_n|}{\sqrt{a_{n-1}a_n}}\right) =: C.$$

Now, we will show that $C < \infty$, or, in other words, the product is convergent, and this will end the proof. Indeed, since

$$\frac{|b_n|}{a_{n-1}} = \frac{a_n}{a_{n-1}} \frac{|b_n|}{a_n} \leq \frac{a_{n-1}}{a_{n-2}} \frac{|b_n|}{a_n} \leq \ldots \leq \frac{a_1}{a_0} \frac{|b_n|}{a_n},$$

the following inequality

$$\sum_{n=1}^{\infty} \frac{k+|b_n|}{\sqrt{a_{n-1}a_n}} \leq 2 \sum_{n=1}^{\infty} \frac{|b_n|}{\sqrt{a_{n-1}a_n}} \leq 2 \left(\sum_{n=1}^{\infty} \frac{|b_n|}{a_{n-1}} \sum_{n=1}^{\infty} \frac{|b_n|}{a_n} \right)^{1/2} < \infty$$

holds. Therefore, $C < \infty$ and the theorem is proved. □

Now we will try to explain the situation about generalized selfadjointness of an operator $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$. From Theorem 1 it follows that generalized selfadjointness of the operator $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ is equivalent to selfadjointness of the operator $\mathbf{JA}J : \ell_2 \to \ell_2$.

Let us consider the operator $\mathbf{JA}J$. Let $f \in D(\mathbf{JA}J)$. Then for all $n \in \mathbb{N}_0$ we have

$$\begin{aligned}(\mathbf{JA}Jf)_n = (JAJf)_n &= p_n^{-1/2}\left(a_{n-1}p_{n-1}^{-1/2}f_{n-1} + b_n p_n^{-1/2} f_n + a_n p_{n+1}^{-1/2} f_{n+1}\right) \\ &= a_{n-1}p_{n-1}^{-1/2}p_n^{-1/2}f_{n-1} + b_n p_n^{-1} f_n + a_n p_n^{-1/2} p_{n+1}^{-1/2} f_{n+1},\end{aligned} \tag{5}$$

where $f_{-1} := 0$. Let us consider the Jacobi matrix of type (4), namely

$$A_p = \begin{pmatrix} b_0 p_0^{-1} & a_0 p_0^{-1/2} p_1^{-1/2} & 0 & 0 & 0 & \ldots \\ a_0 p_0^{-1/2} p_1^{-1/2} & b_1 p_1^{-1} & a_1 p_1^{-1/2} p_2^{-1/2} & 0 & 0 & \ldots \\ 0 & a_1 p_1^{-1/2} p_2^{-1/2} & b_2 p_2^{-1} & a_2 p_2^{-1/2} p_3^{-1/2} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{6}$$

This matrix generates an operator in ℓ_2 in the following way. Let $A'_p : \ell_{\text{fin}} \to \ell_{\text{fin}}$ be an operator in ℓ_2, which acts on finite vectors as follows $A'_p f = A_p f$, $f \in \ell_{\text{fin}}$. Let us denote by $\mathcal{A}_p : \ell_2 \to \ell_2$ the closure of operator A'_p in ℓ_2.

The following results take place.

Theorem 3. *Operators* $\mathbf{JA}J : \ell_2 \to \ell_2$ *and* $\mathcal{A}_p : \ell_2 \to \ell_2$ *are equal.*

Proof. From construction of $\mathcal{A}_p$ and (5) it follows that operators $\mathbf{JA}J$ and $\mathcal{A}_p$ act in the same way on their domains. So, it is sufficient to show that domains of these operators are equal.

At first, we consider view of these domains. Note that $D(\mathbf{JA}J) = \left\{f \in \ell_2 : Jf \in D(\mathbf{A})\right\}$. From the construction of the operators $\mathbf{A}$ and $\mathcal{A}_p$ it follows that $D(\mathbf{A}) = \ell_{\text{fin}} \cup \Omega_1$ and $D(\mathcal{A}_p) = \ell_{\text{fin}} \cup \Omega_2$, where

$$\begin{aligned}\Omega_1 &:= \left\{u \in \ell_2(p) \setminus \ell_{\text{fin}} : Au \in \ell_2\left(p^{-1}\right)\right\}, \\ \Omega_2 &:= \{f \in \ell_2 \setminus \ell_{\text{fin}} : A_p f \in \ell_2\}.\end{aligned}$$

Now, let us show that $D(\mathbf{JA}J) = D(\mathcal{A}_p)$.

Let $f \in D(\mathcal{A}_p)$. If $f \in \ell_{\text{fin}}$, then $Jf \in \ell_{\text{fin}}$. So, $f \in \mathbf{JA}J$. Let $f \in \Omega_2$. Since $f \in \ell_2 \setminus \ell_{\text{fin}}$, $Jf \in \ell_2(p) \setminus \ell_{\text{fin}}$. Show that $AJu \in \ell_2\left(p^{-1}\right)$.

We have

$$\begin{aligned}\sum_{n=0}^{\infty}\left|(AJf)_n\right|^2 p_n^{-1} &= \sum_{n=0}^{\infty}\left|a_{n-1}(Jf)_{n-1}+b_n(Jf)_n+a_n(Jf)_{n+1}\right|^2 p_n^{-1}\\ &= \sum_{n=0}^{\infty}\left|a_{n-1}p_{n-1}^{-1/2}f_{n-1}+b_np_n^{-1/2}f_n+a_np_{n+1}^{-1/2}f_{n+1}\right|^2 p_n^{-1} \\ &= \sum_{n=0}^{\infty}\left|(A_pf)_n\right|^2<\infty.\end{aligned} \tag{7}$$

Therefore $f \in D(\mathbf{JA}J)$.

Now, let $u \in D(\mathbf{JA}J)$. Let $Ju \in \ell_{\text{fin}}$. So, $u \in \ell_{\text{fin}}$ and, therefore, $u \in D(\mathcal{A}_p)$. Let $Ju \in \Omega_1$. So, $Ju \in \ell_2(p) \setminus \ell_{\text{fin}}$ and $AJu \in \ell_2\left(p^{-1}\right)$. Thus, $u \in \ell_2 \setminus \ell_{\text{fin}}$ and from (7) it follows that $A_pu \in \ell_2$. Therefore, $u \in D(\mathcal{A}_p)$. □

Theorem 4. *For any Jacobi matrix A of view* (4) *there exists a weight $p = (p_n)_{n=0}^{\infty}$, such that the operator $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ is generalized selfadjoint.*

Proof. According to Theorem 1 and Theorem 3 it is sufficient to show that the operator $\mathcal{A}_p : \ell_2 \to \ell_2$ is selfadjoint.

Let the weight $p = (p_n)_{n=0}^{\infty}$ is such that $p_n := a_{n-1} + a_n + 1$. Then for all $n \in \mathbb{N}_0$ we have

$$\frac{1}{a_np_n^{-1/2}p_{n+1}^{-1/2}} = \frac{\left(a_{n-1}+a_n+1\right)^{1/2}\left(a_n+a_{n+1}+1\right)^{1/2}}{a_n} > 1.$$

So, $\sum\limits_{n=0}^{\infty} \frac{1}{a_np_n^{-1/2}p_{n+1}^{-1/2}} = \infty$, and therefore, from condition *a)* of Proposition 3 it follows that $\mathcal{A}_p : \ell_2 \to \ell_2$ is selfadjoint. □

Remark 1. *From Theorem 4 we can make the following conclusion: in spite of situation with selfadjointness of operator $\mathcal{A} : \ell_2 \to \ell_2$, i.e. either $\mathcal{A}$ is selfadjoint or not (see, Proposition 3 and Theorem 2), there exists Hilbert space rigging* (3), *such that $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ is generalized selfadjoint.*

Theorem 5. *Let A be an arbitrary matrix of the form* (4). *Then there exists a weight $p = (p_n)_{n=0}^{\infty}$, or, in other words, there exists a Hilbert space rigging* (3), *such that the operator $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ is bounded and generalized selfadjoint, and $D(\mathbf{A}) = \ell_2(p)$.*

Proof. For any $f \in \ell_{\text{fin}}$ we get

$$\begin{aligned}\left|(\mathbf{A}f)_n\right|^2 p_n^{-1} &= \left|a_{n-1}f_{n-1}+b_np_n+a_np_{n+1}\right|^2 p_n^{-1}\\ &\le a_{n-1}^2p_{n-1}^{-1}p_n^{-1}\left|f_{n-1}\right|^2 p_{n-1}+b_n^2p_n^{-2}\left|f_n\right|^2 p_n+a_n^2p_n^{-1}p_{n+1}^{-1}\left|f_{n+1}\right|^2 p_{n+1}.\end{aligned} \tag{8}$$

Let $u \in \ell_2(p)$. Then $\sum\limits_{n=0}^{\infty} |u_n|_n^2 p_n < \infty$. Let us consider a weight $p = (p_n)_{n=0}^{\infty}$, such that $p_n := a_{n-1} + |b_n| + a_n + 1$, $n \in \mathbb{N}_0$. Thus, from (8) we obtain

$$\begin{aligned}\|\mathbf{A}u\|_{\ell_2(p^{-1})}^2 &= \sum_{n=0}^{\infty}\left|(\mathbf{A}u)_n\right|^2 p_n^{-1}\\ &\le \sum_{n=0}^{\infty} a_{n-1}^2p_{n-1}^{-1}p_n^{-1}|u_{n-1}|^2p_{n-1}+\sum_{n=0}^{\infty} b_n^2p_n^{-2}|u_n|^2p_n+\sum_{n=0}^{\infty} a_n^2p_n^{-1}p_{n+1}^{-1}|u_{n+1}|^2p_{n+1}\\ &\le 3\sum_{n=0}^{\infty}|u_n|^2p_n = 3\,\|u\|_{\ell_2(p)}^2<\infty.\end{aligned}$$

From this inequality it follows that $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ is bounded and $D(\mathbf{A}) = \ell_2(p)$.

Generalized selfadjointness of the operator $\mathbf{A}$ can be proved in the same way as in Theorem 4, because the weight considered in the theorem also satisfies the conditions of previous theorem. $\square$

Theorem 6. *The operator $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ is generalized selfadjoint if any one of conditions a), b) or c) of Proposition 3 holds.*

Proof. From Theorems 1 and 3 it follows that an operator $\mathbf{A}$ is generalized selfadjoint if and only if an operator $\mathcal{A}_p$ is selfadjoint. We will use this fact in the proof.

So, let *a)* takes place. Since $\frac{1}{a_n p_n^{-1/2} p_{n+1}^{-1/2}} \geq \frac{1}{a_n}$, we get $\sum\limits_{n=0}^{\infty} \frac{1}{a_n p_n^{-1/2} p_{n+1}^{-1/2}} = \infty$. Therefore, from condition *a)* of Proposition 3 and above mentioned it follows that $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ is generalized selfadjoint.

Let now *b)* holds true. Since

$$\sum_{n=0}^{\infty} \left| \frac{b_{n+1}}{a_n a_{n+1}} \frac{p_n^{1/2} p_{n+1}^{1/2} p_{n+1}^{1/2} p_{n+2}^{1/2}}{p_{n+1}} \right| \geq \sum_{n=0}^{\infty} \left| \frac{b_{n+1}}{a_n a_{n+1}} \right| = \infty,$$

then from condition *b)* of Proposition 3 it follows that $\mathcal{A}_p$ is selfadjoint.

Let *c)* is fulfilled. Since

$$\max\left\{a_0 p_0^{-1/2} p_1^{-1/2}, a_1 p_1^{-1/2} p_2^{-1/2}, \ldots, a_{n-1} p_{n-1}^{-1/2} p_n^{-1/2}\right\} < \max\left\{a_0, a_1, \ldots, a_{n-1}\right\},$$

then from condition *c)* of Proposition 3 it follows that $\mathcal{A}_p : \ell_2 \to \ell_2$ is selfadjoint and, therefore, $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$ is generalized selfadjoint. $\square$

Remark 2. *From Theorem 6 it follows that sufficient conditions a) – c) from Proposition 3 of selfadjointness of an operator $\mathcal{A}$ are also sufficient conditions for an operator $\mathbf{A}$ generalized selfadjointness.*

4 About one example

In article [3], authors were trying to construct an example of the following type (it was constructed there but it contains a mistake, which was admitted later by the authors): in some Hilbert space rigging (1) they were looking for an operator $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ with dense domain $D(\mathbf{A})$ in $\mathcal{H}_+$, which action in space $\mathcal{H}_0$ is selfadjoint, i.e. $\mathbf{A} : \mathcal{H}_0 \to \mathcal{H}_0$ is selfadjoint, but the operator $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ is not generalized selfadjoint. Let us show that such an example can not be constructed.

At first, we consider an example.

Example 1. *Let us consider Hilbert space rigging* (3) *with weight $p = (p_n)_{n=0}^{\infty}$, $p_n := n+1$, $n \in \mathbb{N}_0$. In this chain, let us consider an operator $\mathbf{A} : \ell_2(p) \to \ell_2\left(p^{-1}\right)$, constructed from the matrix* (4) *by the procedure described in Section 3, namely*

$$a_n := n+1, \quad b_n := -\frac{(n+1)^{5/2}}{(n+2)^{3/2}}, \quad n \geq 0.$$

Also let us consider a sequence $u = (u_n)_{n=0}^{\infty}$, $u_n := (n+1)^{-3/2}$, $n \geq 0$, which is the element of $\ell_2(p)$. The operator $\mathbf{A}$ acts on this element in following manner:

$$(\mathbf{A}u)_0 = 0;$$
$$(\mathbf{A}u)_n = n \cdot n^{-3/2} - \frac{(n+1)^{5/2}}{(n+2)^{3/2}}(n+1)^{-3/2} + (n+1)(n+2)^{-3/2} = n^{-1/2}, \quad n \geq 1.$$

Thus, $u \in D(\mathbf{A})$, because

$$\|\mathbf{A}u\|^2_{\ell_2(p^{-1})} = \sum_{n=0}^{\infty} \left|(\mathbf{A}u)_n\right|^2 p_n^{-1} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Also, $\mathbf{A}u$ does not belong to ℓ_2, because $\|\mathbf{A}u\|^2_{\ell_2} = \sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$.

From this example the next statement follows.

Proposition 5. *Let $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ be an operator in some Hilbert space rigging* (1) *with domain $D(\mathbf{A})$ dense in $\mathcal{H}_+$. The operator $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ can not be considered in sense $\mathbf{A} : \mathcal{H}_0 \to \mathcal{H}_0$ in general case, i.e. its considering in sense of action in $\mathcal{H}_0$ is not correct.*

Theorem 7. *Let $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ be an operator in some Hilbert space rigging* (1) *with domain $D(\mathbf{A})$ dense in $\mathcal{H}_+$ and with range of values $R(\mathbf{A})$ belonging to $\mathcal{H}_0$. If the operator $\mathbf{A} : \mathcal{H}_0 \to \mathcal{H}_0$, i.e. the operator $\mathbf{A}$ in sense of action in $\mathcal{H}_0$, is selfadjoint, then the operator $\mathbf{A} : \mathcal{H}_+ \to \mathcal{H}_-$ is generalized selfadjoint.*

Proof. Let us denote for clearness operator $\mathbf{A} : \mathcal{H}_0 \to \mathcal{H}_0$, which we understand as an operator in $\mathcal{H}_0$, by $\mathcal{A} : \mathcal{H}_0 \to \mathcal{H}_0$. So, $\mathbf{A}$ and $\mathcal{A}$ are the same operators but we understand them in different action sense.

Since $\mathcal{A}$ is selfadjoint, $D(\mathcal{A}) = D(\mathbf{A})$ and they act in the same way on their domains, the operator $\mathbf{A}$ is generalized Hermitian. Therefore, $\mathbf{A} \subset \mathbf{A}^+$.

The definition implies that domain $D(\mathbf{A}^+)$ of operator $\mathbf{A}^+$ consists of such $\psi \in \mathcal{H}_+$ that the functional $\varphi \mapsto (\mathbf{A}\varphi, \psi)_{\mathcal{H}_0}$, $\varphi \in D(\mathbf{A})$, is continuous. On the other hand, the domain $D(\mathcal{A}^*)$ of operator $\mathcal{A}^*$ consists of such $g \in \mathcal{H}_0$ that the functional $f \mapsto (\mathcal{A}f, g)_{\mathcal{H}_0}$, $f \in D(\mathcal{A})$, is continuous. Since $D(\mathcal{A}) = D(\mathbf{A})$ and $\mathcal{H}_0 \supset \mathcal{H}_+$, we get $D(\mathcal{A}^*) \supset D(\mathbf{A}^+)$. Therefore, $\mathbf{A}^+ \subset \mathcal{A}^* = \mathcal{A} = \mathbf{A}$. So, $\mathbf{A}$ is generalized selfadjoint. $\square$

So, from Proposition 5 and Theorem 7 it follows that an example described at the beginning of this section can not be constructed.

Also, it is necessary to admit that earlier we tried to construct such an example in terms of differentiation operator $-i\frac{d}{dt}$ in weight Hilbert space rigging of $L^2(\mathbb{R}, dx)$. We could not construct such example at that time (as now we know it can not be constructed), but all obtained in that process results were published in article [5]. The main result of that paper states that an operator, which is generated by $-i\frac{d}{dt}$, is generalized selfadjoint as soon as respective operator is selfadjoint. Thus, the article [5] corroborate obtained result.

5 Ordinary and generalized selfadjointness of operators generated by generalized Jacobi Hermitian matrix

Now, we consider a generalized selfadjointness of generalized Jacobi Hermitian matrices introduced in [4].

Let us consider complex Hilbert space

$$\mathbf{l}_2 = H_0 \oplus H_1 \oplus H_2 \oplus \cdots, \quad H_i = \mathbb{C}^{i+1}, \quad i \in \mathbb{N}_0,$$

of vectors $\mathbf{l}_2 \ni f = (f_n)_{n=0}^{\infty}$, where $f_n = (f_{n;j})_{j=0}^{n} \in H_n$; $f = \sum\limits_{n=0}^{\infty} \sum\limits_{j=0}^{n} f_{n;j} e_{n;j}$ (here $e_{n;j}$, $n, j \in \mathbb{N}_0$, are elements of standard basis in $\mathbf{l}_2$) with scalar product

$$(f, g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty} (f_n, g_n)_{H_n}; \quad f, g \in \mathbf{l}_2.$$

Consider the following Hilbert space rigging of type (1)

$$\mathbf{l}_2(p^{-1}) \supset \mathbf{l}_2 \supset \mathbf{l}_2(p), \tag{9}$$

where $\mathbf{l}_2(p)$ is space of infinite vectors with scalar product

$$(f, g)_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} (f_n, g_n)_{H_n} p_n; \quad f, g \in \mathbf{l}_2(p),$$

with a given weight $p = (p_n)_{n=0}^{\infty}$, $p_n \geq 1$. In this case the operator J acts as follows

$$\mathbf{l}_2 \ni u \mapsto Ju \in \mathbf{l}_2(p): \ (Ju)_n = p_n^{-1/2} u_n.$$

The operator $\mathbf{J} : \mathbf{l}_2(p^{-1}) \to \mathbf{l}_2$ acts in the same way on elements of space $\mathbf{l}_2(p^{-1})$.

Let us consider in the space $\mathbf{l}_2$ a Hermitian matrix $G = (G_{j,k})_{j,k=0}^{\infty}$ with an operator-valued complex elements $G_{j,k}\colon H_k \to H_j$, $G_{j,k} = (G_{j,k;\alpha,\beta})_{\alpha=0}^{j}{}_{\beta=0}^{k}$, of the following block Jacobi structure

$$G = \begin{pmatrix} b_0 & c_0 & 0 & 0 & \dots \\ a_0 & b_1 & c_1 & 0 & \dots \\ 0 & a_1 & b_2 & c_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{where} \quad \begin{array}{l} a_i = G_{i+1,i} : H_i \to H_{i+1}, \\ b_i = G_{i,i} : H_i \to H_i, \\ c_i = G_{i,i+1} : H_{i+1} \to H_i. \end{array} \tag{10}$$

For Hermitianess of matrix G it is necessary and sufficient that $b_i = b_i^*$, $a_i = c_i^*$, where "$*$" denotes adjoint to matrix.

Let $f \in \mathbf{l}_2$. Then the matrix G acts on f in the following manner

$$(Gf)_n = a_{n-1} f_{n-1} + b_n f_n + c_n f_{n+1}, \quad \text{with} \quad f_{-1} = 0. \tag{11}$$

Let us consider an operator $G' : \mathbf{l}_{\text{fin}} \to \mathbf{l}_{\text{fin}}$, which acts on finite sequences $f \in \mathbf{l}_{\text{fin}}$ as in (11), i.e. $G'f = Gf$, $f \in \mathbf{l}_{\text{fin}}$. Operator G' is Hermitian. So, we can consider operator $\mathcal{G} : \mathbf{l}_2 \to \mathbf{l}_2$, which is equal to closure of the operator G' in $\mathbf{l}_2$. In the same way as in Section 3 we can also define an operator $\mathbf{G} : \mathbf{l}_2(p) \to \mathbf{l}_2(p^{-1})$.

Let us consider for some complex number z the recurrence relation

$$\big(G\varphi(z)\big)_n = a_{n-1}\varphi_{n-1}(z) + b_n \varphi_n(z) + c_n \varphi_{n+1}(z) = z\varphi_n(z), \quad n \in \mathbb{N}_0, \tag{12}$$

where $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in H_n$, is a such sequence that $\varphi_{-1}(z) := 0$. In article [4], there was considered problem of selfadjointness of the operator $\mathcal{G}$ and there were obtained the following results.

Proposition 6. *Operator $\mathcal{G}$ is selfadjoint if and only if for any non-zero solution of system* (12) *the condition* $\sum\limits_{n=0}^{\infty} \|\varphi_n(z)\|^2_{H_n} = \infty$ *holds, where* $z \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 7. *Let a matrix G be such that* $\sum\limits_{n=0}^{\infty} \left(\|a_n\|_{n;n+1} + \|c_n\|_{n+1;n} \right)^{-1} = \infty$, *where* $\|\cdot\|_{k;l}$ *defines norm of* $(l+1) \times (k+1)$*-matrix or respective operator which acts from* H_k *to* H_l*. Then operator* $\mathcal{G}$ *is selfadjoint.*

The situation about generalized selfadjointness of $\mathbf{G}$ is similar to described one in Section 3. So, from Theorem 1 it follows that generalized selfadjointness of $\mathbf{G} : \mathbf{l}_2(p) \to \mathbf{l}_2\left(p^{-1}\right)$ is equivalent to selfadjointness of operator $\mathbf{JG}J : \mathbf{l}_2 \to \mathbf{l}_2$, where $\mathbf{J}$ and J are respective operators for Hilbert space rigging (9). Operator $\mathbf{JG}J$ acts as follows

$$
\begin{aligned}
(\mathbf{JG}Jf)_n = (JGJf)_n &= p_n^{-1/2} \left(a_{n-1} p_{n-1}^{-1/2} f_{n-1} + b_n p_n^{-1/2} f_n + c_n p_{n+1}^{-1/2} f_{n+1} \right) \\
&= a_{n-1} p_{n-1}^{-1/2} p_n^{-1/2} f_{n-1} + b_n p_n^{-1} f_n + c_n p_n^{-1/2} p_{n+1}^{-1/2} f_{n+1},
\end{aligned}
$$

where $f \in D(\mathbf{JG}J)$ and $f_{-1} := 0$. Let us consider generalized Jacobi matrix of type (10) of view

$$
G_p = \begin{pmatrix}
b_0 p_0^{-1} & c_0 p_0^{-1/2} p_1^{-1/2} & 0 & 0 & 0 & \dots \\
a_0 p_0^{-1/2} p_1^{-1/2} & b_1 p_1^{-1} & c_1 p_1^{-1/2} p_2^{-1/2} & 0 & 0 & \dots \\
0 & a_1 p_1^{-1/2} p_2^{-1/2} & b_2 p_2^{-1} & c_2 p_2^{-1/2} p_3^{-1/2} & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}.
$$

In the same way as in Section 3 we can generate operator $\mathcal{G}_p : \mathbf{l}_2 \to \mathbf{l}_2$ by matrix G_p. Similarly to the proof of Theorem 3 it is easy to show that the following its analog takes place.

Theorem 8. *Operators* $\mathbf{JG}J : \mathbf{l}_2 \to \mathbf{l}_2$ *and* $\mathcal{G}_p : \mathbf{l}_2 \to \mathbf{l}_2$ *are equal.*

Also, some other analogs of Theorems 4 and 5 take place.

Theorem 9. *For any generalized Jacobi matrix G of view* (10) *there exists a weight* $p = (p_n)_{n=0}^{\infty}$, *such that the operator* $\mathbf{G} : \mathbf{l}_2(p) \to \mathbf{l}_2\left(p^{-1}\right)$ *is generalized selfadjoint.*

Proof. Let the weight $p = (p_n)_{n=0}^{\infty}$ be such that

$$
p_n := \|a_{n-1}\|_{n-1;n} + \|a_n\|_{n;n+1} + \|c_{n-1}\|_{n;n-1} + \|c_n\|_{n+1;n} + 1, \quad n \in \mathbb{N}_0.
$$

Proceeding in the same way as in Theorem 4, from Proposition 7 we obtain that $\mathbf{G}$ is generalized selfadjoint. □

Theorem 10. *Let G be an arbitrary matrix of the form* (10)*. Then there exists a Hilbert space rigging* (9)*, such that* $\mathbf{G} : \ell_2(p) \to \ell_2(p^{-1})$ *is bounded and generalized selfadjoint and* $D(\mathbf{G}) = \mathbf{l}_2(p)$.

Proof. The proof of this theorem is similar to proof of Theorem 5, if we consider a weight $p = (p_n)_{n=0}^{\infty}$, such that

$$
p_n := \|a_{n-1}\|_{n-1;n} + \|a_n\|_{n;n+1} + \|b_n\|_{n;n} + \|c_{n-1}\|_{n;n-1} + \|c_n\|_{n+1;n} + 1, \quad n \in \mathbb{N}_0.
$$

□

Remark 3. *Let us notice that the sufficient condition of the selfadjointness of the operator G from Proposition 7 is also sufficient for the generalized selfadjointness of G.*

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Івасюк І.Я. *Узагальнена самоспряженість операторів, породжених ермітовими якобієвими матрицями* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 203–214.

Досліджується самоспряженість у сенсі гільбертового оснащення і пов'язані з цим питання. Доведено, що ця узагальнена самоспряженість довільного оператора, який діє з позитивного в негативний простір, еквівалентна звичайній самоспряженості певним чином перетвореного цього оператора у базовому ("нульовому") просторі.

Також розглянуто оператори породжені класичними і узагальненими якобієвими ермітовими матрицями, їхня самоспряженість і узагальнена самоспряженість в сенсі вагового гільбертового оснащення. Доведено певні достатні умови узагальненої самоспряженості цих операторів. Використовуючи отримані результати пояснено можливість побудови прикладу узагальненої самоспряженості оператора, який не самоспряжений в класичному сенсі.

Ключові слова і фрази: оснащення гільбертового простору, ермітовий оператор, самоспряжений оператор, трохдіагональна блочна матриця, матриця Якобі.