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A note on the Banach lattice $c_0(\ell_2^n)$, its dual and its bidual

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The main purpose of this paper is to study some geometric and topological properties of c_0-sum of the finite dimensional Banach lattice ℓ_2^n, its dual and its bidual. Among other results, we show that the Banach lattice $c_0(\ell_2^n)$ has the strong Gelfand-Philips property, but does not have the positive Grothendieck property. We also prove that the closed unit ball of $l_\infty(\ell_2^n)$ is an almost limited set.

Key words and phrases: Banach lattice, Dunford-Pettis property, Gelfand-Phillips property, weak Dunford-Pettis property, weak Grothendieck property, positive Grothendieck property, strong Gelfand-Phillips property.

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Introduction

Throughout this paper, X and Y will denote Banach spaces, E and F will denote Banach lattices. We denote by B_X the closed unit ball of X. In a Banach lattice, the additional lattice structure provides a large number of tools that are not available in more general Banach spaces. This fact facilitates the study of geometric and topological properties of Banach lattices. It is extremely important to add more examples of Banach lattices which satisfy or do not some geometric or topological properties. Our objective here is to study the Banach lattices given by $(\bigoplus_{n=1}^\infty \ell_2^n)_0$, $(\bigoplus_{n=1}^\infty \ell_2^n)_1$ and $(\bigoplus_{n=1}^\infty \ell_2^n)_\infty$ and describe which properties each of them satisfies. The importance of such Banach spaces is due to the fact that the result presented by C. Stegall in [14], where he showed that $(\bigoplus_{n=1}^\infty \ell_2^n)_\infty$ does not have Dunford-Pettis property, but its predual, $(\bigoplus_{n=1}^\infty \ell_2^n)_1$, has it.

We will start by recalling concepts of specific sets in Banach spaces and their consequences on their geometric or topological properties.

A bounded set $A \subset X$ is *Dunford-Pettis* (resp. *limited*) if every weakly null sequence in X' converges uniformly to zero on A (resp. if every weak* null sequence in X' converges uniformly to zero on A). Concerning these sets, we can consider a few properties in the class of Banach spaces. A Banach space X has the *DP property* (resp. *DP* property*) if every relatively weakly compact subset of X is Dunford-Pettis (resp. if every relatively weakly compact subset of X is limited). Or equivalently, $x_n'(x_n) \to 0$ for every $x_n \overset{\omega}{\to} 0$ in X and $x_n' \overset{\omega}{\to} 0$ in X' (resp. $x_n'(x_n) \to 0$ for every $x_n \overset{\omega}{\to} 0$ in X and $x_n' \overset{\omega^*}{\to} 0$ in X'). We say that X has *Gelfand-Phlips property* (*GP property* for short), if every limited subset of X is relatively compact.

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Of course the DP* property implies the DP. On the other hand, $L_1[0,1]$ and c_0 are Banach spaces with the DP property without the DP*. Schur spaces have all three properties listed above. Separable and reflexive spaces are examples of Banach spaces with the GP property. For more information concerning those properties we refer the reader to [1,4,6,9].

In the class of Banach lattices, the lattice structure allows us to consider disjoint sequences. A sequence $(x_n) \subset E$ is *disjoint* if $|x_n| \wedge |x_m| = 0$ for every $n \neq m$. A bounded subset $A \subset E$ is *almost Dunford-Pettis* (resp. *almost limited*) if every disjoint weakly null sequence in E' converges uniformly to zero on A (resp. if every disjoint weak* null sequence in E' converges uniformly to zero on A). Next, we will give some properties in Banach lattices that the definitions given for the sets above appear naturally. A Banach lattice E has the *weak DP property* (wDP for short) if every relatively weakly compact subset of E is almost Dunford-Pettis. Or equivalently, if for all Banach space Y, every weakly compact operator $T : F \to Y$ is an *almost Dunford-Pettis* operator, that means, T maps disjoint weakly null sequences of F onto norm null sequences in Y. We say that E has the *weak DP* property* (wDP* for short) if every relatively weakly compact subset of E is almost limited, or equivalently, if $x'_n(x_n) \to 0$ for each weakly null sequence $(x_n) \subset F$ and each disjoint weak* null sequence $(x'_n) \subset F'$. A Banach lattice E is said to have the *strong GP property* (sGP for short) if every almost limited subset of E is relatively compact.

Of course, the DP and the DP* properties imply the wDP and the wDP*, respectively. In [10], D.H. Leung gave the first example of a Banach lattice with the wDP property and without the DP. In [7], the authors showed that $L_1[0,1]$ has the wDP* property even though it does not have the DP*. Note that the sGP property is stronger than the GP. For instance, $L_1[0,1]$ does not have the GP property. We refer to [2,3,7] for more details concerning those properties.

Recall that a Banach space X has the *Grothendieck property* if every weak* null sequence in X' is weakly null. For example, ℓ_∞ has the Grothendieck property. For a Banach lattice, we can consider the weak Grothendieck property and the positive Grothendieck property. From [15], E is said to have the *positive Grothendieck property* if every positive weak* null sequence in E' is weakly null. Following [12], E has the *weak Grothendieck property* if every disjoint weak* null sequence in E' is weakly null. Clearly, the Grothendieck property implies both the positive Grothendick and the weak Grothendieck properties. For instance, ℓ_1 has the weak Grothendieck property, but it fails to have the positive Grothendieck property, and c is a Banach lattice with the positive Grothendieck property without the weak Grothendieck property.

1 Results

First, we are going to fix some notations. Denote by ℓ_2^n the Banach lattice $\mathbb{R}^n$ with Euclidean norm and let

$$E = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_0 = c_0(\ell_2^n), \quad E' = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_1 = l_1(\ell_2^n) \quad \text{and} \quad E'' = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_\infty = l_\infty(\ell_2^n). \quad (1)$$

We can consider in E a natural structure of Banach lattice induced by its unconditional basis $(e_j^i)_{i,j}$, where $e_j^i = (0,\dots,0,\overbrace{e_j}^{i},0,\dots)$ with $e_j = (0,\dots,0,1_{(j)},0,\dots)$. Thus E' and E'' also are Banach lattices with their dual structures.

Our goal in this section is to study the topological properties of such Banach lattices. In the following, E, E' and E'' will be fixed as in (1). It is known that E' is a Schur space (see [14]), and consequently, E' has the DP property. However, its dual E'' does not have it. This was the first example of a Banach space with the DP property whose dual space does not have it.

Since E' has the Schur property, it has the DP, the DP* and the sGP properties, then E has the DP and wDP properties. In the next proposition, we will show that E does not have the wDP* property.

Proposition 1. *The Banach lattice E does not have the wDP* property.*

Proof. If (e_n) is the Schauder basis in c_0 and if $T : c_0 \to E$ is the positive diagonal operator given by

$$T(\alpha_j)_j = \begin{pmatrix} \alpha_1 & 0 & 0 & \dots \\ & \alpha_2 & 0 & \dots \\ & & \alpha_3 & \dots \\ & & & \end{pmatrix},$$

we have that $Te_n \overset{\omega}{\to} 0$ in E. On the other hand, the sequence $e'_{n,n} = (0, \dots, 0, e_n, 0, \dots) \in E'$ is disjoint and weak* null with $e'_{n,n}(Te_n) = 1$ for every n. So, E does not have the wDP* property. □

As E'' does not have the DP property, it is natural to ask if E'' has the wDP property. And here we will show that E'' does not have it. To do this, we need the next two lemmas.

Lemma 1. *Let F and G be Banach lattices such that F has the wDP property. If $T : F \to G$ is a surjective lattice isomorphism, then G also has the wDP property.*

Proof. Let X be a Banach space and $S : G \to X$ be a weakly compact operator. So $S \circ T : F \to X$ is a weakly compact operator. As F has the wDP property, then $S \circ T$ is an almost DP operator operator. Now we show that the operator S is an almost DP operator. Let $(y_n) \subset G$ be a disjoint weakly null sequence, so there exists $(x_n) \subset F$ a disjoint weakly null sequence such that $Tx_n = y_n$ for all n. Therefore $S(y_n) = S\big(T(x_n)\big)$ and we have that $S(y_n) \to 0$ in X, so S is an almost DP and the result follows. □

Let F be a Banach lattice and $G \subset F$ a sublattice. We say that G is *a complemented sublattice* of F if there is a bounded projection $P : F \to F$ such that $P(F) = G$.

Lemma 2. *Let F a Banach lattice and let G be a complemented sublattice of F. If F has the wDP property, then G has wDP property.*

Proof. Consider a Banach space X and a weakly compact operator $T : G \to X$. Let $P : F \to F$ be a bounded projection such that $P(F) = G$. So $T \circ P : F \to X$ is also a weakly compact operator. Thus, $T \circ P$ is an almost DP operator. If (z_n) is a disjoint weakly null sequence in G, since G is a sublattice of F, it follows that (z_n) is a disjoint weakly null sequence in F. As $T(z_n) = T\big(P(z_n)\big)$ we get that $T(z_n) \to 0$ and the result is true. □

Now we can prove that E'' does not have the wDP property.

Proposition 2. *The Banach lattice E'' does not have the wDP property.*

Proof. Consider the bounded linear operator $R : E' \to \ell_2$ given by

$$R(x) = (x_{1,1} + x_{2,1} + \cdots, x_{2,2} + x_{3,2} + \cdots, \cdots).$$

By C. Stegall [14], we have that $R' : \ell_2 \to E''$ is an isomorphism on $R'(\ell_2)$, and $R'(\ell_2)$ is a complemented subspace of E'', as Banach spaces. It is easy to verify that R' is a lattice isomorphism on $R'(\ell_2)$ and $R'(\ell_2)$ is a complemented sublattice of E''. Since ℓ_2 does not have the wDP property, it follows from Lemma 1 that $R'(\ell_2)$ cannot have the wDP property. By Lemma 2, it follows that E'' cannot have the wDP property. □

Since every almost limited set is almost DP set, as a consequence of Proposition 2, E'' cannot have the wDP* property. We observe that E has the GP property, because E is separable.

Now, we will prove that E has the sGP property. First we need the following assertion.

Lemma 3. *The Banach lattice E is Dedekind complete.*

Proof. Let $A \subset E$ such that $a \leq x$ for every $a \in A$ and some $x \in E^+$. In particular,

$$a = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \cdots \\ & a_{2,2} & a_{3,2} & \cdots \\ & & a_{3,3} & \cdots \\ & & & \cdots \end{pmatrix} \leq \begin{pmatrix} x_{1,1} & x_{2,1} & x_{3,1} & \cdots \\ & x_{2,2} & x_{3,2} & \cdots \\ & & x_{3,3} & \cdots \\ & & & \cdots \end{pmatrix} = x$$

holds for every $a \in A$. So, $a_{i,j} \leq x_{i,j}$ in $\mathbb{R}$ for every $i \in \mathbb{N}$ and $j = 1, \ldots, i$. As $\mathbb{R}$ is Dedekind complete, there are $z_{i,j} = \sup \{a_{i,j} : a = (a_{k,l})_{l \leq k} \in A\}$ for all $i \in \mathbb{N}$ and $j = 1, \ldots, i$. Now, let $z = (z_{i,j})_{j \leq i}$. Since $x_{i,j} \leq z_{i,j}$ holds for every $i \in \mathbb{N}$ and $j = 1, \ldots, i$, it follows that $z \in E$. Now we prove that $z = \sup A$. In fact, if $y \in E$ is such that $a \leq y$ for every $a \in A$, so $a_{i,j} \leq y_{i,j}$ for every $i \in \mathbb{N}$ and $j = 1, \ldots, i$. Thus $z_{i,j} \leq y_{i,j}$ for every $i \in \mathbb{N}$ and $j = 1, \ldots, i$, hence z is the supremum of A in E. □

As a consequence of the above lemma, we have that E has order continuous norm. We will conclude that E has the sGP property showing that E is a discrete Banach lattice. First, we recall that an element x belongs to a Banach lattice F is *discrete* if $x > 0$ and $|y| \leq x$ implies $y = tx$ for some real number t. If every order interval $[0, y]$ in F contains a discrete element, then F is said to be a *discrete Banach lattice*.

Theorem 1. *The Banach lattice E has the sGP property.*

Proof. By [2, Theorem 2.1], it suffices to prove that E is a discrete Banach lattice. Let $[0, y]$ be an order interval in E, take

$$y = \begin{pmatrix} y_{1,1} & y_{2,1} & y_{3,1} & \cdots \\ & y_{2,2} & y_{3,2} & \cdots \\ & & y_{3,3} & \cdots \\ & & & \cdots \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} y_{1,1} & 0 & 0 & \cdots \\ & 0 & 0 & \cdots \\ & & 0 & \cdots \\ & & & \cdots \end{pmatrix}.$$

So $x \in [0, y]$. If $|z| \leq x$ in E, it follows that

$$z = \begin{pmatrix} z_{1,1} & 0 & 0 & \cdots \\ & 0 & 0 & \cdots \\ & & 0 & \cdots \\ & & & \cdots \end{pmatrix}$$

with $|z_{1,1}| \leq y_{1,1}$ in $\mathbb{R}$. This implies that there exists a real number t such that $z_{1,1} = ty_{1,1}$. Thus $z = tx$. The result follows. □

We claim that E'' does not have the GP property. Indeed, consider the positive operator $S : \ell_\infty \to E''$ given by

$$S(\alpha_j)_j = \begin{pmatrix} \alpha_1 & 0 & 0 & \ldots \\ & \alpha_2 & 0 & \ldots \\ & & \alpha_3 & \ldots \\ & & & \end{pmatrix}.$$

As $(e_n)_n \subset \ell_\infty$ is a weakly null limited sequence, this implies that (Se_n) is a weakly null limited sequence in E'' such that $\|Se_n\|_\infty = 1$ for all n. That means E'' does not have the Gelfand-Phillips property.

Next, we study the Grothendieck type properties in E and E'.

Proposition 3. *The Banach lattice E does not have the weak Grothendieck property and does not have the positive Grotendieck property.*

Proof. Let $(e'_{n,n}) \subset E'$ as given in the proof of Proposition 1. This sequence is positive, disjoint and weak* null in E', however, $(e'_{n,n})$ is not weakly null. Indeed, if $x'' \in E''$ given by

$$x'' = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ & 1 & 0 & 0 & \cdots \\ & & 1 & 0 & \cdots \\ & & & 0 & \cdots \end{pmatrix}.$$

then $x''(e'_{n,n}) = 1$ for all n. □

Proposition 4. *The Banach lattice E' has the weak Grothendieck property, however it does not have the positive Grothendieck property.*

Proof. Let $e \in E''$ be given by

$$e = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ & 1 & 1 & \cdots \\ & & 1 & \cdots \\ & & & \cdots \end{pmatrix}$$

It is easy to see that e is an order unit of E'', that means $B_{E''} = [-e, e]$. Let $(x''_n) \subset E''$ be a disjoint weak* null sequence. In particular, (x''_n) is bounded, and so there exists $M > 0$ such that $x''_n \in [-Me, Me]$ for every $n \in \mathbb{N}$. Consequently, (x''_n) is a disjoint order bounded sequence in E'', hence $x''_n \stackrel{\omega}{\to} 0$ in E'' (see [1, p. 192]). Therefore E' has the weak Grothendieck property.

Consider the diagonal operator $T : \ell_1 \to E'$, given by

$$T(\alpha_j)_j = \begin{pmatrix} \alpha_1 & 0 & 0 & \dots \\ & \alpha_2 & 0 & \dots \\ & & \alpha_3 & \dots \\ & & & \end{pmatrix}.$$

It is easy to see thar T is a lattice isometry and for each $\alpha = (\alpha_j)_j \in \ell_1$, we have $\|T(\alpha)\| = \|\alpha\|$, hence T is a lattice embedding. By [13, Proposition 2.3.11], it follows that ℓ_1 is isomorphic to a positively complemented sublattice in E', then E' does not have the positive Grothendieck property. □

Corollary 1. *The norm in E'' is not order continuous.*

Proof. By [12, Proposition 4.9], if F is a Banach lattice which has the weak Grothendieck property and F' has order continuous norm, then F also has the positive Grothendieck property. So, by Proposition 4, it follows that E'' does not have order continuous norm. □

Now we want to give a version of Phillip's Lemma for E''. To do this we use Dixmeir's Theorem, that is, if X is a Banach space, then X' is complemented in X'' (see [8]). Despite being a known result, we decided to state it in the next lemma and present a proof for the specific case of the Banach lattice E'''.

Lemma 4. *Consider $E^{\perp} = \{f \in E''' : f(x) = 0, \forall\, x \in E\}$. Then $E''' = E' \oplus E^{\perp}$ and $E^{\perp}$ is an ideal in E'''.*

Proof. Let $f \in E'''$ and put $a_{i,j} = f(e_{i,j})$ for all $i \in \mathbb{N}$ and $j = 1, \dots, i$. We claim that

$$a' = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \cdots \\ & a_{2,2} & a_{3,2} & \cdots \\ & & a_{3,3} & \cdots \\ & & & \cdots \end{pmatrix} \in E'.$$

Indeed, since

$$\sum_{i=1}^{\infty} \|a_i\|_2 \leq \sum_{i=1}^{\infty}\sum_{j=1}^{i} |a_{i,j}| = \sum_{i=1}^{\infty}\sum_{j=1}^{i} f(\epsilon_{i,j}e_{i,j}) = f\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{i} \epsilon_{i,j}e_{i,j}\Big) \leq \|f\|.$$

On the other hand, if $x = (x_{i,j})_{1\leq j \leq i} \in E$, then

$$f(x) - a'(x) = \lim_{n\to\infty} \left[\sum_{i=1}^{n}\sum_{j=1}^{i} f_{i,j}(x_{i,j}e_{i,j}) - \sum_{i=1}^{n}\sum_{j=1}^{i} a_{i,j}(x_{i,j}) \right] = 0.$$

As $E' \cap E^{\perp} = \{0\}$, we get $E''' = E' \oplus E^{\perp}$.

By Lemma 3, E has order continuous norm, as a consequence it is an ideal in E''. We claim that $E^{\perp}$ is an ideal in E'''. Indeed, let $x''', y''' \in E'''$ with $|y'''| \leq |x'''|$ and $x''' \in E^{\perp}$. If $x \in E^+$ and $|y''| \leq x$ in E'', then $|y''| \in E$, this implies that $|x'''|(x) = \sup\{|x'''(y)| : |y| \leq x\} = 0$. Finally, if $x \in E$,

$$|y'''(x)| \leq |y'''|(|x|) \leq |x'''|(|x|) = 0.$$

Therefore $y''' \in E^{\perp}$. □

Now we give a version of Phillip's Lemma for E''. The proof follows the same idea of [1, Theorem 4.67]. We remark that in [5] the authors has showed that B_E is a limited set in E'', but they used another technique in another context.

Proposition 5. *Every weak* null sequence in E'' converges uniformly to zero on B_E. Consequently, B_E is a limited set in E''.*

Proof. Let $(f_n) \subset E'''$ be a weak* null sequence. By Lemma 4 we write $f_n = x_n + g_n$ with $(x_n) \subset E'$ and $(g_n) \subset E^{\perp}$. As E' has order continuous norm then E' is an ideal in E'''. On the other hand, since $E^{\perp}$ also is an ideal in E''', by [1, Theorem 1.41], we have that E' is a projection band in E''', which yields that $x_n \overset{\omega^*}{\to} 0$ in E''' (see [1, Theorem 4.46]). Then $x_n \overset{\omega}{\to} 0$ in E', and since E' has the Schur property, $x_n \to 0$ in E'. As a consequence,

$$\|f_n\|_{B_E} = \sup_{x \in B_E} |x_n(x)| \leq \|x_n\| \to 0.$$

□

Proposition 6. *The Banach lattice E'' has the weak and the positive Grothendieck properties.*

Proof. Let $(x'_n) \subset E'''$ be a positive weak* null sequence. Since e is an order unit of E'', that means $B_{E''} = [-e, e]$, we get that

$$\|x'_n\| = \sup_{x \in [-e,e]} |x_n(x)'| = x'_n(e) \to 0.$$

So $x'_n \overset{\omega}{\to} 0$, and E'' has the positive Grothendieck property.

Now, as every disjoint weak* null sequence $(f_n) \subset E'''$ implies $|f_n| \overset{\omega^*}{\to} 0$ in E''. So, E'' has the positive Grothendieck property. □

In the next result we classify the closed unit balls of E, E' and bounded subset E'' concerning if they are (or not) almost Dunford-Pettis or almost limited. As E' has the Schur property, B_E is a Dunford-Pettis set.

Proposition 7.

(1) The closed unit ball of E is not almost limited.

(2) The closed unit ball of E' is not almost Dunford-Pettis.

(3) Every norm bounded subset in E'' is almost limited.

Proof.

(1) Let $T : c_0 \to E$ and $(e'_{n,n}) \subset E'$ be the positive operator given in the proof of Proposition 1. Since $\|e'_{n,n}\|_{B_E} = \sup_{x \in B_E} |e'_{n,n}(x)| \geq e'_{n,n}(Te_n) = 1$ for all n, where $(e_n)_n$ is the canonical basis in c_0, we have that B_E is not almost limited.

(2) The unit diagonal sequence $e''_{n,n} = (0, \ldots, 0, e_n, 0, \ldots)$ for all n is weakly null and disjoint in E''. Since $\sup_{x \in B_{E'}} |e''_{n,n}(x)| \geq \sup_{x \in B_{E'}} |e''_{n,n}(e'_{n,n})| = 1$ for all n, we have that $B_{E'}$ is not almost Dunford-Pettis. As a consequence $B_{E'}$ it is not almost limited.

(3) Consider a norm bounded subset $A \subset E''$. Then there exists $M > 0$ such that $A \subset M \cdot B_{E''} = [-Me, Me] = \text{sol}(Me)$. By [11, Lemma 2.1], we have that $M \cdot B_{E''}$ is almost limited. Consequently, A is almost limited as well.

□

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Головним завданням цієї статті є дослідження деяких геометричних та топологічних властивостей c_0-суми скінченно вимірної банахової ґратки ℓ_2^n, її спряженої та другої спряженої. Серед іншого ми показуємо, що банахова ґратка $c_0(\ell_2^n)$ володіє сильною властивістю Гельфанда-Філіпса, але не володіє додатньою властивістю Ґротендіка. Ми також доводимо, що замкнута одинична куля простору $l_\infty(\ell_2^n)$ є майже граничною множиною.

Ключові слова і фрази: банахова ґратка, властивість Данфорда-Петтіса, властивість Гельфанда-Філіпса, слабка властивість Данфорда-Петтіса, слабка властивість Ґротендіка, додатна властивість Ґротендіка, сильна властивість Гельфанда-Філіпса.