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# Inverse problem with two unknown time-dependent functions for $2b$-order differential equation with fractional derivative

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We study the inverse problem for a differential equation of order $2b$ with a Riemann-Liouville fractional derivative over time and given Schwartz-type distributions in the right-hand sides of the equation and the initial condition. The generalized (time-continuous in a certain sense) solution $u$ of the Cauchy problem for such an equation, the time-dependent continuous young coefficient and a part of a source in the equation are unknown.

In addition, we give the time-continuous values $\Phi_j(t)$ of desired generalized solution $u$ of the problem on a fixed test functions $\varphi_j(x)$, $x \in \mathbb{R}^n$, namely $(u(\cdot,t), \varphi_j(\cdot)) = \Phi_j(t)$, $t \in [0, T]$, $j = 1, 2$.

We find sufficient conditions for the uniqueness of the generalized solution of the inverse problem throughout the layer $Q := \mathbb{R}^n \times [0, T]$ and the existence of a solution in some layer $\mathbb{R}^n \times [0, T_0]$, $T_0 \in (0, T]$.

*Key words and phrases:* distribution, fractional derivative, inverse problem, Green vector-function.

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## Introduction

Equations with fractional derivatives [1] and inverse problems to them appear in different branches of science and engineering, and the range of the applicability of the generated models increase considerable. The conditions of classical solvability of the Cauchy and boundary value problems to equations with a time fractional derivative were obtained, for example, in [2–8]. The inverse boundary value problems to a time fractional diffusion equation with different unknown functions or parameters were investigated, for example, in [9–19]. Most papers were devoted to inverse problems with an unknown right-hand sides, mainly under regular data.

In this paper, for the equation

$$u_t^{(\beta)} - A(D)u - r(t)u = g(t)F_0(x,t) + F(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T] := Q, \tag{1}$$

with the Riemann-Liouville fractional derivative of order $\beta \in (0,1)$ we study the inverse problem

$$u(x,0) = F_1(x), \quad x \in \mathbb{R}^n, \tag{2}$$

$$(u(\cdot,t), \varphi_1(\cdot)) = \Phi_1(t), \quad (u(\cdot,t), \varphi_2(\cdot)) = \Phi_2(t), \quad t \in [0,T], \tag{3}$$

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of determination the triple $(u,r,g)$ where

$$A(D)u = \sum_{|\gamma|\le 2b} A_\gamma D^\gamma u,$$

$\dfrac{\partial u}{\partial t} - A(D)u$ is the parabolic differential expression [3], $F_1$ and $F$ are given Schwartz type distributions [20], $\Phi_1, \Phi_2$ are given continuous functions, the symbol $(u(\cdot,t), \varphi_j(\cdot))$ stands for the value of an unknown distribution $u$ on the given test function $\varphi_j$ for every $t \in [0,T]$.

Note that source inverse boundary value problems for time-fractional diffusion equations under regular given data in the right-hand sides and similar (integral) over-determination conditions were studied, for example, in [9–11]. The over-determination condition of kind (3), but with the scalar product $(u, \varphi_0)$ in abstract Hilbert space, was used in [12]. An overview of individual results on inverse problems for anomalous diffusion processes is given in [13]. The inverse boundary value problem with an unknown only young coefficient in the case $b = 1$, one over-determination condition of kind (3) and regular data was studied in [14], and in [15] in the case of given distributions with compact supports in the right-hand sides. Source inverse problem for a $2b$-order equation with a time-fractional derivative was studied in [16]. Inverse problems for a time-fractional differential equations with a time-integral over-determination conditions were investigated, for example, in [17–19].

In this paper, the result [16] is generalized to the case when, in addition to the solution of the Cauchy problem, we look for two unknown, time-dependent functions (part of the right-hand side and the lowest coefficient in the equation).

## 1 Notations, definitions and auxiliary results

We use the following notations: $Q = \mathbb{R}^n \times (0,T]$, $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\bar{\alpha} = (\alpha_0,\alpha)$, $\alpha_j \in \mathbb{Z}_+$, $j \in \{0,1,\ldots,n\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$, $D^\alpha v(x,t) = D_x^\alpha v(x,t) = \frac{\partial^{|\alpha|} v(x,t)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, $D^{\bar{\alpha}} v(x,t) = (\frac{\partial}{\partial t})^{\alpha_0} D^\alpha v(x,t)$. $S(\mathbb{R}^n)$ is the space of indefinitely differentiable functions $v$ in $\mathbb{R}^n$ such that $x^\gamma D^\alpha v$ are bounded in $\mathbb{R}^n$ for all multi-indexes $\alpha$, $\gamma$ (the Schwartz space of smooth rapidly decreasing functions), $S_\gamma(\mathbb{R}^n)$, $\gamma > 0$, is the space of type $S(\mathbb{R}^n)$ (see [20, p. 201]), defined as

$$S_\gamma(\mathbb{R}^n) = \big\{ v \in S(\mathbb{R}^n) : |D^\alpha v(x)| \le C_\alpha e^{-a|x|^{\frac{1}{\gamma}}}, \;\; x \in \mathbb{R}^n, \quad \forall\, \alpha \big\}$$

with some positive constants $C_\alpha = C_\alpha(v)$ and $a = a(v)$. For the fixed $a > 0$ define

$$S_{\gamma,(a)}(\mathbb{R}^n) = \big\{ v \in S_\gamma(\mathbb{R}^n) : |D^\alpha v(x)| \le C_{\alpha,\delta}(v) e^{-(a-\delta)|x|^{\frac{1}{\gamma}}}, \quad x \in \mathbb{R}^n, \quad \forall \alpha, \;\; \forall\, \delta > 0 \big\},$$

$C^{\infty,(0)}(\bar{Q}) = \{ v \in C^\infty(\overline{Q}) : (\frac{\partial}{\partial t})^k v|_{t=T} = 0, \; k \in \mathbb{Z}_+ \}$. $S(\bar{Q})$ ($S_\gamma(\bar{Q})$, $S_{\gamma,(a)}(\bar{Q})$) is the space of functions $v \in C^{\infty,(0)}(\bar{Q})$ such that $(\frac{\partial}{\partial t})^s v(\cdot,t) \in S(\mathbb{R}^n)$ ($S_\gamma(\mathbb{R}^n)$, $S_{\gamma,(a)}(\mathbb{R}^n)$, respectively) for each $t \in [0,T]$, $s \in \mathbb{Z}_+$. The symbol $(f,\varphi)$ stands for the value of the distribution $f$ on the test function $\varphi$. Similarly to [21, p. 209], we introduce the space

$$S'_{\gamma,(a),C}(\bar{Q}) = \big\{ f \in S'_{\gamma,(a)}(\bar{Q}) : \; \big(f(x,\cdot), \varphi(x)\big) \in C[0,T] \quad \forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n) \big\}, \; a > 0.$$

We denote by $(g\hat{*}\varphi)(x) = \big(g(\xi), \varphi(x+\xi)\big)$ the convolution of the distribution $g$ and the test function $\varphi$, by $f*g$ the convolution of the distributions $f$ and $g$, namely $(f*g, \varphi) = (f, g\hat{*}\varphi)$ for any test function $\varphi$. We use the function

$$f_\lambda(t) = \frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)} \text{ for } \lambda > 0 \quad \text{and} \quad f_\lambda(t) = f'_{1+\lambda}(t) \text{ for } \lambda \le 0,$$

where $\Gamma(\lambda)$ is the Gamma-function, $\theta(t)$ is the Heaviside function. Note that $f_\lambda * f_\mu = f_{\lambda+\mu}$ and $f_\lambda \hat{*} f_\mu = f_{\lambda+\mu}$.

The Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta > 0$ is defined by the formula

$$v^{(\beta)}(t) = f_{-\beta}(t) * v(t),$$

the Djrbashian-Caputo (regularized) fractional derivative of order $\beta \in (0,1)$ is defined by

$$D^\beta v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} v'(\tau)\, d\tau,$$

and therefore

$$D^\beta v(t) = v^{(\beta)}(t) - f_{1-\beta}(t) v(0).$$

We denote

$$(Lv)(x,t) \equiv v_t^{(\beta)}(x,t) - (Av)(x,t),$$
$$(L^{reg}v)(x,t) \equiv D_t^\beta v(x,t) - (Av)(x,t),$$
$$(\widehat{L}v)(x,t) \equiv f_{-\beta}(t)\hat{*}v(x,t) - (Av)(x,t), \quad (x,t) \in Q.$$

The following Green formula

$$\int_Q v(x,\tau)(\widehat{L}\psi)(x,\tau)\, dx\, d\tau = \int_Q (L^{reg}v)(x,\tau)\psi(x,\tau)\, dx\, d\tau + \int_Q v(x,0) f_{1-\beta}(\tau)\psi(x,\tau)\, dx\, d\tau$$

holds, where $v, \psi \in S(\bar{Q})$.

**Definition 1.** *The function $u \in S'_{\gamma,(a),C}(\bar{Q})$ is called a solution of the Cauchy problem (1), (2) if the identity*

$$\begin{aligned} &\int_0^T \Big[\big(u(\cdot,t), (\widehat{L}\psi)(\cdot,t)\big) + r(t)\big(u(\cdot,t), \psi(\cdot,t)\big)\Big]\, dt \\ &\qquad = \int_0^T \big[g(t)\big(F_0(\cdot,t), \psi(\cdot,t)\big) + \big(F(\cdot,t), \psi(\cdot,t)\big)\big]\, dt + \big(F_1(y) f_{1-\beta}(t), \psi(y,t)\big) \end{aligned} \tag{4}$$

*holds for all $\psi \in S_{\gamma,(a)}(\bar{Q})$.*

**Definition 2.** *The triple $(u, r, g) \in S'_{\gamma,(a),C}(\bar{Q}) \times C[0,T] \times C[0,T]$ is called a solution of the problem (1)–(3) if the identity (4) and the condition (3) hold.*

It follows from (2) and (3) the compatibility conditions

$$(F_1, \varphi_1) = \Phi_1(0), \quad (F_1, \varphi_2) = \Phi_2(0). \tag{5}$$

**Definition 3.** *The vector-function $\big(G_0(x,t), G_1(x,t)\big)$ is called a Green vector-function of the Cauchy problem (2) to the equation $(Lu)(x,t) = \Phi(x,t)$, $(x,t) \in Q$, if under rather regular $\Phi$, $F_1$ the function*

$$u(x,t) = \int_0^t d\tau \int_{\mathbb{R}^n} G_0(x-y, t-\tau)\Phi(y,\tau)\,dy + \int_{\mathbb{R}^n} G_1(x-y,t)F_1(y)\,dy, \quad (x,t) \in \overline{Q},$$

*is the regular solution of this problem.*

Such Green vector-function exists (see, for example, [6]) and has the following bounds:

$$|G_0(x,t)| \le Ct^{-\frac{\beta n}{2b}+\beta-1} e^{-c(|x|t^{-\frac{\beta}{2b}})^{\frac{2b}{2b-\beta}}} \Psi_{n-2b}(|x|t^{-\frac{\beta}{2b}}),$$
$$|G_1(x,t)| \le Ct^{-\frac{\beta n}{2b}} e^{-c(|x|t^{-\frac{\beta}{2b}})^{\frac{2b}{2b-\beta}}} \Psi_{n-2b}(|x|t^{-\frac{\beta}{2b}}),$$

where $\Psi_m(z) = \begin{cases} 1, & m<0, \\ 1+|ln|z||, & m=0, \\ |z|^{-m}, & m>0 \end{cases}$ for $|z|<1$ and $\Psi_m(z) = \Psi_m(1)$ for $|z|>1$.

In what follows, $c$, $C$, $c_k$, $\widehat{c}_k$, $d_k$, $\widehat{d}_k$, $C_k$, $\widehat{C}_k$ are positive constants for $k \in \mathbb{Z}_+$. Let

$$(\widehat{\mathcal{G}}_0\varphi)(y,\tau) = \int_\tau^T dt \int_{\mathbb{R}^n} \varphi(x,t)G_0(x-y,t-\tau)\,dx, \quad (y,\tau) \in \bar{Q},$$
$$(\widehat{\mathcal{G}}_1\varphi)(y) = \int_0^T dt \int_{\mathbb{R}^n} \varphi(x,t)G_1(x-y,t)\,dx, \quad y \in \mathbb{R}^n,$$
$$(\widehat{G_j}\varphi)(y,t) = \int_{\mathbb{R}^n} \varphi(x)G_j(x-y,t)\,dx, \quad (y,t) \in \bar{Q}, \ \ j = \overline{0,1}.$$

**Lemma 1** ([16]). *If $a>0$, $\gamma \ge 1-\frac{\beta}{2b}$, $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, then there exist numbers $C>0$, $a' \in (0,a]$ ($a' = a$ if $0 < aT^{\frac{\beta}{2b\gamma}} \le c$) such that for all $k \in \mathbb{Z}_+$, multi-index $\kappa$, $|\kappa| = k$, $\delta>0$ the following bounds hold:*

$$\left|D_y^\kappa(\widehat{G_0}\varphi)(y,t)\right| \le c_k t^{\beta-1} e^{-(a'-\delta)|y|^{\frac{1}{\gamma}}} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha\varphi(x)| e^{(a-\delta)|x|^{\frac{1}{\gamma}}}, \quad (y,t) \in Q,$$
$$\left|D_y^\kappa(\widehat{G_1}\varphi)(y,t)\right| \le c_k e^{-(a'-\delta)|y|^{\frac{1}{\gamma}}} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha\varphi(x)| e^{(a-\delta)|x|^{\frac{1}{\alpha}}}, \quad (y,t) \in \bar{Q}.$$

## 2 Existence and uniqueness theorems

**Lemma 2.** *Assume that $\gamma \ge 1$, $0 < aT^{\frac{\beta}{2b\gamma}} \le c$, $F, F_0 \in S'_{\gamma,(a),C}(\bar{Q})$, $F_1 \in S'_{\gamma,(a)}(\mathbb{R}^n)$, $\varphi_j, A\varphi_j \in S_{\gamma,(a)}(\mathbb{R}^n)$, $\Phi_j, \Phi_j^{(\beta)} \in C[0,T]$, $j = 1,2$, the conditions* (5) *hold,*

$$d(t) := \Phi_1(t)(F_0,\varphi_2) - \Phi_2(t)(F_0,\varphi_1) \ne 0, \quad t \in [0,T].$$

*The triple $(u,r,g) \in S'_{\gamma,(a),C}(\bar{Q}) \times C[0,T] \times C[0,T]$ is the solution of the problem* (1)–(3) *if and only if $u$ is the solution of the equation*

$$\begin{aligned}\Big(u(\cdot,t),\varphi(\cdot)\Big) = &-\int_0^t \Big[\big(F_0(\cdot,\tau),\varphi_2(\cdot)\big)\big(u(\cdot,\tau),A\varphi_1(\cdot)\big)\\
&\qquad - \big(F_0(\cdot,\tau),\varphi_1(\cdot)\big)\big(u(\cdot,\tau),A\varphi_2(\cdot)\big)\Big]\big(u(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\frac{d\tau}{d(\tau)}\\
&+\int_0^t h(\tau)\big(u(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\frac{d\tau}{d(\tau)}\\
&+\int_0^t \Big[\Phi_2(\tau)\big(u(\cdot,\tau),A\varphi_1(\cdot)\big)\\
&\qquad - \Phi_1(\tau)\big(u(\cdot,\tau),A\varphi_2(\cdot)\big)\Big]\big(F_0(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\frac{d\tau}{d(\tau)}\\
&+\big(u_0(\cdot,t),\varphi(\cdot)\big) \qquad \forall\varphi\in S_{\gamma,(a)}(\mathbb{R}^n),\ t\in[0,T]\end{aligned} \tag{6}$$

*where*

$$\begin{aligned}\Big(u_0(\cdot,t),\varphi(\cdot)\Big) = &\int_0^t \big(F(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\,d\tau + \big(F_1(\cdot),(\widehat{G}_1\varphi)(\cdot,t)\big)\\
&+\int_0^t h_0(\tau)\big(F_0(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\frac{d\tau}{d(t)} \qquad \forall\varphi\in S_{\gamma,(a)}(\mathbb{R}^n),\ t\in[0,T],\end{aligned}$$

$$h_0(\tau)=\Phi_1(\tau)\big[\Phi_2^{(\beta)}(\tau)-\big(F(\cdot,\tau),\varphi_2(\cdot)\big)\big]-\Phi_2(\tau)\big[\Phi_1^{(\beta)}(\tau)-\big(F(\cdot,\tau),\varphi_1(\cdot)\big)\big],\quad \tau\in[0,T],$$

$$h(\tau)=(F_0,\varphi_2)\big[\Phi_1^{(\beta)}(\tau)-\big(F(\cdot,\tau),\varphi_1(\cdot)\big)\big]-(F_0,\varphi_1)\big[\Phi_2^{(\beta)}(\tau)-\big(F(\cdot,\tau),\varphi_2(\cdot)\big)\big],\quad \tau\in[0,T]$$

*and*

$$\begin{aligned}r(t) &= \frac{1}{d(t)}\Big\{\big(F_0(\cdot,t),\varphi_2(\cdot)\big)\big[\Phi_1^{(\beta)}(t)-\big(u(\cdot,t),A\varphi_1(\cdot)\big)-\big(F(\cdot,t),\varphi_1(\cdot)\big)\big]\\
&\qquad -\big(F_0(\cdot,t),\varphi_1(\cdot)\big)\big[\Phi_2^{(\beta)}(t)-\big(u(\cdot,t),A\varphi_2(\cdot)\big)-\big(F(\cdot,t),\varphi_2(\cdot)\big)\big]\Big\},\\
g(t) &= \frac{1}{d(t)}\Big\{\Phi_1(t)\big[\Phi_2^{(\beta)}(t)-\big(u(\cdot,t),A\varphi_2(\cdot)\big)-\big(F(\cdot,t),\varphi_2(\cdot)\big)\big]\\
&\qquad -\Phi_2(t)\big[\Phi_1^{(\beta)}(t)-\big(u(\cdot,t),A\varphi_1(\cdot)\big)-\big(F(\cdot,t),\varphi_1(\cdot)\big)\big]\Big\}, \qquad t\in[0,T].\end{aligned} \tag{7}$$

*Proof.* We wright the equation (1) as follows

$$u(x,t)=f_\beta(t)*(A(x,D)u)(x,t)+f_\beta(t)*\big(r(t)u(x,t)\big)+(f_\beta*g)(t)F_0(x)+f_\beta(t)*F(x,t).$$

By using the condition (3) we get

$$\begin{aligned}\Phi_j(t)=&f_\beta(t)*\big(u(\cdot,t),A\varphi_j(\cdot)\big)+f_\beta(t)*\big(r(t)\Phi_j(t)\big)\\
&+f_\beta*\big(g(t)F_0(\cdot,t),\varphi_j(\cdot,t)\big)+f_\beta(t)*\big(F(\cdot,t),\varphi_j(\cdot)\big), \qquad j=1,2,\end{aligned}$$

and therefore,

$$\Phi_j^{(\beta)}(t)=\big(u(\cdot,t),A\varphi_j(\cdot)\big)+r(t)\Phi_j(t)+g(t)\big(F_0(\cdot,t),\varphi_j(\cdot)\big)+\big(F(\cdot,t),\varphi_j(\cdot)\big), \qquad t\in[0,T].$$

Under the condition $d(t)\neq 0$, $t\in[0,T]$, from here we find $r(t), g(t)$, as in (7).

By assumptions, $r, g \in C[0,T]$ for each $u \in S'_{\gamma,(a),C}(\bar{Q})$. Repeating the proof of Theorem 1 from [16] we get that $u \in S'_{\gamma,(a),C}(\bar{Q})$ is the solution of the Cauchy problem (1), (2) if and only if it is the solution of the equation

$$\Big(u(\cdot,t),\varphi(\cdot)\Big)=\int_0^t\Big(F(\cdot,\tau)+r(\tau)u(\cdot,\tau)+g(\tau)F_0(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\Big)d\tau+\Big(F_1(\cdot),(\widehat{G}_1\varphi)(\cdot,t)\Big)$$

for all $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, $t \in [0,T]$, that is

$$\begin{aligned}\Big(u(\cdot,t),\varphi(\cdot)\Big) &= \int_0^t r(\tau)\Big(u(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\Big)\,d\tau \\ &\quad+\int_0^t \Big(F(\cdot,\tau)+g(\tau)F_0(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\Big)\,d\tau+\Big(F_1(\cdot),(\widehat{G}_1\varphi)(\cdot,t)\Big)\end{aligned} \tag{8}$$

for all $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, $t \in [0,T]$. Substituting (7) in (8) we get (6) for the solution of the problem (1)–(3). □

It was shown in [16] that $f \in S'_{\gamma,(a)}(\mathbb{R}^n)$ if and only if there exist constants $C_a > 0$, $k = k(a) \in \mathbb{N}$, $k \geq 2$, such that

$$|(f,\varphi)| \leq C_a\|\varphi\|_{k,a} \quad \forall\varphi \in S_{\gamma,(a)}(\mathbb{R}^n),$$

where $\|\varphi\|_{k,a} = \sup\limits_{|\alpha|\leq k,\ x\in\mathbb{R}^n} e^{a(1-\frac{1}{k})|x|^{\frac{1}{\gamma}}}|D^\alpha\varphi(x)| < +\infty$.

The number $k$ can be called the order of $f \in S'_{\gamma,(a)}(\mathbb{R}^n)$.

**Theorem 1.** *In assumptions of Lemma 2 there exists $T > 0$ and the solution $(u,r,g) \in S'_{\gamma,(a),C}(\bar{Q}) \times C[0,T] \times C[0,T]$ of the problem* (1)–(3) *such that $u$ is the solution of the equation* (6)*, $r(t)$ and $g(t)$ are defined by* (7)*.*

*Proof.* According to Lemma 2, in order to prove the existence of the problem (1)–(3) it is sufficient to prove the solvability of the equation (6) in $S'_{\gamma,(a),C}(\bar{Q})$.

In assumptions of the theorem, using Lemma 1, for all $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, $t \in [0,T]$ we get

$$\begin{aligned}\int_0^t \big|\big(F(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\big|\,d\tau &\leq C_a\int_0^t \|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\hat{k},a}\,d\tau \\ &\leq C_1\int_0^t (t-\tau)^{\beta-1}\,d\tau\,\|\varphi\|_{k_0,a} = \frac{C_1}{\beta}t^\beta\,\|\varphi\|_{\hat{k},a},\end{aligned}$$

$$\int_0^t |h_0(\tau)\big(F_0(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)|\,d\tau \leq C_2\,\|\varphi\|_{k_0,a},$$

$$\big|\big(F_1(\cdot),(\widehat{G}_1\varphi)(\cdot,t)\big)\big| \leq C_3\|(\widehat{G}_1\varphi)(\cdot,t)\|_{k_1,a} \leq C_4\,\|\varphi\|_{k_1,a}$$

if $F(\cdot,t)$, $F_0(\cdot,t)$ have the orders $\hat{k}, k_0$, respectively, for each $t \in [0,T]$, $F_1$ has the order $k_1$ as distributions from $S'_{\gamma,(a)}(\mathbb{R}^n)$. Then $u_0 \in S'_{\gamma,(a),C}(\bar{Q})$ and it has the order $K = \max\{\hat{k}, k_0, k_1\}$ for each $t \in [0,T]$. So, the solution of the equation (6) (if exists) has the order $K$. We put $k \geq K$,

$$M_{k,(a)} = M_{k,(a)}(T) = \Big\{v \in S'_{\gamma,(a),C}(\bar{Q}) : \|v\|_{M_{k,(a)}} = \max_{t\in[0,T]}\ \sup_{\substack{\varphi\in S_{\gamma,(a)}(\mathbb{R}^n),\\ \|\varphi\|_{k,(a)}=1}} \big|\big(v(\cdot,t),\varphi(\cdot)\big)\big| < +\infty\Big\},$$

$$M_{k,(a),R}(T) = \{v \in M_{k,(a)}(T) : \|v\|_{M_{k,(a)}} \leq R\}$$

and prove the unique solvability of the equation (6) in the space $M_{k,a,R}(T) \subset M_{k,a}(T) \subset S'_{\gamma,(a),C}(\bar{Q})$ for each $k \geq K$ with some $R > 0$, $T > 0$.

On $M_{k,(a)}$ we consider the operator $P$:

$$\begin{aligned}
\big((Pv)(\cdot,t),\varphi(\cdot)\big) = &-\int_0^t \Big[\big(F_0(\cdot,\tau),\varphi_2(\cdot)\big)\big(v(\cdot,\tau),A\varphi_1(\cdot)\big) - \big(F_0(\cdot,\tau),\varphi_1(\cdot)\big)\big(v(\cdot,\tau),A\varphi_2(\cdot)\big)\Big] \\
&\qquad \times \big(v(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\frac{d\tau}{d(\tau)} \\
&-\int_0^t h(\tau)\big(v(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\frac{d\tau}{d(\tau)} \\
&+\int_0^t \big[\Phi_2(\tau)\big(v(\cdot,\tau),A\varphi_1(\cdot)\big) - \Phi_1(\tau)\big(v(\cdot,\tau),A\varphi_2(\cdot)\big)\big] \\
&\qquad \times \big(F_0(\cdot,\tau),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\frac{d\tau}{d(\tau)} \\
&+\big(u_0(\cdot,t),\varphi(\cdot)\big) \qquad \forall\varphi \in S_{\gamma,(a)}(\mathbb{R}^n),\ \ v \in M_{k,(a)},\ t \in [0,T].
\end{aligned}$$

Then for each $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, $v \in M_{k,(a),R}$, $t,\tau \in [0,T]$, by using Lemma 1 we obtain

$$\begin{aligned}
\frac{|\big((Pv)(\cdot,t),\varphi(\cdot)\big)|}{\|\varphi\|_{k,a}} &\leq C_5 t^\beta\big[(R^2+R)\big(\|A\varphi_1\|_{k,a}+\|A\varphi_2\|_{k,a}\big)+R\big]+\|u_0\|_{M_{k,(a)}} \\
&\leq C_6 t^\beta(R^2+R)+\|u_0\|_{M_{k,(a)}}.
\end{aligned}$$

We first choose $R > 2\|u_0\|_{M_{k,(a)}}$ and $t_1 \in (0,T]$, $t_1^\beta < \frac{1}{4C_6}$, such that

$$C_6 t^\beta R + \|u_0\|_{M_{k,(a)}} < \frac{R}{2} \quad \forall t \in [0,t_1].$$

Then for such $R$, $t \in [0,t_1]$ we obtain

$$\frac{|\big((Pv)(\cdot,t),\varphi(\cdot)\big)|}{\|\varphi\|_{k,a}} < C_6 t^\beta R^2 + \frac{R}{2}.$$

Now we choose $R > 2\|u_0\|_{M_{k,(a)}}$ and $t_2 \leq t_1$ such that $C_6 t^\beta R^2 \leq \frac{R}{2}$, that is $2C_6 R t^\beta \leq 1$. Then for each $v \in M_{k,(a),R}$, $0 < t \leq \min\{t_1,t_2\}$ we get $\|Pv\|_{M_{k,(a)}} < R$.

Similarly, for each $v_1, v_2 \in M_{k,(a),R}$ we have

$$\begin{aligned}
\frac{|\big(Pv_1)(\cdot,t)-\big(Pv_2)(\cdot,t),\varphi(\cdot)\big)|}{\|\varphi\|_{k,a}} &\leq C_7 t^\beta R\big(\|A\varphi_1\|_{k,a}+\|A\varphi_2\|_{k,a}\big)\|v_1-v_2\|_{M_{k,(a)}} \\
&\leq C_8 t^\beta R\|v_1-v_2\|_{M_{k,(a)}}
\end{aligned}$$

and for $0 < t \leq t_3 \leq \min\{t_1,t_2\}$ we get $\|Pv_1 - Pv_2\|_{M_{k,(a)}} < \|v_1 - v_2\|_{M_{k,(a)}}$.

So, $P$ is the contraction operator on $M_{k,(a),R}(T)$ with some $T \leq t_3$, and by the Banach theorem we obtain the unique solvability of the equation (6) in $M_{k,(a),R}(T)$. □

**Theorem 2.** *Let* $\varphi_j$, $A\varphi_j \in S_{\gamma,(a)}(\mathbb{R}^n)$, $\Phi_j$, $\Phi_j^{(\beta)} \in C[0,T]$, $j=1,2$, $F_0 \in S'_{\gamma,(a)}(\mathbb{R}^n)$ *and* $d(t)\neq 0$, $t\in[0,T]$. *Then the solution of the problem* (1)–(3) *is unique.*

*Proof.* Let $(u_1,r_1,g_1)$, $(u_2,r_2,g_2)$ be two solutions of the problem (1)–(3). Then for $u=u_1-u_2$, $r=r_1-r_2$, $g=g_1-g_2$ we obtain the problem

$$Lu(x,t)=r(t)u_1+r_2(t)u+g(t)F_0(x,t),\quad (x,t)\in Q,$$
$$u(x,0)=0,\quad x\in\mathbb{R}^n,$$
$$\big(u(\cdot,t),\varphi_1(\cdot)\big)=0,\quad \big(u(\cdot,t),\varphi_2(\cdot)\big)=0,\quad t\in[0,T].$$

As in the proof of Theorem 1, using (3), we find

$$\begin{aligned} r(t)&=-\frac{1}{d(t)}\big[\big(F_0(\cdot,t),\varphi_2(\cdot)\big)\big(u(\cdot,t),A\varphi_1(\cdot)\big)-\big(F_0(\cdot,t),\varphi_1(\cdot)\big)\big(u(\cdot,t),A\varphi_2(\cdot)\big)\big],\\ g(t)&=\frac{1}{d(t)}\big[\Phi_2(t)\big(u(\cdot,t),A\varphi_1(\cdot)\big)-\Phi_1(t)\big(u(\cdot,t),A\varphi_2(\cdot)\big)\big],\quad t\in[0,T]. \end{aligned}\tag{9}$$

By Definitions 1 and 2 the identity

$$\int_0^T \big(u(\cdot,t),(\widehat{L}\psi)(\cdot,t)+r_2(t)\psi(\cdot,t)\big)\,dt=\int_0^T \big(r(t)u_1(\cdot,t)+g(t)F_0(\cdot,t),\psi(\cdot,t)\big)\,dt \tag{10}$$

holds for all $\psi\in S_{\gamma,(a)}(\bar{Q})$. According to [16, Lemma 4], for each $\rho\in S_{\gamma,(a)}(\bar{Q})$ there exists $\psi=\widehat{\mathcal{G}}_0\rho\in S_{\gamma,(a)}(\bar{Q})$ such that $(\widehat{L}\psi)(x,t)=\rho(x,t)$, $(x,t)\in\bar{Q}$. Then (10) implies

$$\int_0^T \big(u(\cdot,t),\rho(\cdot,t)+r_2(t)(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big)\,dt=\int_0^T \big(r(t)u_1(\cdot,t)+g(t)F_0(\cdot),(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big)\,dt$$

for all $\rho\in S_{\gamma,(a)}(\bar{Q})$.

Using (9), from here we get

$$\begin{aligned} \int_0^T \big(u(\cdot,t),\rho(\cdot,t)+r_2(t)(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big)\,dt&=-\int_0^T \big[\big(F_0(\cdot,t),\varphi_2(\cdot)\big)\big(u(\cdot,t),A\varphi_1(\cdot)\big)\\ &\quad-\big(F_0(\cdot,t),\varphi_1(\cdot)\big)\big(u(\cdot,t),A\varphi_2(\cdot)\big)\big]\big(u_1(\cdot,t),(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big)\frac{dt}{d(t)}\\ &\quad+\int_0^T \big[\Phi_2(t)\big(u(\cdot,t),A\varphi_1(\cdot)\big)-\Phi_1(t)\big(u(\cdot,t),A\varphi_2(\cdot)\big)\big]\\ &\quad\times\big(F_0(\cdot,t),(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big)\frac{dt}{d(t)} \end{aligned}$$

for all $\rho\in S_{\gamma,(a)}(\bar{Q})$, that is

$$\begin{aligned} \int_0^T \Big\{\Big(u(\cdot,t),\rho(\cdot,t)&+r_2(t)(\widehat{\mathcal{G}}_0\rho)(\cdot,t)+\frac{1}{d(t)}\big[\big(F_0(\cdot,t),\varphi_2(\cdot)\big)A\varphi_1(\cdot)\\ &-\big(F_0(\cdot,t),\varphi_1(\cdot)\big)A\varphi_2(\cdot)\big]\big(u_1(\cdot,t),(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big)\\ -\frac{1}{d(t)}\big[\Phi_2(t)A\varphi_1(\cdot)&-\Phi_1(t)A\varphi_2(\cdot)\big]\big(F_0(\cdot),(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big)\Big)\Big\}dt=0 \end{aligned}\tag{11}$$

for all $\rho\in \mathcal{S}_{\gamma,(a)}(\bar{Q})$.

By assumptions of the theorem, we have $A\varphi_j \in S_{\gamma,(a)}(\mathbb{R}^n)$, $j = 1,2$, $\widehat{\mathcal{G}}_0\rho \in S_{\gamma,(a)}(\bar{Q})$, $\big(u_1(\cdot,t),(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big) := p_1(t,\rho)$, $\big(F_0(\cdot,t),(\widehat{\mathcal{G}}_0\rho)(\cdot,t)\big) := p_2(t,\rho)$ are the known functions from $C[0,T]$. Then

$$\begin{aligned}\rho(\cdot,t) + r_2(t)(\widehat{\mathcal{G}}_0\rho)(\cdot,t) + \frac{1}{d(t)}\Big\{ &\big[(F_0(\cdot,t),\varphi_2(\cdot)\big)A\varphi_1(\cdot)\\ &- \big(F_0(\cdot,t),\varphi_1(\cdot)\big)A\varphi_2(\cdot)\big]\, p_1(t,\rho)\\ &+ \big[\Phi_2(t)A\varphi_1(\cdot) - \Phi_1(t)A\varphi_2(\cdot)\big]\, p_2(t,\rho)\Big\} \in S_{\gamma,(a)}(\mathbb{R}^n)\end{aligned}$$

for all $t \in [0,T]$. For each $\psi(x,t) = \varphi(x)\mu(t)$ with $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, $\mu \in C^{\infty,(0)}[0,T]$ the linear Volterra equation

$$\begin{aligned}\rho(\cdot,t) + r_2(t)(\widehat{\mathcal{G}}_0\rho)(\cdot,t) + \frac{1}{d(t)}\Big\{ &\big[(F_0(\cdot,t),\varphi_2(\cdot)\big)A\varphi_1(\cdot) - \big(F_0(\cdot,t),\varphi_1(\cdot)\big)A\varphi_2(\cdot)\big]\, p_1(t,\rho)\\ &+ \big[\Phi_2(t)A\varphi_1(\cdot) - \Phi_1(t)A\varphi_2(\cdot)\big]\, p_2(t,\rho)\Big\} = \varphi(x)\mu(t)\end{aligned}$$

has the unique solution $\rho \in S_{\gamma,(a)}(\bar{Q})$. Then from (11) we get

$$\int_0^T \big(u(\cdot,t),\varphi(\cdot)\big)\mu(t)\,dt = 0 \quad \forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n),\ \mu \in C^{\infty,(0)}[0,T].$$

By Du Bois-Reymond lemma we obtain $u = 0$ in $S'_{\gamma,(a),C}(\bar{Q})$. Then (9) implies $r(t) = 0$, $g(t) = 0$, $t \in [0,T]$. $\square$

## Conclusions

We established the uniqueness and local solvability of the inverse problem of the determining dependent on time unknown young coefficient and component in the right-hand side of $2b$-order differential equation having fractional derivative of order $\beta \in (0,1)$ with respect to time and given Schwartz type distributions in the right-hand sides of the equation and the initial condition.

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Лопушанський А.О., Лопушанська Г.П. *Обернена задача з двома невідомими залежними від часу функціями для диференціального рівняння порядку* 2*b* *з дробовою похідною за часом* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 213–222.

Ми вивчаємо обернену задачу для диференціального рівняння порядку $2b$ з дробовою похідною Рімана-Ліувіля за часом і заданими узагальненими функціями типу Шварца у правих частинах рівняння і початкової умови. Невідомими є узагальнений розв'язок $u$ задачі Коші для такого рівняння (неперервний за часом у певному сенсі) і залежні від часу неперервний молодший коефіцієнт та компонента правої частини рівняння.

Додатково ми задаємо неперервні за часом значення $\Phi_j(t)$ шуканого узагальненого розв'язку $u$ задачі Коші на фіксованих основних функціях $\varphi_j(x)$, $x \in \mathbb{R}^n$, а саме $(u(\cdot,t),\varphi_j(\cdot)) = \Phi_j(t)$, $t \in [0,T]$, $j = 1,2$.

Знаходимо достатні умови єдиності узагальненого розв'язку оберненої задачі у всьому шарі $Q := \mathbb{R}^n \times (0,T]$ й існування розв'язку в деякому шарі $\mathbb{R}^n \times (0,T_0]$, $T_0 \in (0,T]$.

*Ключові слова і фрази:* розподіл, дробова похідна, обернена задача, вектор-функція Ґріна.