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Ulam type stability analysis for generalized proportional fractional differential equations

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The main aim of the current paper is to be appropriately defined several types of Ulam stability for non-linear fractional differential equation with generalized proportional fractional derivative of Riemann-Liouville type. In the new definitions, the initial values of the solutions of the given equation and the corresponding inequality could not coincide but they have to be closed enough. Some sufficient conditions for three types of Ulam stability for the studied equations are obtained, namely Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. Some of them are applied to a fractional generalization of a biological model.

Key words and phrases: generalized proportional fractional derivative, Mittag-Leffler function, Ulam type stability.

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1 Introduction

Fractional calculus has recently acquired plentiful circulation and great significance because of its applications in fields of science and engineering. Fractional differential equations appear strongly in the diffusion process, the process of dynamics, signal and image processing, etc. For instance, see the books [8, 23–26, 29, 30].

In recent years, there are honorable efforts for obtaining new classes of fractional operators by introducing more general or new kernels. F. Jarad et al. [18] introduced a new generalized proportional derivative which is well-behaved and has several advantages over the classical derivatives as meaning that it generalizes formerly known derivatives in the literature. For recent contributions relevant to fractional differential equations via generalized proportional derivatives, see [1–3, 10, 15].

On the other hand, stability analysis is one of the most important areas of interest by researchers of fractional differential equations. Stability allows us to contrast the comportment of solutions beginning at various points. The concept of Ulam stability was initially launched by S.M. Ulam [32], and then the contributions continued by D.H. Hyers [17] and Th.M. Rassias [27] in order to obtain significant improvements in this field. For more details on the recent advances on the topic, we refer to the monographs [16, 20] and the research papers [1, 3–5, 7, 9, 11, 12, 14, 21, 28, 31, 33, 34].

The aforementioned works inspires us, in the current paper, to discuss the stability analysis of an initial value problem (IVP for short) for nonlinear generalized proportional fractional

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differential equations of Riemann-Liouville fractional type (PIVP for short):

$$\begin{cases} \left({}^{R}_{a}\mathscr{D}^{\alpha,\rho}u\right)(t) = \lambda u(t) + \mathcal{G}\big(t, u(t)\big), & t \in [a,b], \\ \left({}_{a}\mathscr{I}^{1-\alpha,\rho}u\right)(a) = u_0, \end{cases} \tag{1}$$

where $u(\cdot) : [a,b] \to \mathbb{R}$, and $\rho \in (0,1]$, $\alpha \in (0,1)$, λ, u_0 are real constants, ${}^{R}_{a}\mathscr{D}^{\alpha,\rho}$ denotes the generalized proportional fractional derivative of Riemann-Liouville type of order α, ${}_{a}\mathscr{I}^{1-\alpha,\rho}$ denotes the generalized proportional fractional integral of order $1-\alpha$, and $\mathcal{G} : [a,b] \times \mathbb{R} \to \mathbb{R}$ is given continuous function.

We study three types of Ulam stability: Ulam-Hyers ($\mathcal{UH}$) stability, Ulam-Hyers-Rassias ($\mathcal{UHR}$) stability and generalized Ulam-Hyers-Rassias ($\mathcal{GUHR}$) stability for the PIVP (1).

Note Ulam type stability is studied in [31] for generalized proportional fractional differential equations of Caputo type. As it is well known in the literature, the initial conditions of any types of Caputo fractional differential equations are similar to the case of ordinary derivatives. But it is not the situation with the Riemann-Liouville type as the studied in this paper.

Following are the main contributions of the paper:

- definitions of the classical ones of Ulam type stabilities for nonlinear generalized proportional fractional differential equations, including closeness between the initial value of the solutions of the equation and the corresponding inequality, are generalized;

- some sufficient conditions for the generalized types of Ulam-Hyers stability, Ulam-Hyers-Rassia stability and generalized Ulam-Hyers-Rassias stability are obtained;

- application of the theoretical results for obtaining bounds of the solutions of a fractional generalization of a biological model is presented.

The paper is structured as follows. In Section 2, we recall some useful preliminaries and auxiliary results. In Section 3, three definitions for Ulam type stability are derived for nonlinear generalized proportional fractional differential equation. Finally, in order to confirm the validity of the theoretical findings, some of the obtained results are applied to a fractional generalization of a biological model in Section 4.

2 Preliminaries and auxiliary results

In this section, we recall some definitions and notations for generalized proportional fractional derivative and integral from [18,19].

Definition 1. *Take $\rho \in (0,1]$ and $\alpha > 0$. The generalized proportional fractional integral of order α of a function $v \in L^1([a,b])$ is defined by*

$$\left({}_{a}\mathscr{I}^{\alpha,\rho}v\right)(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\alpha-1} v(s)\, ds.$$

Definition 2. *Take $\rho \in (0,1]$ and $\alpha \in (0,1)$. The generalized proportional fractional derivative of Riemann-Liouville type of order α of a function v is defined by*

$$\begin{aligned}\left({}_a^R\mathscr{D}^{\alpha,\rho}v\right)(t) &= \frac{\mathscr{D}_t^{1,\rho}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}v(s)\,ds \\ &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\Bigg((1-\rho)\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}v(s)\,ds \\ &\qquad + \frac{d}{dt}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}v(s)\,ds\Bigg),\end{aligned}$$

where $\left(\mathscr{D}^{1,\rho}v\right)(t) = (1-\rho)v(t) + \rho v'(t)$.

We will provide the following preliminary result which is similar to [23, Lemma 3.2] for Riemann-Liouville fractional derivative.

Lemma 1 ([15]). *Let $\rho \in (0,1], \alpha \in (0,1)$ and $y(t) \in L^1([a,b],\mathbb{R})$. Then*

(i) if there exists a limit $\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}t}(t-a)^{1-\alpha}y(t)\right) = c \in \mathbb{R}$, then also exists a limit

$$\left({}_a\mathscr{I}^{1-\alpha,\rho}y\right)(a) := \lim_{t\to a+}\left({}_a\mathscr{I}^{1-\alpha,\rho}y\right)(t) = c\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}e^{\frac{\rho-1}{\rho}a};$$

(ii) if $\left({}_a\mathscr{I}^{1-\alpha,\rho}y\right)(a) = k \in \mathbb{R}$, and there exists the limit $\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}t}(t-a)^{1-\alpha}y(t)\right)$, then

$$\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}t}(t-a)^{1-\alpha}y(t)\right) = \frac{k\rho^{1-\alpha}e^{\frac{1-\rho}{\rho}a}}{\Gamma(\alpha)}.$$

Remark 1. *According to Lemma 1 the initial value condition in* (1) *could be replaced by*

$$\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}u(t)\right) = \frac{u_0\rho^{1-\alpha}}{\Gamma(\alpha)}.$$

Define the set

$$C_{1-\alpha,\rho}([a,b]) = \left\{x(t):\ (a,b]\to\mathbb{R}:\ x\in C((a,b],\mathbb{R}),\ \lim_{t\to a+} e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}x(t) < \infty\right\}$$

with the norm $\|x\|_{C_{1-\alpha,\rho}} = \max_{t\in[a,b]}\left|e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}x(t)\right|$. Note $C_{1-\alpha,\rho}([a,b])$ is a Banach space. If $u_n \in C_{1-\alpha,\rho}([a,b]), n = 1,2,\ldots,$ and $\|u_n - u\|_{C_{1-\alpha,\rho}[a,b]} \to 0$, then $u \in C_{1-\alpha,\rho}([a,b])$.

Consider the linear scalar fractional equation with generalized proportional fractional derivative and initial value conditions

$$\begin{cases}\left({}_a^R\mathscr{D}^{\alpha,\rho}u\right)(t) = \lambda u(t) + f(t), \quad t\in[a,b],\\ \left({}_a\mathscr{I}^{1-\alpha,\rho}u\right)(a) = u_0,\end{cases} \tag{2}$$

where $u(\cdot):[a,b]\to\mathbb{R}$, $f\in C([a,b])$ and $\rho\in(0,1]$, $\alpha\in(0,1)$, $\lambda,\ u_0$ are real constants.

Lemma 2 ([15]). *The PIVP* (2) *has a unique solution* $u \in C_{1-\alpha,\rho}([a,b])$ *given by*

$$u(t) = u_0 e^{(\rho-1)\frac{t-a}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1} + \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right) f(s)\, ds, \quad t \in (a,b],$$

where $E_{p,q}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jp+q)}$ *is the Mittag-Leffler function with two parameters (see, e.g., [23]).*

Based on Lemma 2, it follows the following integral presentation of a solution of PIVP (1).

Lemma 3. *Let the function* $x \in C_{1-\alpha,\rho}([a,b])$ *be a solution of PIVP* (1)*. Then it satisfies the integral equality*

$$x(t) = u_0 e^{(\rho-1)\frac{t-a}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1} + \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right) \mathcal{G}(s, x(s))\, ds, \quad t \in (a,b]. \tag{3}$$

Proposition 1. *Let* λ *be a real number, and* $\alpha \in (0,1), \rho \in (0,1]$*. Then the inequality*

$$\int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right) ds \le \frac{\rho^{\alpha}}{|\lambda|}\left| E_{\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right) - 1\right|, \quad t \in [a,b],$$

holds.

Proof. Using the definition for the Mittag-Leffler function with two parameters and $0 < \rho < 1$, we obtain

$$\begin{aligned}\int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right) ds &\le \int_a^t (t-s)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right)^n}{\Gamma((n+1)\alpha)} ds \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n \int_a^t (t-s)^{(n+1)\alpha-1} ds}{\rho^{n\alpha}\Gamma((n+1)\alpha)} = \sum_{n=0}^{\infty} \frac{\lambda^n (t-a)^{(n+1)\alpha}}{\rho^{n\alpha}(n+1)\alpha\Gamma((n+1)\alpha)} \\ &= \sum_{n=0}^{\infty} \frac{\rho^{\alpha}\lambda^{n+1}(t-a)^{(n+1)\alpha}}{\lambda\rho^{n\alpha+\alpha}\Gamma((n+1)\alpha+1)} = \frac{\rho^{\alpha}}{\lambda}\sum_{n=1}^{\infty} \frac{\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)^n}{\Gamma(n\alpha+1)} \le \frac{\rho^{\alpha}}{|\lambda|}\left| E_{\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right) - 1\right|.\end{aligned}$$

The case of $\rho = 1$ is obvious. □

Proposition 2 ([23]). *For* $q \in (0,1)$*, the following properties*

$$0 \le E_{q,q}(-\lambda t^q) \le \frac{1}{\Gamma(q)}, \quad t \ge 0,\ \lambda \ge 0,$$

$$\lim_{t \to 0+} E_{q,q}(-\lambda t^q) = E_{q,q}(0) = \frac{1}{\Gamma(q)}$$

hold.

Our main proofs are based on the appropriate modification of the classical Gronwall inequality for generalized proportional fractional integral. We will set up only the results without the proof one of the inequality in [6, Corollary 3].

Proposition 3. *Let $\alpha > 0, \rho \in (0,1]$. Let $u(t), v(t)$ be nonnegative locally integrable on $[a,b]$ functions, and the function $v(t)$ is nondecreasing. Let $B > 0$ be a real constant and*

$$u(t) \leq v(t) + B \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\alpha-1} u(s)\, ds, \quad t \in [a,b].$$

Then

$$u(t) \leq v(t) E_\alpha \big(B\Gamma(\alpha)(t-a)^\alpha \big), \quad t \in [a,b].$$

3 Ulam type stability

We study three types of Ulam stability: Ulam-Hyers ($\mathcal{UH}$) stability, Ulam-Hyers-Rassias ($\mathcal{UHR}$) stability and generalized Ulam-Hyers-Rassias ($\mathcal{GUHR}$) stability for the PIVP (1). The definitions for the studied differential equation with generalized proportional fractional derivative are appropriately changed comparatively to those for ordinary differential equations given in [28]. Note in the case of ordinary derivatives as well as Caputo type fractional derivatives, the initial value of the corresponding equation could be the same as the value of the approximating function (the solution of the inequality), but in the case of Riemann-Liouville type fractional derivatives the initial value of the solution has to be enough close to the one of the solution of the inequality.

We will consider the following assumptions.

(A1) Let $\mathcal{G} \in C\left([a,b] \times \mathbb{R}, \mathbb{R}\right)$ and there exists a constant $L > 0$ such that

$$\big|\mathcal{G}(t,u_1) - \mathcal{G}(t,u_2)\big| \leq L|u_1 - u_2|, \quad t \in [a,b], \ u_1, u_2 \in \mathbb{R}.$$

(A2) For any initial value $u_0 \in \mathbb{R}$ the PIVP (1) has a solution.

Remark 2. *The existence of PIVP* (1) *is studied in [22] and some conditions for existence and uniqueness are obtained there (see [22, Theorem 4.3 and Remark 4.8]).*

Let $\varepsilon > 0$ and $\varphi : [a,b] \to [0,\infty)$ be non-decreasing function such that for any $t \in [a,b]$ the inequality $\int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds < \infty$ holds.

Definition 3. *The PIVP* (1) *is Ulam-Hyers stable stable if there exists a real number $a_{\mathcal{G}} > 0$ such that for each $\varepsilon > 0$ and for each solution $w \in \mathcal{C}_{1-\alpha,\rho}\big([a,b]\big)$ of the inequality*

$$\Big| \left({}_a^R\mathscr{D}^{\alpha,\rho} w \right)(t) - \lambda w(t) - \mathcal{G}\big(t, w(t)\big) \Big| \leq \varepsilon, \tag{4}$$

there exists a solution $u \in \mathcal{C}_{1-\alpha,\rho}\big([a,b]\big)$ of PIVP (1) *with $|u_0 - w_0| \leq a_{\mathcal{G}}\varepsilon$, where $w_0 = \lim\limits_{t\to a+} e^{\frac{1-\rho}{\rho}(t-a)} (t-a)^{1-\alpha} w(t)$, such that*

$$\big|w(t) - u(t)\big| \leq a_{\mathcal{G}}\varepsilon, \quad t \in [a,b].$$

Definition 4. *The PIVP* (1) *is Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $a_{\mathcal{G},\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $w \in \mathcal{C}_{1-\alpha,\rho}([a,b])$ of the inequality*

$$\left| \left({}_a^R\mathscr{D}^{\alpha,\rho} w \right)(t) - \lambda w(t) - \mathcal{G}\big(t, w(t)\big) \right| \le \varepsilon \varphi(t) \tag{5}$$

there exists a solution $u \in \mathcal{C}_{1-\alpha,\rho}([a,b])$ of PIVP (1) with $|u_0 - w_0| \le \varepsilon a_{\mathcal{G},\varphi}\, \varphi(a)$, where $w_0 = \lim_{t\to a+} e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha} w(t)$, such that

$$\big|w(t) - u(t)\big| \le \varepsilon\, a_{\mathcal{G},\varphi}\, \varphi(t), \quad t \in [a,b].$$

Definition 5. *The PIVP* (1) *is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists a real number $a_{\mathcal{G},\varphi} > 0$ such that for each solution $w \in \mathcal{C}_{1-\alpha,\rho}([a,b])$ of the inequality*

$$\left| \left({}_a^R\mathscr{D}^{\alpha,\rho} w \right)(t) - \lambda w(t) - \mathcal{G}\big(t, w(t)\big) \right| \le \varphi(t) \tag{6}$$

there exists a solution $u \in \mathcal{C}_{1-\alpha,\rho}([a,b])$ of PIVP (1) *with $|u_0 - w_0| \le a_{\mathcal{G},\varphi}\varphi(a)$, where $w_0 = \lim_{t\to a+} e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha} w(t)$, such that*

$$\big|w(t) - u(t)\big| \le a_{\mathcal{G},\varphi}\varphi(t), \quad t \in [a,b].$$

Remark 3. *A function $w \in \mathcal{C}_{1-\alpha,\rho}([a,b])$ is a solution of the inequality* (4) *if and only if there exist a function $h \in C([a,b],\mathbb{R})$, which depend on w, such that*

(i) $\big|h(t)\big| \le \varepsilon$,

(ii) $\left({}_a^R\mathscr{D}^{\alpha,\rho} w \right)(t) = \lambda w(t) + \mathcal{G}\big(t, w(t)\big) + h(t)$

for all $t \in [a,b]$.

Surely, similar remarks can be observed for the inequalities (5) and (6). For inequality (4) we have the following result.

Lemma 4. *Let $\lambda \in \mathbb{R}$, $\alpha \in (0,1)$, $\rho \in (0,1]$, and the function $w \in \mathcal{C}_{1-\alpha,\rho}([a,b])$ be a solution of inequality* (4)*. Then it satisfies the inequality*

$$\begin{aligned} &\Bigg| w(t) - w_0\, e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha,\alpha}\left(\lambda \left(\frac{t-a}{\rho}\right)^{\alpha}\right) \left(\frac{t-a}{\rho}\right)^{\alpha-1} \\ &\quad - \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{\frac{\rho-1}{\rho}(t-s)} E_{\alpha,\alpha}\left(\lambda \left(\frac{t-s}{\rho}\right)^{\alpha}\right) \mathcal{G}\big(s, w(s)\big)\, ds \Bigg| \\ &\qquad \le \frac{\varepsilon}{|\lambda|\Gamma(\alpha)} \left| E_\alpha\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right) - 1 \right|, \qquad t \in [a,b]. \end{aligned} \tag{7}$$

Proof. According to Remark 3 and Definition 3 the function $w \in C_{1-\alpha,\rho}([a,b])$ satisfies the integral equality

$$\begin{aligned} w(t) &= w_0\, e^{(\rho-1)\frac{t-a}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1} \\ &\quad + \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right) \mathcal{G}\big(s, x(s)\big)\, ds \\ &\quad + \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right) h(s)\, ds, \quad t \in (a,b], \end{aligned} \tag{8}$$

where $w_0 = \lim\limits_{t\to a+} e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}w(t)$.

According to *(i)*, $0<\rho<1$ and Proposition 1, from equality (8) we obtain

$$\begin{aligned}\Big| w(t) &- w_0 e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1} \\ &- \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{\frac{\rho-1}{\rho}(t-s)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\mathcal{G}(s,w(s))\,ds\Big| \\ &\le \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{\frac{\rho-1}{\rho}(t-s)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right)|h(s)|\,ds \\ &\le \frac{\varepsilon}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{\frac{\rho-1}{\rho}(t-s)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right) ds \\ &\le \frac{\varepsilon}{\rho^{\alpha}\Gamma(\alpha)}\frac{\rho^{\alpha}}{|\lambda|}\left|E_{\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)-1\right|, \quad t\in[a,b].\end{aligned}$$

□

Similarly the following result could be proved.

Lemma 5. *Let $\lambda<0$, $\alpha\in(0,1)$, $\rho\in(0,1]$. Let a function $\varphi\in C([a,b],\mathbb{R}_+)$ be nondecreasing such that $\int_a^t (t-s)^{\alpha-1}\varphi(s)\,ds \le \Lambda_{\varphi}\varphi(t)$, where $\Lambda_{\varphi}>0$ is a constant. Let a function $w\in\mathcal{C}_{1-\alpha,\rho}([a,b])$ be a solution of inequality* (5). *Then it satisfies the inequality*

$$\begin{aligned}\Big| w(t) &- w_0\, e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1} \\ &- \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{\frac{\rho-1}{\rho}(t-s)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\mathcal{G}(s,w(s))\,ds\Big| \\ &\le \frac{\varepsilon\Lambda_{\varphi}\varphi(t)}{\rho^{\alpha}\Gamma(\alpha)}\max_{t,s\in[a,b]:\,t\ge s} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right), \quad t\in[a,b].\end{aligned}$$

Now, we will study Ulam type stability of the PIVP (1).

Theorem 1 ($\mathcal{UH}$ stability)**.** *Let $\lambda\in\mathbb{R}$, $\alpha\in(0,1)$, $\rho\in(0,1]$. Suppose that*

1) *conditions (A1) and (A2) are satisfied;*

2) *for any $\varepsilon>0$ the inequality* (4) *has at least one solution.*

Then the PIVP (1) *is Ulam-Hyers stable with*

$$a_{\mathcal{G}} = \left[\frac{1}{|\lambda|\Gamma(\alpha)}\left|E_{\alpha}\left(\lambda\left(\frac{b-a}{\rho}\right)^{\alpha}\right)-1\right| + \mathscr{E}\left(\frac{b-a}{\rho}\right)^{\alpha-1}\right] E_{\alpha}\left(\frac{L\mathscr{M}}{\rho^{\alpha}}(b-a)^{\alpha}\right),$$

where $\mathscr{E} = \max_{t\in[a,b]} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)$ and $\mathscr{M} = \max_{t,s\in[a,b]:\,t\ge s} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right)$.

Proof. Let $\varepsilon>0$ be an arbitrary number and $y\in C_{1-\alpha,\rho}([a,b])$ be a solution of the inequality (4), which exists according to condition *(A1)*.

Let $y_0 = \lim_{t\to a+} e^{\frac{\rho-1}{\rho}(t-a)^{1-\alpha}} y(t)$ and $u_0 \in \mathbb{R} : |u_0 - y_0| \le \varepsilon$. According to condition *(A2)* PIVP (1) has an unique solution $x \in C_{1-\alpha,\rho}\big([a,b]\big)$. The function $x(t)$ satisfies the integral equality (3).

According to Lemma 4 we obtain

$$\begin{aligned}
\big|x(t)-y(t)\big| \le & \left|(u_0-y_0)\, e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1}\right| \\
& + \left|\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\Big[\mathcal{G}\big(s,x(s)\big)-\mathcal{G}\big(s,y(s)\big)\Big]\, ds\right| \\
& + \left|y(t) - y_0 e^{(\rho-1)\frac{t-a}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1}\right. \\
& \left. - \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\mathcal{G}\big(s,y(s)\big)\, ds\right| \\
\le & \frac{L}{\rho^{\alpha}\Gamma(\alpha)} \max_{t,s\in[a,b]:\, t\ge s} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)}\big|x(s)-y(s)\big|\, ds \\
& + \frac{\varepsilon}{|\lambda|\Gamma(\alpha)}\left|E_{\alpha}\left(\lambda\left(\frac{b-a}{\rho}\right)^{\alpha}\right)-1\right| + \varepsilon\mathscr{E}\left(\frac{b-a}{\rho}\right)^{\alpha-1}.
\end{aligned}$$

According to Proposition 3 with

$$u(t) = \big|x(t)-y(t)\big|, \qquad v(t) = \varepsilon\left[\frac{1}{|\lambda|\Gamma(\alpha)}\left|E_{\alpha}\left(\lambda\left(\frac{b-a}{\rho}\right)^{\alpha}\right)-1\right| + \mathscr{E}\left(\frac{b-a}{\rho}\right)^{\alpha-1}\right]$$

and $B = \frac{L\mathscr{M}}{\rho^{\alpha}\Gamma(\alpha)}$ we get

$$\big|x(t)-y(t)\big| \le \varepsilon\left[\frac{1}{|\lambda|\Gamma(\alpha)}\left|E_{\alpha}\left(\lambda\left(\frac{b-a}{\rho}\right)^{\alpha}\right)-1\right| + \mathscr{E}\left(\frac{b-a}{\rho}\right)^{\alpha-1}\right] E_{\alpha}\left(\frac{L\mathscr{M}}{\rho^{\alpha}}(t-a)^{\alpha}\right)$$

for $t \in [a,b]$. The above inequality proves the claim. □

Theorem 2 ($\mathcal{UHR}$ stability)**.** *Let $\lambda < 0$, $\alpha, \rho \in (0,1)$. Suppose that*

1) *conditions (A1) and (A2) are satisfied;*
2) *there exists a nondecreasing function $\varphi \in C\big([0,T],[0,\infty)\big)$ such that*

$$\int_0^t (t-s)^{\alpha-1}\varphi(s)ds \le \Lambda_{\varphi}\varphi(t),$$

   where $\Lambda_{\varphi} > 0$ is a constant;

3) *for any $\varepsilon > 0$ the inequality* (5) *has at least one solution.*

Then the PIVP (1) *is Ulam-Hyers-Rassias stable with respect to φ with*

$$a_{\mathcal{G}} = \left[\frac{\Lambda_{\varphi}\varphi(b)}{\rho^{\alpha}\Gamma(\alpha)}\mathscr{M} + \varphi(a)\mathscr{E}\left(\frac{b-a}{\rho}\right)^{\alpha-1}\right] E_{\alpha}\left(\frac{L\mathscr{M}}{\rho^{\alpha}}(b-a)^{\alpha}\right), \tag{9}$$

where $\mathscr{E}$ and $\mathscr{M}$ are defined in Theorem 1.

Proof. Let $\varepsilon > 0$ be a real number and $y \in C_{1-\alpha,\rho}\big([a,b]\big)$ be a solution of the inequality (5) with the function $\varphi(t)$ defined in the condition *2)* of the theorem.

Similarly to inequalities (5) and (7) there exists a function $h \in \big(C\big([a,b]\big),\mathbb{R}\big)$ such that $\big|h(t)\big| \le \varepsilon\varphi(t)$, $t \in [a,b]$, and according to Lemma 5 we obtain

$$\begin{aligned}\big|x(t)-y(t)\big| \le{}& \frac{L}{\rho^\alpha\Gamma(\alpha)}\mathscr{M}\int_a^t (t-s)^{\alpha-1}e^{(\rho-1)\left(\frac{t-s}{\rho}\right)}\big|x(s)-y(s)\big|\,ds\\ &+\frac{\varepsilon\Lambda_\varphi\varphi(t)}{\rho^\alpha\Gamma(\alpha)}\max_{t,s\in[a,b]:\,t\ge s} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^\alpha\right)+\varepsilon\varphi(a)\mathscr{E}\left(\frac{b-a}{\rho}\right)^{\alpha-1}.\end{aligned}$$

According to Proposition 3 with

$$u(t)=\big|x(t)-y(t)\big|,\qquad v(t)=\varepsilon\left[\frac{\Lambda_\varphi\varphi(t)}{\rho^\alpha\Gamma(\alpha)}\mathscr{M}+\varphi(a)\mathscr{E}\left(\frac{b-a}{\rho}\right)^{\alpha-1}\right]$$

and $B=\frac{L\mathscr{M}}{\rho^\alpha\Gamma(\alpha)}$ we get

$$\big|x(t)-y(t)\big| \le \varepsilon\left[\frac{\Lambda_\varphi\varphi(t)}{\rho^\alpha\Gamma(\alpha)}\mathscr{M}+\varphi(a)\mathscr{E}\left(\frac{b-a}{\rho}\right)^{\alpha-1}\right]E_\alpha\left(\frac{L\mathscr{M}}{\rho^\alpha\Gamma(\alpha)}\Gamma(\alpha)(t-a)^\alpha\right)$$

for $t \in [a,b]$. The above inequality proves the claim. □

Theorem 3 ($\mathcal{GUHR}$ stability)**.** *Let* $\lambda < 0$*,* $\alpha,\rho \in (0,1)$*. Suppose that*

1) conditions 1) and 2) of Theorem 2 are satisfied;

2) the inequality (6) *has at least one solution.*

Then the PIVP (1) *is generalized Ulam-Hyers-Rassias stable with respect to* φ *with* $a_{\mathcal{G}}$*, defined by* (9)*.*

4 Application to a biological model

In this section, we will apply the obtained results to a biological model and its fractional generalizations. Also, we will illustrate the non-uniqueness of the bounds in Ulam type stability.

Consider the population model, where the newborns are randomly distributed over M sites and offspring landing in an occupied site die. With constant per capita birth and death rates, B and μ respectively, $B > \mu$, we obtain the equation (see [13]): $N'(t) = BN(t)\left(1-\frac{N(t)}{M}\right)-\mu N(t)$ (this site-limited model is analogous to the Levins metapopulation model).

Now, consider the fractional generalization of the model with $\alpha \in (0,1)$:

$$\left({}_0^R\mathscr{D}^{\alpha,\rho}N\right)(t) = BN(t)\left(1-\frac{N(t)}{M}\right)-\mu N(t) \tag{10}$$

with the initial conditions

$$\lim_{t\to0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-\alpha}N(t)\right)=N_0.$$

In this case $\lambda = -\mu < B, \mathcal{G}(t,x) = -\frac{B}{M}x^2 + Bx$. Then

$$\big|\mathcal{G}(t,x) - \mathcal{G}(t,u)\big| = \left| -\frac{B}{M}\left(x^2 - u^2\right) + B(x-u)\right| \le \left(B + \frac{B}{M}(x+u)\right)|x-u|.$$

We will consider the case of bounded population $N \le W,\ W > 0$. Therefore, $L = B + \frac{2WB}{M}$ and assumption *(A1)* is satisfied.

Consider the partial case $B = 0.7, \mu = 0.01, \rho = 0.1, a = 0, b = 6, \alpha = 0.2, \beta = 2, M = 1000, W = 150, \mu = 0.1$. Then

$$\frac{L\gamma\left(\alpha, \frac{1-\rho}{\rho}(b-a)\right)}{\Gamma(\alpha)(1-\rho)^{\alpha}} = \frac{\frac{2\cdot 300\cdot 0.7}{100}\gamma\left(0.2, \frac{1-0.1}{0.1}(6-0)\right)}{\Gamma(0.2)(1-0.1)^{0.2}} \approx 0.929379 < 1,$$

where $\gamma(\alpha, t)$ is the lower incomplete Gamma function.

According to [22, Theorem 4.2] the PIVP (10) has a solution in $C_{1-\alpha,\rho}\big([a,b]\big)$. According to Lemma 3 this solution has the integral presentation

$$N(t) = N_0 e^{(\rho-1)\frac{t-a}{\rho}} E_{\alpha,\alpha}\left(-\mu\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1}$$
$$+ \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(-\mu\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\left(B\,N(s) - \frac{B}{M}N^2(s)\right)ds, \quad t \in (a,b].$$

Consider the function $y(t) = e^{\frac{\rho-1}{\rho}t}t^2$. Then $\lim\limits_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-\alpha}y(t)\right) = \lim\limits_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-\alpha}e^{\frac{\rho-1}{\rho}t}t^2\right) = 0.$

Apply the equality

$$\left({}_0^R\mathscr{D}^{\alpha,\rho}x^{\beta-1}e^{\frac{\rho-1}{\rho}x}\right)(t) = \frac{\rho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha-1}e^{\frac{\rho-1}{\rho}t} \tag{11}$$

and get

$$\left({}_0^R\mathscr{D}^{\alpha,\rho}y\right)t = {}_0^R\mathscr{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}t^2 = \frac{\rho^{\alpha}\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}e^{\frac{\rho-1}{\rho}t}.$$

Thus, for any real number $\varepsilon > 0$ for the partial values of the parameters given above, we consider the function $y(t) = 334\,\varepsilon\, e^{\frac{\rho-1}{\rho}t}t^2$. This function satisfies the inequality

$$\left|\left({}_a^R\mathscr{D}^{\alpha,\rho}y\right)(t) + \mu y(t) - By(t) + \frac{B}{M}y^2(t)\right|$$
$$\le \left|\frac{0.1^{0.2}\Gamma(3)}{\Gamma(3-0.2)}334\varepsilon t^{2-0.2}e^{-9t} + (0.01-0.7)e^{-9t}334\varepsilon t^2 + \frac{0.7}{1000}e^{-18t}334^2\varepsilon^2t^4\right| \tag{12}$$
$$= 334\,\varepsilon\, e^{-9t}\left|\frac{0.1^{0.2}\Gamma(3)}{\Gamma(3-0.2)}t^{2-0.2} + (0.01-0.7)t^2 + \frac{0.7}{1000}e^{-9t}334\varepsilon t^4\right| \le \varepsilon, \quad t \in [0,6].$$

The graphs of the functions $g(t) = 334\,\varepsilon\, e^{-9t}\left|\frac{0.1^{0.2}\Gamma(3)}{\Gamma(3-0.2)}t^{2-0.2} + (0.01-0.7)t^2 + \frac{0.7}{1000}e^{-9t}334\,\varepsilon\, t^4\right|$ and $g_2(t) = 334\varepsilon e^{-9t}\frac{0.7}{1000}e^{-9t}334^2t^4$ are plotted on Figures 1 and 2, respectively.

It could be seen the function $g_2(t)$ has very small values and therefore the inequality (12) holds.

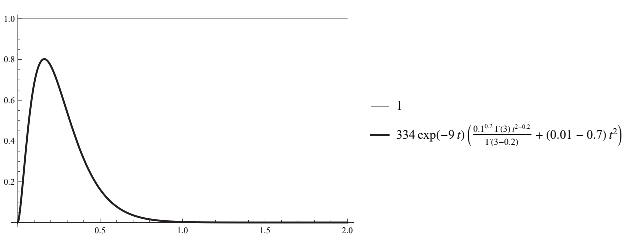

Figure 1. Graph of the function $g(t)$

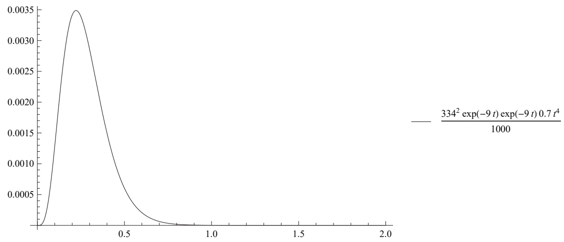

Figure 2. Graph of the function $g_2(y)$

According to Theorem 1, the solution $N(t)$ of (10) is $\mathcal{UH}$ stable with

$$a_{\mathcal{G}} = \left[\frac{1}{|-0.1|\Gamma(0.2)}\left|E_{0.2}\left(-0.1\left(\frac{6}{0.1}\right)^{0.2}\right)-1\right| + \mathscr{E}\left(\frac{6}{0.1}\right)^{0.2-1}\right] E_{0.2}\left(\frac{L\mathscr{E}}{0.1^{0.2}}6^{0.2}\right) \approx 0.8395,$$

where $\mathscr{E} = \max_{t\in[0,6]} E_{0.2,0.2}\left(-0.1\left(\frac{t}{0.1}\right)^{0.2}\right) \approx 0.217825.$

Therefore, the solution $N(t)$ with $0 < N_0 \leq 0.8395\,\varepsilon$ satisfies the inequality

$$\left|N(t) - 334\varepsilon e^{\frac{\rho-1}{\rho}t}t^2\right| \leq 0.8395\,\varepsilon \qquad \text{or} \qquad 0 \leq N(t) \leq \varepsilon\left(334 e^{\frac{\rho-1}{\rho}t}t^2 + 0.8395\right),$$

for $t \in [0,6]$, i.e. by $\mathcal{UH}$ stability we obtain bounds of the $N(t)$. It could be seen that this bound is better than W for $\varepsilon < 47$.

Since in the studied example $\lambda < 0$, we could study also $\mathcal{UHR}$ stability. Consider the function $y(t) = e^{\frac{\rho-1}{\rho}t}t^{1-\alpha}$. Then $\lim_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-\alpha}y(t)\right) = \lim_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-\alpha}e^{\frac{\rho-1}{\rho}t}t^{1-\alpha}\right) = 0.$

Apply the equality (11) with $\beta = 2-\alpha$ and get

$$\left({}_0^R\mathscr{D}^{\alpha,\rho}y\right)(t) = {}_0^R\mathscr{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}t^{1-\alpha} = \frac{\rho^\alpha\Gamma(2-\alpha)}{\Gamma(2-2\alpha)}t^{1-2\alpha}e^{\frac{\rho-1}{\rho}t}.$$

Thus for any real number $\varepsilon > 0$ for the partial values of the parameters given above, we consider the function $y(t) = \varepsilon e^{\frac{\rho-1}{\rho}t}t^{1-\alpha}$. This function satisfies the inequality (see Figure 3)

$$\begin{aligned}
&\left|\varepsilon\frac{\rho^\alpha\Gamma(2-\alpha)}{\Gamma(2-2\alpha)}t^{1-2\alpha}e^{\frac{\rho-1}{\rho}t} + \varepsilon(\mu-B)e^{\frac{\rho-1}{\rho}t}t^{1-\alpha} + \frac{B}{M}\varepsilon^2 e^{\frac{\rho-1}{\rho}2t}t^{2-2\alpha}\right| \\
&\quad \leq \varepsilon e^{\frac{\rho-1}{\rho}t}t^{1-\alpha}\left|\frac{\rho^\alpha\Gamma(2-\alpha)}{\Gamma(2-2\alpha)}t^{-\alpha} + (\mu-B) + \varepsilon\frac{B}{M}e^{\frac{\rho-1}{\rho}t}t^{1-\alpha}\right| \\
&\quad \leq \varepsilon e^{\frac{0.1-1}{0.1}t}t^{1-0.2}\left|\frac{0.1^{0.2}\Gamma(2-0.2)}{\Gamma(2-0.4)}t^{-0.2} + (0.01-0.7) + \varepsilon\frac{0.7}{1000}e^{\frac{0.1-1}{0.1}t}t^{1-0.2}\right| \leq \varepsilon\varphi(t)
\end{aligned} \tag{13}$$

for $t \in [0,6]$ with $\varphi(t) = t^{0.8} + 0.01$.

Consider the integral in condition *2)* of Theorem 2 (see Figure 4)

$$\int_0^t (t-s)^{\alpha-1}\varphi(s)\,ds = \int_0^t (t-s)^{0.2-1}(s^{0.8}+0.01)ds \leq 7\varphi(t), \quad t \in [0,6], \tag{14}$$

i.e. the condition *2)* of Theorem 2 is satisfied with $\Lambda_\varphi = 7$.

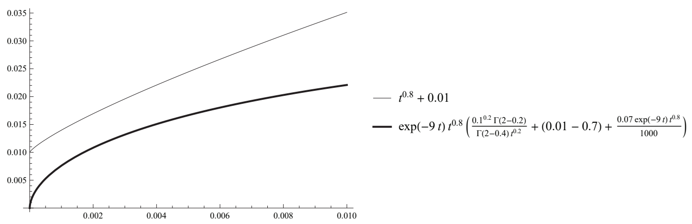

Figure 3. Illustration of the inequality (13)

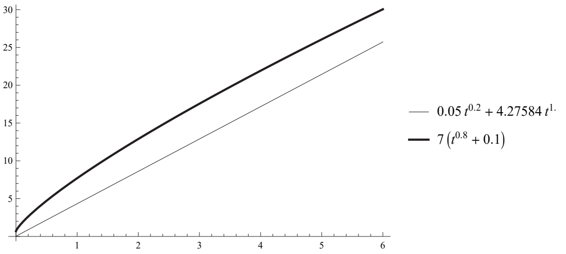

Figure 4. Illustration of the inequality (14)

According to Theorem 2 the PIVP (10) is Ulam-Hyers-Rassias stable with respect to $\varphi(t) = t^{0.8} + 0.01$ and the coefficient

$$a_{\mathcal{G}} = \left[\frac{7\left(6^{0.8} + 0.01\right)}{0.1^{0.2}\Gamma(0.2)} \max_{t,s\in[0,6]:\ t\geq s} E_{0.2,0.2}\left(-0.1\left(\frac{t-s}{0.1}\right)^{0.1}\right) + 0.01\mathscr{E}\left(\frac{6}{0.1}\right)^{-0.8}\right]$$
$$\times E_{0.2}\left(\frac{0.7(1+\frac{300}{1000})\mathscr{E}}{0.1^{0.2}}6^{0.2}\right) \approx 4.18953.$$

Therefore,

$$\left|N(t) - e^{-9t}t^{0.8}\right| \leq 4.18953\,\varepsilon, \quad t \in [0,6],$$

where $0 < N_0 \leq 4.18953\,\varepsilon$. Note that $\varepsilon < 35.8036$, because $W = 150$.

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Хрістова С., Аббас М.І. *Аналіз стабільності типу Улама для узагальнених диференціальних рівнянь з пропорційними дробовими похідними* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 114–127.

Основною метою даної роботи є відповідне визначення кількох типів стійкості Улама для нелінійного дробового диференціального рівняння з узагальненою пропорційною дробовою похідною типу Рімана-Ліувіля. У нових визначеннях початкові значення розв'язків даного рівняння та відповідної нерівності не можуть збігатися, але вони мають бути достатньо близькими. Отримано деякі достатні умови для трьох типів стійкості Улама для досліджуваних рівнянь, а саме стійкості Улама-Хайєрса, стійкості Улама-Хайєрса-Рассіаса та узагальненої стійкості Улама-Хайєрса-Рассіаса. Деякі з них застосовуються для дробового узагальнення біологічної моделі.

Ключові слова і фрази: узагальнена пропорційна дробова похідна, функція Міттаг-Лефлера, стійкість типу Улама.