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Riemann solitons on para-Sasakian geometry

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The goal of the present article is to investigate almost Riemann soliton and gradient almost Riemann soliton on 3-dimensional para-Sasakian manifolds. At first, it is proved that if (g, Z, λ) is an almost Riemann soliton on a para-Sasakian manifold M^3, then it reduces to a Riemann soliton and M^3 is of constant sectional curvature -1, provided the soliton vector Z has constant divergence. Besides these, we prove that if Z is pointwise collinear with the characteristic vector field ξ, then Z is a constant multiple of ξ and the manifold is of constant sectional curvature -1. Moreover, the almost Riemann soliton is expanding. Furthermore, it is established that if a para-Sasakian manifold M^3 admits gradient almost Riemann soliton, then M^3 is locally isometric to the hyperbolic space $H^3(-1)$. Finally, we construct an example to justify some results of our paper.

Key words and phrases: para-Sasakian manifold, almost Riemann soliton, gradient almost Riemann soliton.

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Introduction

Since Einstein manifolds perform a significant role in mathematics and physics, the investigation of Einstein manifolds and their generalizations is a fascinating topic in Riemannian and semi-Riemannian geometry. In the last few years, several generalizations of Einstein manifolds such as Ricci soliton, gradient Ricci soliton, gradient Einstein soliton, etc. have been studied. The idea of Ricci flow was introduced by R.S. Hamilton [8] and expressed by $\frac{\partial}{\partial_t} g(t) = -2S(t)$, where S indicates the Ricci tensor.

As a natural generalization, the notion of Riemann flow (see [18]) is expressed by $\frac{\partial}{\partial_t} G(t) = -2Rg(t)$, $G = \frac{1}{2} g \otimes g$, where R is the Riemann curvature tensor and $\otimes$ is *Kulkarni-Nomizu* product, defined as follows (see [1, p. 47])

$$\begin{aligned}(P \otimes Q)(X, Y, U, W) = P(X, W)Q(Y, U) + P(Y, U)Q(X, W)\\ - P(X, U)Q(Y, W) - P(Y, W)Q(X, U),\end{aligned}$$

where P and Q are $(0, 2)$-tensor field.

Similar to Ricci soliton, the interesting concept of Riemann soliton was introduced by I.E. Hirică and C. Udriste [11]. Analogous to I.E. Hirică and C. Udriste [11], a semi-Riemannian metric g on a semi-Riemannian manifold M is said to be a *Riemann soliton* if there exist a C^∞ vector field Z and a real scalar λ such that

$$2R + \lambda g \otimes g + g \otimes \pounds_Z g = 0. \tag{1}$$

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The soliton will be named as *expanding* ($\lambda > 0$), *steady* ($\lambda = 0$) or *shrinking* ($\lambda < 0$), respectively. If Z is gradient of the potential function γ, then the manifold is said to be *gradient Riemann soliton*. Then the previous equation can be written as

$$2R + \lambda g \otimes g + g \otimes \nabla^2 \gamma = 0, \tag{2}$$

where $\nabla^2 f$ indicates the Hessian of γ. If we modify the equations (1) and (2) by fixing the condition on the parameter λ to be a variable function, then it reduces to *ARS* and *gradient ARS* respectively. Here the terminology "almost Riemann solitons" is written as *ARS* which will be applied throughout the article.

Riemann solitons and gradient Riemann solitons on Sasakian manifolds have been discussed in detail by I.E. Hiricǎ and C. Udriste (see [11]). Moreover, Riemann soliton concerning infinitesimal harmonic transformation was investigated in [17]. Here it is appropriate to notice that R. Sharma in [15] investigated almost Ricci soliton in K-contact geometry and in [16] with divergence free soliton vector field. Very recently in [7], the authors studied Riemann soliton within the context of a contact manifold and proved various fascinating results. We may mention [3,6] and the references given there for more information about Riemann soliton.

The above studies motivate us to investigate an *ARS* and the *gradient ARS* in a 3-dimensional para-Sasakian manifolds.

The current article is structured as follows. At first, we recollect a few formulas of para-Sasakian manifolds. Beginning from Section 3, after providing the proof, we will write our main theorems. After that, we construct an example to verify some results of our article.

1 Para-Sasakian manifolds

The very attractive topic of paracontact metric structures were published in [12]. This fascinating subject (*paracontact geometry*), has been investigated in the previous years by various articles concerning the theory of paracontact manifolds and mathematical physics (see [4,9,10]). In this context, we refer the reader to [2,12,13] and references therein.

Let M^n be a C^∞ differentiable manifold equipped with a 1-form η, a unique characteristic vector field ξ and a $(1,1)$-type tensor field ϕ such that

$$\phi^2 E = E - \eta(E)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi E) = 0$$

and the almost paracomplex structure on each fibre of $\mathcal{D} = ker\eta$ is induced by the tensor field ϕ. In other words, the eigendistributions $\mathcal{D}^+_\prec$ and $\mathcal{D}^-_\prec$ of ϕ have the equal dimension corresponding to the eigenvalues 1 and -1, respectively. Then M^n is named as an *almost paracontact manifold*. In addition, if M^n obeys

$$g(\xi, E) = \eta(E), \quad g(\phi E, \phi F) = -g(E,F) + \eta(E)\eta(F), \tag{3}$$

where g indicates a semi-Riemannian metric, then M^n is termed as *almost paracontact metric manifold* [14] with the structure (ϕ, ξ, η, g).

The *Nijenhuis* torsion is defined by

$$[\phi,\phi](E,F) = \phi^2[E,F] + [\phi E, \phi F] - \phi[\phi E, F] - \phi[E, \phi F].$$

The almost paracontact manifold is called normal if the tensor field $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes. The fundamental 2-form of the *almost paracontact metric manifold* is defined by $\Phi(E, F) = g(E, \phi F)$. If $d\eta(E, F) = g(E, \phi F)$, then (M, ϕ, ξ, η, g) is verbally expressed to be *paracontact metric manifold*.

A symmetric trace-free operator $h = \frac{1}{2}\pounds_\xi \phi$ in a paracontact manifold satisfies $h\xi = 0$ and $\nabla_E \xi = -\phi E + \phi h E$. It is to be noted that ξ being Killing vector field is equivalent to the condition $h = 0$ and (ϕ, ξ, η, g) is called K-paracontact structure. If the normality condition is satisfied in a paracontact metric manifold M^n then it is said to be a para-Sasakian manifold. It is well circulated that every para-Sasakian manifold is necessarily K-paracontact. The converse is not true in general, but it holds when the manifold is of dimension three [5].

In M^n the subsequent results hold:

$$R(E, F)\xi = \eta(E)F - \eta(F)E, \tag{4}$$

$$(\nabla_E \phi)F = -g(E, F)\xi + \eta(F)E,$$

$$\nabla_E \xi = -\phi E, \tag{5}$$

$$R(E, \xi)F = g(E, F)\xi - \eta(F)E, \tag{6}$$

$$S(E, \xi) = -2n\eta(E), \quad Q\xi = -2n\xi, \tag{7}$$

for any E, F, where Q is the Ricci operator, i.e. $g(QE, F) = S(E, F)$ on the manifold.

In a 3-dimensional semi-Riemannian manifold the Riemannian curvature tensor is written by

$$R(E, F)Z = g(F, Z)QE - g(E, Z)QF + S(F, Z)E - S(E, Z)F - \frac{r}{2}[g(F, Z)E - g(E, Z)F], \tag{8}$$

for any E, F, Z. Substituting F=Z=ξ in the foregoing equation and utilizing (4) and (7) we get (see [10])

$$QE = \frac{1}{2}[(r + 2)E - (r + 6)\eta(E)\xi], \tag{9}$$

which implies

$$S(E, F) = \frac{1}{2}[(r + 2)g(E, F) - (r + 6)\eta(E)\eta(F)]. \tag{10}$$

Now we write the subsequent results.

Lemma 1 ([10, Lemma 3.3])**.** *For a para-Sasakian manifold $(M^3, \eta, \xi, \phi, g)$, we have*

$$\xi r = 0. \tag{11}$$

Lemma 2 ([7, Lemma 3.8])**.** *For any E, F on M, in a gradient ARS $(M, g, \gamma, m, \lambda)$, we infer*

$$R(E, F)D\gamma = (\nabla_F Q)E - (\nabla_E Q)F + \{F(2\lambda + \triangle\gamma)E - E(2\lambda + \triangle\gamma)F\}, \tag{12}$$

where $\triangle\gamma$ = div $D\gamma$, $\triangle$ is the Laplacian operator.

2 *ARS* on 3-dimensional para-Sasakian manifolds

We consider a para-Sasakian manifold M^3 admitting an *ARS* defined by (1). Using *Kulkarni-Nomizu* product in (1) we write

$$\begin{aligned}
2R(E,F,W,X) &+ 2\lambda\{g(E,X)g(F,W) - g(E,W)g(F,X)\} \\
&+ \{g(E,X)(\pounds_Z g)(F,W) + g(F,W)(\pounds_Z g)(E,X) - g(E,W)(\pounds_Z g)(F,X) \\
&\qquad - g(F,X)(\pounds_Z g)(E,W)\} = 0.
\end{aligned} \tag{13}$$

Contracting (13) over E and X, we lead

$$(\pounds_Z g)(F,W) + 2S(F,W) + (4\lambda + 2divZ)g(F,W) = 0. \tag{14}$$

Utilizing (10) in the foregoing equation we obtain

$$(\pounds_Z g)(F,W) = -(r + 2 + 4\lambda + 2divZ)g(F,W) + (r+6)\eta(F)\eta(W) = 0. \tag{15}$$

Taking covariant derivative along E and applying Z has constant divergence, we infer

$$\begin{aligned}
(\nabla_E \pounds_Z g)(F,W) = &- [(Er) + 4(E\lambda)]g(F,W) + (Er)\eta(F)\eta(W) \\
&- (r+6)[g(\phi E,F)\eta(W) + g(\phi E,W)\eta(F)] = 0.
\end{aligned} \tag{16}$$

Next recall the famous formula by Yano (see [19]):

$$(\pounds_Z \nabla_E g - \nabla_E \pounds_Z g - \nabla_{[Z,E]} g)(F,W) = -g((\pounds_Z \nabla)(E,F),W) - g((\pounds_Z \nabla)(E,W),F).$$

Hence by a simple calculation, we can easily get

$$(\nabla_E \pounds_Z g)(F,W) = g((\pounds_Z \nabla)(E,F),W) + g((\pounds_Z \nabla)(E,W),F). \tag{17}$$

Using symmetric property of $\pounds_F \nabla$, it reveals from (17) that

$$g((\pounds_Z \nabla)(E,F),W) = \frac{1}{2}(\nabla_E \pounds_Z g)(F,W) + \frac{1}{2}(\nabla_F \pounds_Z g)(E,W) - \frac{1}{2}(\nabla_W \pounds_Z g)(E,F). \tag{18}$$

Using (16) in (18) we infer

$$\begin{aligned}
2g((\pounds_Z \nabla)(E,F),W) = &- [(Er) + 4(E\lambda)]g(F,W) + (Er)\eta(F)\eta(W) \\
&- (r+6)[g(\phi E,F)\eta(W) + g(\phi E,W)\eta(F)] \\
&- [(Fr) + 4(F\lambda)]g(E,W) + (Fr)\eta(E)\eta(W) \\
&- (r+6)[g(\phi F,E)\eta(W) + g(\phi F,W)\eta(E)] \\
&+ [(Wr) + 4(W\lambda)]g(E,F) + (Wr)\eta(E)\eta(F) \\
&- (r+6)[g(\phi W,E)\eta(F) + g(\phi W,F)\eta(E)].
\end{aligned} \tag{19}$$

Then substituting $E = F = e_i$ in (19) and removing W, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

$$(\pounds_F \nabla)(e_i, e_i) = -4D\lambda. \tag{20}$$

Again differentiating (14) and utilizing it in (17) we can readily produce

$$g((\pounds_Z\nabla)(E,F),W) = (\nabla_W S)(E,F) - (\nabla_E S)(F,W) - (\nabla_F S)(E,W). \tag{21}$$

Taking $E = F = e_i$ in (21) and summing over i we lead

$$(\pounds_Z\nabla)(e_i, e_i) = 0, \tag{22}$$

where $\{e_i\}$ is an orthonormal frame. Combining (20) and (22) yields

$$D\lambda = 0.$$

This means that λ is constant and leads to the following result.

Theorem 1. *If the soliton vector Z has constant divergence in a 3-dimensional para-Sasakian manifold, then an ARS reduces to a Riemann soliton.*

Applying the above theorem and removing W from (19) gives

$$\begin{aligned} 2(\pounds_Z\nabla)(E,F) = &- (Er)F + (Er)\eta(F)\xi - (r+6)[g(\phi E,F)\xi + \phi E\eta(F)] \\ &- (Fr)E + (Fr)\eta(E)\xi - (r+6)[g(\phi F,E)\xi + \phi F\eta(E)] \\ &+ Dr[g(E,F) - \eta(E)\eta(F)] - (r+6)[\phi E\eta(F) + \phi F\eta(E)]. \end{aligned}$$

Replacing Y by ξ and utilizing (11) yields

$$(\pounds_Z\nabla)(E,\xi) = -(r+6)\phi E. \tag{23}$$

Executing covariant derivative of (23) along F, we infer

$$(\nabla_F\pounds_Z\nabla)(E,\xi) - (\pounds_Z\nabla)(E,\phi F) = -(Fr)\phi E - (r+6)[-g(E,F)\xi + \eta(E)F].$$

If we use the subsequent formula

$$(\pounds_F R)(X,Y)Z = (\nabla_X\pounds_F\nabla)(Y,Z) - (\nabla_Y\pounds_F\nabla)(X,Z)$$

in the foregoing equation, we obtain

$$\begin{aligned} (\pounds_Z R)(E,F)\xi = &-(\pounds_Z\nabla)(F,\phi E) + (\pounds_Z\nabla)(E,\phi F) + (Er)\phi F \\ &- (Fr)\phi E - (r+6)[\eta(E)F - \eta(F)E]. \end{aligned} \tag{24}$$

Setting $F = \xi$ in the previous equation yields

$$(\pounds_Z R)(E,\xi)\xi \;\; = -(r+6)[\eta(E)\xi - E]. \tag{25}$$

Again from (15) we lead

$$(\pounds_Z g)(E,\xi) = (4 - 4\lambda - 2divZ)\eta(E). \tag{26}$$

Lie-differentiating (3) along Z and by virtue of (26) we obtain

$$(\pounds_Z\eta)(E) - g(\pounds_Z\xi, E) + (4\lambda + 2divZ - 4)\eta(E) = 0. \tag{27}$$

Replacing E by ξ in the previous equation gives

$$\eta(\pounds_Z\xi) = (2\lambda + divZ - 2). \tag{28}$$

Executing Lie derivative of (4) along Z we lead

$$(\pounds_Z R)(E,\xi)\xi + g(E,\pounds_Z\xi)\xi - 2\eta(\pounds_Z\xi)E = \{(\pounds_Z\eta)E\}\xi.$$

Utilizing (25), (27) and (28) in the above equation we infer

$$(-r - 10 + 4\lambda + 2divZ)[\eta(E)\xi - E] = 0.$$

Next tracing the foregoing equation yields

$$divZ = \Big(\frac{r}{2} + 5 - 2\lambda\Big).$$

Clearly, contracting the equation (24) leads

$$(\pounds_Z S)(E,\xi) = -g(Dr,\phi E) - 2(r+6)\eta(E). \tag{29}$$

Taking $\pounds_Z$ to (7), recalling (10) gives

$$(\pounds_Z S)(E,\xi) + \Big(\frac{r}{2} + 1\Big)g(E,\pounds_Z\xi) - \Big(\frac{r}{2} + 3\Big)\eta(\pounds_Z\xi)\eta(E) = -2(\pounds_Z\eta)E.$$

Using (29) and (26) in the foregoing equation yields

$$-g(Dr,\phi E) - 2(r+6)\eta(E) + \Big(\frac{r}{2} + 3\Big)g(E,\pounds_Z\xi) - \Big(\frac{r}{2} + 3\Big)\eta(\pounds_Z\xi)\eta(E) + 2\eta(E).$$

By a simple calculation, substituting E by ξ in (30) and using (29) we can easily get $r = -6$. If $r = -6$, then from equation (10) we find that g is an Einstein metric, i.e. $S = -2g$. Therefore, by utilizing equation (8) we infer that the manifold is of constant sectional curvature -1.

Thus we can write the following assertion.

Theorem 2. *If a semi-Riemannian metric of a para-Sasakian manifold M^3 is the ARS, then M^3 is of constant sectional curvature -1, provided the soliton vector Z has constant divergence.*

Now let Z be point-wise collinear with the characteristic vector field ξ, i.e. $Z = b\xi$, where b is a function on M^3. Since $div\xi = 0$ in a para-Sasakian manifold M^3, we have $divZ = (\xi b)$. Therefore from (14) we lead

$$g(\nabla_E b\xi, F) + g(\nabla_F b\xi, E) + 2(\xi b)g(E,F) + 2S(E,F) + 4\lambda g(E,F) = 0. \tag{30}$$

Using (5) in (30), we get

$$(Eb)\eta(F) + (Fb)\eta(E) + 2(\xi b)g(E,F) + 2S(E,F) + 4\lambda g(E,F) = 0. \tag{31}$$

Putting $F = \xi$ in (31) and using (7) yields

$$(Eb) + 3(\xi b)\eta(E) - 4\eta(E) + 4\lambda\eta(E) = 0. \tag{32}$$

Putting $E = \xi$ in (32) we obtain

$$(\xi b) = 1 - \lambda. \tag{33}$$

Putting the value of ξb in (32) yields

$$db = (1-\lambda)\eta. \tag{34}$$

Operating (34) by d and using Poincare lemma $d^2 \equiv 0$, we obtain

$$0 = d^2 b = (1-\lambda)d\eta + d\lambda\eta. \tag{35}$$

Taking wedge product of (35) with η, we have

$$(1-\lambda)\eta \wedge d\eta = 0.$$

Since $\eta \wedge d\eta \neq 0$ in a para-Sasakian manifold M^3, therefore

$$\lambda = 1. \tag{36}$$

Using (36) in (34) gives $db = 0$, i.e. $b =$constant. Therefore from (31) we infer

$$S(E,F) = -2g(E,F),$$

which implies M^3 is an Einstein manifold. Therefore from (8) we conclude that the manifold is of constant sectional curvature -1.

Thus we state the following result.

Theorem 3. *Let $(M^3, \phi, \xi, \eta, g)$ ba a para-Sasakian manifold. If g represents an ARS and Z is pointwise collinear with the characteristic vector field ξ, then Z is a constant multiple of ξ and M^3 is of constant sectional curvature -1. Moreover, the ARS is expanding.*

In particular if $Z = \xi$, we can write the following assertion.

Corollary 1. *If a para-Sasakian manifold M^3 admits an ARS (g, ξ, λ), then the manifold is of constant sectional curvature -1.*

3 Gradient *ARS*

This section is devoted to investigate 3-dimensional para-Sasakian manifolds admitting gradient *ARS*. First we write the subsequent result without proof, since, by a simple calculations, the result can be obtained from (9).

Lemma 3. *For a para-Sasakian manifold $(M^3, \eta, \xi, \phi, g)$ we have*

$$(\nabla_F Q)\xi = \Big(\frac{r}{2} + 3\Big)\phi F, \quad (\nabla_\xi Q)E = 0.$$

Substituting F by ξ in (12) and using the above Lemma 3, we infer

$$R(E,\xi)D\gamma = -\Big(\frac{r}{2} + 3\Big)\phi E + \big\{\xi(2\lambda + \triangle\gamma)E - E(2\lambda + \triangle\gamma)\xi\big\}.$$

Then using (6), we get

$$g(E, D\gamma + D(2\lambda + \triangle\gamma))\xi = -\Big(\frac{r}{2} + 3\Big)\phi E + \big\{(\xi\gamma) + \xi(2\lambda + \triangle\gamma)\big\}E. \tag{37}$$

Executing the inner product of the foregoing equation with ξ yields

$$E(\gamma + (2\lambda + \triangle\gamma)) = \{(\xi\gamma) + \xi(2\lambda + \triangle\gamma)\}\eta(E), \tag{38}$$

from which easily we lead

$$d(\gamma + (2\lambda + \triangle\gamma)) = \{(\xi\gamma) + \xi(2\lambda + \triangle\gamma)\}\eta,$$

where the exterior derivative is denoted by d. From the above equation we conclude that $\gamma + (2\lambda + \triangle\gamma)$ is invariant along the distribution $\mathcal{D}$. In other terms, $E(\gamma + (2\lambda + \triangle\gamma)) = 0$ for any $E \in \mathcal{D}$. Utilizing (38) in (37), we infer

$$\{(\xi\gamma) + \xi(2\lambda + \triangle\gamma)\}[\eta(E)\xi - E] = -\left(\frac{r}{2} + 3\right)\phi E. \tag{39}$$

Contracting the previous equation gives

$$\{(\xi\gamma) + \xi(2\lambda + \triangle\gamma)\} = 0. \tag{40}$$

Using (40) in (39), we lead

$$\left(\frac{r}{2} + 3\right)\phi E = 0.$$

If $\phi E = 0$. Clearly, operating ϕ we can easily obtain $\phi^2 E = 0$, which is a contradiction. Thus we infer $r = -6$. Then from (10) we obtain $S = -2g$. Therefore from (8) we state that M^3 is of constant sectional curvature -1. Hence M^3 is locally isometric to the hyperbolic space $H^3(-1)$.

Hence we write the following result.

Theorem 4. *If a para-Sasakian manifold M^3 admits a gradient ARS (γ, ξ, λ), then the manifold is locally isometric to the hyperbolic space $H^3(-1)$.*

4 Example

We consider the manifold $M^3 = \{(x, y, z) \in \mathbb{R}^3\}$ in which $z \neq 0$. The linearly independent vector fields are

$$u_1 = e^z \frac{\partial}{\partial y}, \quad u_2 = e^z \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad u_3 = \frac{\partial}{\partial z}.$$

Let g be the semi-Riemannian metric expressed by

$$g(u_1, u_1) = 1, \quad g(u_2, u_2) = -1, \quad g(u_3, u_3) = 1, \quad g(u_1, u_2) = g(u_1, u_3) = g(u_2, u_3) = 0.$$

Let η and ϕ are defined by $\eta(E) = g(E, u_3)$ for any vector field $E \in \chi(M)$ and

$$\phi(u_1) = u_2, \quad \phi(u_2) = u_1, \quad \phi(u_3) = 0.$$

Next utilizing the linearity property of ϕ and g we infer

$$\eta(u_3) = 1,$$

$$\phi^2 E = E - \eta(E)u_3,$$

$$g(\phi E, \phi F) = -g(E, F) + \eta(E)\eta(F)$$

for any vector fields $E, F \in \chi(M)$. Thus for $u_3 = \xi$, the structure (η, ξ, ϕ, g) satisfies a paracontact structure on M^3.

Then we infer

$$[u_1, u_2] = 0, \quad [u_1, u_3] = -u_1, \quad [u_2, u_3] = -u_2.$$

Apprehending $u_3 = \xi$ and utilizing Koszul's formula for the semi-Riemannian metric g, we can spontaneously calculate

$$\nabla_{u_1} u_1 = u_3, \quad \nabla_{u_1} u_2 = 0, \quad \nabla_{u_1} u_3 = -u_1,$$

$$\nabla_{u_2} u_1 = 0, \quad \nabla_{u_2} u_2 = -u_3, \quad \nabla_{u_2} u_3 = -u_2,$$

$$\nabla_{u_3} u_1 = 0, \quad \nabla_{u_3} u_2 = 0, \quad \nabla_{u_3} u_3 = 0.$$

Hence $M^3(\eta, \xi, \phi, g)$ is a para-Sasakian manifold. Now it can be easily checked that

$$R(u_1, u_2)u_3 = 0, \quad R(u_2, u_3)u_3 = -u_2, \quad R(u_1, u_3)u_3 = -u_1,$$

$$R(u_1, u_2)u_2 = u_1, \quad R(u_2, u_3)u_2 = -u_3, \quad R(u_1, u_3)u_2 = 0,$$

$$R(u_1, u_2)u_1 = u_2, \quad R(u_2, u_3)u_1 = 0, \quad R(u_1, u_3)u_1 = u_3.$$

Using the above expressions, the Ricci tensor can be obtained as

$$S(u_1, u_1) = -g(R(u_1, u_2)u_2, u_1) + g(R(u_1, u_3)u_3, u_1) = -2.$$

Similarly, we get

$$S(u_2, u_2) = 2, \quad S(u_3, u_3) = -2.$$

Therefore, the scalar curvature r is calculated as

$$r = S(u_1, u_1) - S(u_2, u_2) + S(u_3, u_3) = -6.$$

Let us suppose that the manifold admit *ARS* (g, Z, λ). If we suppose that the soliton vector Z has constant divergence, then from equation (29) we get

$$(Er) = 4(E\lambda).$$

Since here $r = -6$, therefore $\lambda =$ constant. Hence, the *ARS* reduces to a Riemann soliton provided Z has constant divergence. Thus Theorem 1 is verified. Also from the components of the curvature tensor we find that the manifold is of constant sectional curvature -1. Hence, Theorem 2 is verified.

Now from equation (31), we have

$$(\xi b) + r + 6\lambda = 0. \tag{41}$$

Also equation (33) holds for ARS in a para-Sasakian manifold. That is,

$$(\xi b) = 1 - \lambda.$$

From the last two equations, we obtain $\lambda = 1$ and hence the ARS is expanding. Also from (41) using $r = -6$ and $\lambda = 1$, we find $(\xi b) = 0$. Utilizing this result in (32), we infer $b =$ constant. Thus, Theorem 3 is also verified.

Let us suppose that the manifold under consideration admit ARS (g, ξ, λ). Since $\pounds_\xi g = 0$ in a para-Sasakian manifold, equation (13) reduces to

$$2R(E,F)W + 2\lambda\{g(F,W)E - g(E,W)F\} = 0, \tag{42}$$

for all vector fields E, F, W. From the components of the curvature tensor we infer that the manifold is of constant sectional curvature -1. Since the manifold is of constant sectional curvature -1, therefore equation (42) implies that $\lambda = 1$. Hence Corollary 1 is verified.

5 Conclusion

The fascinating idea of Riemann soliton were recently introduced by I.E. Hiricǎ and C. Udriste [11]. This soliton corresponds to a fixed point of the Riemannian flow and they can be viewed as a dynamical system, on the space of Riemannian metric modulo diffeomorphism. In this context, we should mention that the space of constant sectional curvature is generalized by the new idea of the Riemann soliton.

In the present investigation, we study ARS, the natural extension of the Riemann soliton in a 3-dimensional para-Sasakian manifold M^3 and observe that the ARS reduces to a Riemann soliton and M^3 is of constant sectional curvature -1, provided the soliton vector Z has constant divergence. Also, a para-Sasakian manifold M^3 admitting gradient ARS is shown to be locally isometric to the hyperbolic space $H^3(-1)$.

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Де К., Де Ю.Ч. *Рімановi солітони у пара-Сасакяновій геометрії* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 395–405.

Метою цієї статті є дослідити майже рімановий солітон і ґрадієнтний майже рімановий солітон на тривимірному пара-Сасакяновому многовиді. Спочатку доведено, що якщо (g, Z, λ) є майже рімановим солітоном на пара-Сасакяновому многовиді M^3, то він зводиться до ріманового солітону і M^3 має сталу секційну кривину -1, при цьому солітоновий вектор Z має сталу дивергенцію. Поза цим ми доводимо, що якщо Z є поточково колінеарний з характеристичним векторним полем ξ, то Z є стало кратним до ξ і многовид має сталу секційну кривину -1. Більше того, майже рімановий солітон розширюється. Крім того, встановлено, що якщо пара-Сасакяновий многовид M^3 допускає ґрадієнтний майже рімановий солітон, то M^3 є локально ізометричним до гіперболічного простору $H^3(-1)$. Насамкінець ми збудували приклад щоб обґрунтувати деякі результати нашої статті.

Ключові слова і фрази: пара-Сасакяновий многовид, майже рімановий солітон, ґрадієнтний майже рімановий солітон.