ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2021, **13** (3), 587–591
doi:10.15330/cmp.13.3.587-591

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2021, Т.13, №3, С.587–591

Interpolational (L, M)-rational integral fraction on a continual set of nodes

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In the paper, an integral rational interpolant on a continual set of nodes, which is the ratio of a functional polynomial of degree L to a functional polynomial of degree M, is constructed and investigated. The resulting interpolant is one that preserves any rational functional of the resulting form.

Key words and phrases: interpolation, functional polynomial, continual set of nodes, chain fraction, rational fraction.

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1 Introduction

A lot of papers are devoted to problem of approximation of functionals $F : L_1(0,1) \to \mathbb{R}^1$ on a continual set of nodes

$$\begin{gathered} x^n(z, \boldsymbol{\xi}^n) = x_0(z) + \sum_{i=1}^{n} H(z-\xi_i)[x_i(z) - x_{i-1}(z)], \\ \boldsymbol{\xi}^n = (\xi_1, \xi_2, \dots, \xi_n) \in \boldsymbol{\Omega}_{z^n} = \{\boldsymbol{z}^n : 0 \le z_1 \le \dots \le z_n \le 1\}, \end{gathered} \tag{1}$$

for example, see [1–10, 12].

Let $x_i(z) \in Q[0,1]$, $i = 0, 1, \dots$, be arbitrary fixed elements of the space $Q[0,1]$ of piecewise continuous functions on the segment $[0,1]$ with a finite number of discontinuity points of the first kind. The set of such functions is called the interpolant framework. Let $H(t)$ be the Heaviside function.

In the papers [1,2], the approximation of the Urisohn operator by polynomials of Bernstein-type and Stancu-type is investigated. In the papers [3–5], the polynomial approximation of functionals is researched. The works [6–10] are devoted to the representation of functionals by chain fractions.

To simplify a notation of a finite chain fraction we use the following notes

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}} = \mathop{\mathrm{D}}_{i=1}^{n} \frac{a_i}{b_i} = \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \dots + \frac{a_n|}{|b_n}. \tag{2}$$

УДК 519.65
2010 *Mathematics Subject Classification:* 40A15, 41A05, 41A10.

Note, that using of the formula (2) both for practical calculations and for theoretical research is quite inconvenient. Therefore, it will be useful its transformation (see [11, p. 40]).

Lemma 1. *nth suitable fraction*
$$Q_n = q_0 + \mathop{\mathrm{D}}_{i=1}^{n} \frac{q_i}{1}$$
of continued fraction Q_∞ coincides with the fraction $Q_n = A_n/B_n$, where nth numerator A_n and nth denominator B_n are determined by recurrent formulas
$$A_k = A_{k-1} + q_k A_{k-2}, \quad k = 1, 2, \dots, \quad A_{-1} = 1, \quad A_0 = K_0^I,$$
$$B_k = B_{k-1} + q_k B_{k-2}, \quad k = 1, 2, \dots, \quad B_{-1} = 0, \quad B_0 = 1.$$

Note, if $q_i = q_i\big(x(\cdot)\big)$, $i = 0, 1, \dots, n$, are defined by formula (2), then A_n, B_n are the functional polynomials of variable $x(z)$ of $[(n+1)/2][(n+2)/2]$th and $[n/2][(n+3)/2]$th degrees respectively and $Q_n\big(x(\cdot)\big)$ is rational functional interpolant. Here and subsequently, square brackets denote the integer part of a real. However, the total functional degree of the numerator and denominator is $[(n+1)/2][(n+2)/2] + [n/2][(n+3)/2]$, and the number of nodes in the framework of continuous interpolation nodes is equal to $n+1$.

2 Formulation of the problem and solution

Let us consider the problem of constructing an approximation rational fraction for the functional $F : L_1(0,1) \to \mathbb{R}^1$.

We can find a rational interpolation functional fraction in the following form
$$R_{l,m}(x(\cdot)) = \frac{A_l(x(\cdot))}{B_m(x(\cdot))}, \tag{3}$$
where
$$A_l(x(\cdot)) = F(x_0(\cdot)) + \sum_{j=1}^{l} \int_0^1 \int_{z_1}^1 \dots \int_{z_{j-1}}^1 K_{1,j}^I(\mathbf{z}^j) \prod_{p=1}^{j} (x(z_p) - x_{p-1}(z_p))\, dz_j \dots dz_1, \tag{4}$$
$$B_m(x(\cdot)) = 1 + \sum_{j=1}^{m} \int_0^1 \int_{z_1}^1 \dots \int_{z_{j-1}}^1 K_{2,j}^I(\mathbf{z}^j) \prod_{p=1}^{j} (x(z_p) - x_{p-1}(z_p))\, dz_j \dots dz_1. \tag{5}$$

In the general case, all kernels $K_{1,j}^I(\mathbf{z}^j)$ and $K_{2,j}^I(\mathbf{z}^j)$ should be found from interpolation conditions. However, we will do otherwise. First, we consider the following theorem from [12].

Theorem 1. *In order for the functional $F : L_1(0,1) \to \mathbb{R}^1$ to have a representation*
$$F(x(\cdot)) = F(x_0(\cdot)) + \sum_{j=1}^{n} \int_0^1 \int_{z_1}^1 \dots \int_{z_{j-1}}^1 K_j^I(\mathbf{z}^j) \prod_{p=1}^{j} (x(z_p) - x_{p-1}(z_p))\, dz_j \dots dz_1 + R_n(x(\cdot)), \tag{6}$$
it is necessary and sufficient to satisfy the substitution rule
$$\begin{aligned} &\frac{\partial^p}{\partial z_1 \partial z_2 \dots \partial z_p} \Big[F(x^{p+1}(\cdot; \mathbf{z}^{p+1}))\big|_{z_{p+1}=z_p} \Big] \\ &= \Big[\frac{\partial^p}{\partial z_1 \partial z_2 \dots \partial z_p} F(x^{p+1}(\cdot; \mathbf{z}^{p+1})) \Big]\Big|_{z_{p+1}=z_p} \frac{x_{p+1}(z_p) - x_{p-1}(z_p)}{x_p(z_p) - x_{p-1}(z_p)}, \quad p = 1, \dots, n, \end{aligned} \tag{7}$$

where the kernels are defined by formulas

$$K_j^I(z^j) = (-1)^j \prod_{p=1}^{j} (x_p(z_p) - x_{p-1}(z_p))^{-1} \frac{\partial^j F(x^j(\cdot; z^j))}{\partial z_1 \dots \partial z_j}, \quad j = 1, \dots, n, \tag{8}$$

and residue has the following form

$$\begin{aligned} R_n(x(\cdot)) = (-1)^{n+1} \int_0^1 \int_{z_1}^1 \cdots \int_{z_n}^1 \frac{\partial^{n+1} F(x^{n+1}(\cdot; z^{n+1}))}{\partial z_1 \partial z_2 \dots \partial z_{n+1}} \\ \times \prod_{p=1}^{n} \frac{(x(z_p) - x_{p-1}(z_p))}{(x_p(z_p) - x_{p-1}(z_p))} dz_{n+1} dz_n \dots dz_1, \quad x_{n+1}(z) = x(z). \end{aligned} \tag{9}$$

Let us consider the case, when $l \leq m$ and $A_l(x(\cdot))$ from (4) coincides with the polynomial expansion of the functional $F(x(\cdot))$ from (6) and the kernels $K_{1,j}^I(z^j)$ are defined by formulas (8). In this case we can find the residue by formula (9) and the substitution rule (7) holds.

To find the kernels $K_{2,j}^I(z^j)$ from (5) we can use the interpolation conditions

$$R_{l,m}(x^j(\cdot; z^j)) = F(x^j(\cdot; z^j)), \quad j = 1, 2, \dots, m,$$

where $x^j(t; z^j) = x_0(t) + \sum_{p=1}^{j} H(t - z_p)[x_p(t) - x_{p-1}(t)]$, $j = 1, 2, \dots, m$.

Thus, we obtain the kernel

$$K_{2,j}^I(z^j) = (-1)^j \prod_{p=1}^{j} (x_j(z_p) - x_{p-1}(z_p))^{-1} \frac{\partial^j}{\partial z_1 \dots \partial z_j} \frac{A_l(x^j(\cdot; z^j))}{F(x^j(\cdot; z^j))}. \tag{10}$$

Note, that from the Theorem 1 it follows that the equality

$$\frac{\partial^j}{\partial z_1 \dots \partial z_j} \frac{A_k(x^j(\cdot; z^j))}{F(x^j(\cdot; z^j))} = 0$$

holds for all $j \leq k$. Therefore, the ratio (5) will have the following form

$$B_m(x(\cdot)) = 1 + \sum_{j=l+1}^{m} \int_0^1 \int_{z_1}^1 \cdots \int_{z_{j-1}}^1 K_{2,j}^I(z^j) \prod_{p=1}^{j} (x(z_p) - x_{p-1}(z_p)) \, dz_j \dots dz_1. \tag{11}$$

In this case the following theorem holds.

Theorem 2. *In order for there to be an unique rational interpolation functional* (3), (4), (8), (10), (11), *on a continual set of nodes* (1)

$$x^j(t; z^j) = x_0(t) + \sum_{p=1}^{j} H(t - z_p)[x_p(t) - x_{p-1}(t)], \quad j = 1, 2, \dots, m,$$

it is necessary and sufficient for the functional $F(x(\cdot))$ *to satisfy the substitution rule* (7).

Note, that the interpolant (3), (4), (8), (10), (11) is the one that holds any rational functional of the form (3).

Example. *Let the functional $F\big(x(\cdot)\big)$ has the form*

$$F(x(\cdot)) = arctg \int_0^1 x(t)dt$$

and $l = 1$, $m = 2$. Then from (3), (4) *and* (11) *we obtain*

$$x^1(z;\xi_1) = x_0(z) + H(z-\xi_1)(x_1(z) - x_0(z)),$$
$$x^2(z;\boldsymbol{\xi}^2) = x_0(z) + H(z-\xi_1)(x_1(z) - x_0(z)) + H(z-\xi_2)(x_2(z) - x_1(z)),$$
$$R^I_{1,2}(x(\cdot)) = \frac{A_1(x(\cdot))}{B_2(x(\cdot))},$$

where

$$A_1(x(\cdot)) = F(x_0(\cdot)) - \int_0^1 \frac{dF(x^1(\cdot;z^1))}{dz} \frac{(x(z) - x_0(z))}{(x_1(z) - x_0(z))}\, dz,$$
$$B_2(x(\cdot)) = 1 + \int_0^1 \int_0^1 \frac{\partial^2}{\partial z_1 \partial z_2} \frac{A_1(x^2(\cdot;z^2))}{F(x^2(\cdot;z^2))} \prod_{i=1}^{2} \frac{(x(z_i) - x_{i-1}(z_i))}{(x_i(z_i) - x_{i-1}(z_i))}\, dz_2\, dz_1.$$

Therefore, the rational interpolation functional fraction on a continual set of nodes $x^2(z;\boldsymbol{\xi}^2)$, $\boldsymbol{\xi}^2 \in \boldsymbol{\Omega}_2$, is constructed for a functional $F(x(\cdot))$. For the framework of interpolation nodes we can take, for example, $x_0(t) = 0$, $x_1(t) = t$, $x_2(t) = t^2$. Then it is easy to verify that

$$\frac{d}{d\xi_1}\Big[F(x^2(\cdot;\boldsymbol{\xi}^2))\big|_{\xi_2=\xi_1}\Big] = \frac{-\xi_1^2}{1+(1/3-\xi_1^3/3)^2}, \quad \Big[\frac{\partial}{\partial\xi_1}F(x^2(\cdot;\boldsymbol{\xi}^2))\Big]\Big|_{\xi_2=\xi_1} = \frac{-\xi_1}{1+(1/3-\xi_1^3/3)^2},$$

and the substitution rule holds. The results of the calculations we record in the Table 1, where $R^I_{1,2}\big(x(\cdot)\big)$ is rational interpolation functional, and

$$d = arctg \int_0^1 x(t)dt, \quad dP = R^I_{1,2} - d.$$

	$x(t) = \sqrt{t}$	$x(t) = \sin(t)$
$R^I_{1,2}$	0.6179360142	0.4223479311
d	0.5880026036	0.4308892024
dP	0.0299334106	0.0085412713

Table 1. The results of calculations.

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Received 09.06.2021
Revised 03.10.2021

Баранецький Я.О., Демків І.І., Копач М.І., Соломко А.В. *Інтерполяційний (L, M)-раціональний інтегральний дріб на континуальній множині вузлів* // Карпатські матем. публ. — 2021. — Т.13, №3. — C. 587–591.

У статті будується та досліджується інтегральний раціональний інтерполянт на континуальній множині вузлів, який є відношенням функціонального полінома степеня L до функціонального полінома степеня M. Одержаний інтерполянт є таким, що зберігає будь-який раціональний функціонал одержаного вигляду.

Ключові слова і фрази: інтерполяція, функціональний поліном, континуальна множина вузлів, ланцюговий дріб, раціональний дріб.