Bases in finite groups of small order

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A subset B of a group G is called a *basis* of G if each element $g \in G$ can be written as $g = ab$ for some elements $a, b \in B$. The smallest cardinality $|B|$ of a basis $B \subseteq G$ is called the *basis size* of G and is denoted by $r[G]$. We prove that each finite group G has $r[G] > \sqrt{|G|}$. If G is Abelian, then $r[G] \geq \sqrt{2|G| - |G|/|G_2|}$, where $G_2 = \{g \in G : g^{-1} = g\}$. Also we calculate the basis sizes of all Abelian groups of order ≤ 60 and all non-Abelian groups of order ≤ 40.

Key words and phrases: finite group, Abelian group, basis, basis size, basis characteristic.

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Introduction

A subset B of a group G is called a *basis* of G if each element $g \in G$ can be written as $g = ab$ for some $a, b \in B$. The smallest cardinality of a basis of G is called the *basis size* of G and is denoted by $r[G]$. The problem of estimating $r[G]$ for cyclic groups was first proposed by I. Schur and various bounds were obtained by H. Rohrbach [16], L. Moser [12], A. Stöhr [18], W. Klotz [10] and others. Bases for arbitrary groups were investigated by H. Rohrbach [17] and lately by J. Cherly [7], E.A. Bertram, M. Herzog [6], M.B. Nathanson [14], G. Kozma and A. Lev [19].

The definition of a basis B for a group G implies that $|G| \leq |B|^2$ and hence $r[G] \geq \sqrt{|G|}$. The fraction

$$\delta[G] := \frac{r[G]}{\sqrt{|G|}} \geq 1$$

is called the *basis characteristic* of G.

For a class $\mathcal{G}$ of finite groups the number

$$\delta[\mathcal{G}] = \sup_{G \in \mathcal{G}} \delta[G]$$

is called the *basis characteristic* of the class $\mathcal{G}$. By Groups we shall denote the class of all finite groups.

In this paper we are interested in evaluating the basis characteristics of finite groups. In fact, this problem has been studied in the literature. In particular, G. Kozma and A. Lev [19] proved that each finite group G has basis characteristic $\delta[G] \leq \frac{4}{\sqrt{3}} \approx 2.3094$.

A standard model of a cyclic group of order n is the multiplicative group

$$C_n = \{z \in \mathbb{C} : z^n = 1\}$$

of n-th roots of 1. The group C_n is isomorphic to the additive group of the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

For a real number x we put

$$\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\} \quad \text{and} \quad \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}.$$

1 Known results

In this section we recall some known results on bases in finite groups. The following fundamental fact was proved by G. Kozma and A. Lev [19] (using the classification of finite simple groups).

Theorem 1 ([19]). *Each finite group G has the basis characteristic $\delta[G] \leq \frac{4}{\sqrt{3}}$.*

The following proposition was proved by E.A. Bertram and M. Herzog.

Proposition 1 ([6]). *Let G be a finite group. Then*

1) $r[G] \leq |G : H| \cdot r[H]$ for any subgroup $H \subseteq G$;

2) $r[G] \leq r[G/H] \cdot |H|$ for any normal subgroup $H \subseteq G$;

3) $r[G] \leq 2r[G/H] \cdot r[H]$ for any normal subgroup $H \subseteq G$;

4) $r[G] \leq r[G/H] \cdot r[H]$ for any subgroup $H \subseteq Z(G)$;

5) $r[G] \leq |H| + |G : H| - 1$ for any subgroup $H \subseteq G$.

In Proposition 1 by $Z(G) = \{z \in G : \forall x \in G \ (xz = zx)\}$ the *center* of G is denoted.

The following theorem on lower bounds of basis size of Abelian groups was proved by E.A. Bertram and M. Herzog.

Theorem 2 ([6]). *Each finite Abelian group G has the basis size*

$$r[G] \geq \frac{1}{2}\left(\sqrt{8|G| + 1} - 1\right).$$

The following upper bound on the basis size of cyclic groups was obtained by E.A. Bertram and M. Herzog in [6].

Theorem 3 ([6]). *For any $n > 2$ the cyclic group C_n has the basis size*

$$r[C_n] < 2\sqrt{n - 1}.$$

The upper bound in Theorem 3 can be asymptotically improved using the old result of A. Mrose.

Theorem 4 ([13]). *For a sufficiently large n, the cyclic group C_n has basis size*

$$r[C_n] \leq \sqrt{\frac{7}{2}n} \approx 1.87\sqrt{n}.$$

The constant $\sqrt{\frac{7}{2}} \approx 1.87$ was recently improved by J. Kohonen in [11].

Theorem 5 ([11]). *For a sufficiently large n, the cyclic group C_n has basis size*

$$r[C_n] \leq \sqrt{\frac{294}{85}n} \approx 1.86\sqrt{n}.$$

Bases in dihedral and Boolean groups were investigated in [8].

In evaluating the basis characteristics of groups we can use their difference characteristics. A subset B of a group G is called a *difference basis* of G if each element $g \in G$ can be written as $g = xy^{-1}$ for some $x, y \in B$. The smallest cardinality of a difference basis of G is called the *difference size* of G and is denoted by $\Delta[G]$. The definition of a difference basis B for a group G implies that $|G| \leq |B|^2$ and hence $\Delta[G] \geq \sqrt{|G|}$. The fraction

$$\eth[G] := \frac{\Delta[G]}{\sqrt{|G|}} \geq 1$$

is called the *difference characteristic* of G.

Difference bases and difference characteristics in dihedral and Abelian groups were investigated in [3, 4]. Difference bases have applications in the study of structure of superextensions of groups, see [1, 5, 9].

The following facts on the difference characteristics of cyclic groups were proved in [2].

Theorem 6 ([2]). *For any $n \in \mathbb{N}$ the cyclic group C_n has the difference characteristic:*

1) $\eth[C_n] \leq \eth[C_4] = \frac{3}{2}$;

2) $\eth[C_n] \leq \eth[C_2] = \eth[C_8] = \sqrt{2}$ *if $n \neq 4$;*

3) $\eth[C_n] \leq \frac{12}{\sqrt{73}} < \sqrt{2}$ *if $n \geq 9$;*

4) $\eth[C_n] \leq \frac{24}{\sqrt{293}} < \frac{12}{\sqrt{73}}$ *if $n \geq 9$ and $n \neq 292$;*

5) $\eth[C_n] < \frac{2}{\sqrt{3}}$ *if $n \geq 2 \cdot 10^{15}$.*

The following Table 1 of difference sizes and characteristics of cyclic groups C_n for ≤ 100 is taken from [2].

n	$\Delta[C_n]$	$\eth[C_n]$	n	$\Delta[C_n]$	$\eth[C_n]$	n	$\Delta[C_n]$	$\eth[C_n]$	n	$\Delta[C_n]$	$\eth[C_n]$
1	1	1	26	6	1.1766...	51	8	1.1202...	76	10	1.1470...
2	2	1.4142...	27	6	1.1547...	52	9	1.2480...	77	10	1.1396...
3	2	1.1547...	28	6	1.1338...	53	9	1.2362...	78	10	1.1322...
4	3	1.5	29	7	1.2998...	54	9	1.2247...	79	10	1.1250...
5	3	1.3416...	30	7	1.2780...	55	9	1.2135...	80	11	1.2298...
6	3	1.2247...	31	6	1.0776...	56	9	1.2026...	81	11	1.2222...
7	3	1.1338...	32	7	1.2374...	57	8	1.0596...	82	11	1.2147...
8	4	1.4142...	33	7	1.2185...	58	9	1.1817...	83	11	1.2074...
9	4	1.3333...	34	7	1.2004...	59	9	1.1717...	84	11	1.2001...
10	4	1.2649...	35	7	1.1832...	60	9	1.1618...	85	11	1.1931...
11	4	1.2060...	36	7	1.1666...	61	9	1.1523...	86	11	1.1861...
12	4	1.1547...	37	7	1.1507...	62	9	1.1430...	87	11	1.1793...
13	4	1.1094...	38	8	1.2977...	63	9	1.1338...	88	11	1.1726...
14	5	1.3363...	39	7	1.1208...	64	9	1.125	89	11	1.1659...
15	5	1.2909...	40	8	1.2649...	65	9	1.1163...	90	11	1.1595...
16	5	1.25	41	8	1.2493...	66	10	1.2309...	91	10	1.0482...
17	5	1.2126...	42	8	1.2344...	67	10	1.2216...	92	11	1.1468...
18	5	1.1785...	43	8	1.2199...	68	10	1.2126...	93	12	1.2443...
19	5	1.1470...	44	8	1.2060...	69	10	1.2038...	94	12	1.2377...
20	6	1.3416...	45	8	1.1925...	70	10	1.1952...	95	12	1.2311...
21	5	1.0910...	46	8	1.1795...	71	10	1.1867...	96	12	1.2247...
22	6	1.2792...	47	8	1.1669...	72	10	1.1785...	97	12	1.2184...
23	6	1.2510...	48	8	1.1547...	73	9	1.0533...	98	12	1.2121...
24	6	1.2247...	49	8	1.1428...	74	10	1.1624...	99	12	1.2060...
25	6	1.2	50	8	1.1313...	75	10	1.1547...	100	12	1.2

Table 1: Difference sizes and characteristics of cyclic groups C_n for $n \leq 100$.

2   Basis sizes and characteristics of finite groups

Theorem 7. *For any non-trivial finite group G we have $r[G] \leq 2 \cdot \Delta[G] - 1$. If G has even order, then $r[G] \leq 2 \cdot \Delta[G] - 2$.*

Proof. Choose any difference basis $D \subseteq G$ of cardinality $|D| = \Delta[G]$. Then $G = DD^{-1}$. Replacing D by $x^{-1}D$ for any element $x \in D$, we can assume that D contains the identity element e of the group G.

The set $A = D \cup D^{-1}$ has cardinality $|A| \leq 2|D| - 1$ and has $AA \supseteq DD^{-1} = G$, which means that A is a basis and $r[G] \leq |A| \leq 2|D| - 1 = 2 \cdot \Delta[G] - 1$.

Now assume that G has even order. Then G contains an element a or order 2, according to the Sylow Theorem [15, 1.6.16]. Since $DD^{-1} = G$, there are elements $b, c \in D$ with $a = bc^{-1}$. Then $cb^{-1} = a^{-1} = a = bc^{-1}$ and for the set $B = Db^{-1} \cup bD^{-1}$ we have

$$\{e, a\} \subseteq Db^{-1} \cap bD^{-1}$$

and hence $|B| \leq 2 \cdot |D| - 2$. On the other hand,

$$BB \supseteq (Db^{-1}) \cdot (bD^{-1}) = DD^{-1} = G$$

and hence

$$r[G] \le |B| \le 2 \cdot |D| - 2 = 2 \cdot \Delta[G] - 2.$$

$\square$

In Table 2 we present the results of computer calculations of basis sizes and characteristics of cyclic groups of order ≤ 60. In this table

$$lb[C_n] := \left\lceil \sqrt{2n + 0.25} - 0.5 \right\rceil$$

is the lower bound given in Theorem 2.

n	$lb[C_n]$	$r[C_n]$	$\delta[C_n]$	n	$lb[C_n]$	$r[C_n]$	$\delta[C_n]$	n	$lb[C_n]$	$r[C_n]$	$\delta[C_n]$
1	1	1	1	21	6	7	1.5275...	41	9	10	1.5617...
2	2	2	1.4142...	22	7	8	1.7056...	42	9	10	1.5430...
3	2	2	1.1547...	23	7	8	1.6681...	43	9	10	1.5249...
4	3	3	1.5	24	7	8	1.6329...	44	9	11	1.6583...
5	3	3	1.3416...	25	7	8	1.6	45	9	11	1.6397...
6	3	4	1.6329...	26	7	8	1.5689...	46	10	11	1.6218...
7	4	4	1.5118...	27	7	8	1.5396...	47	10	11	1.6045...
8	4	4	1.4142...	28	7	9	1.7008...	48	10	11	1.5877...
9	4	4	1.3333...	29	8	8	1.4855...	49	10	11	1.5714...
10	4	5	1.5811...	30	8	8	1.4605...	50	10	12	1.6970...
11	5	5	1.5075...	31	8	9	1.6164...	51	10	11	1.5403...
12	5	5	1.4433...	32	8	9	1.5909...	52	10	12	1.6641...
13	5	5	1.3867...	33	8	9	1.5666...	53	10	12	1.6483...
14	5	6	1.6035...	34	8	9	1.5434...	54	10	12	1.6329...
15	5	6	1.5491...	35	8	9	1.5212...	55	10	12	1.6180...
16	6	6	1.5	36	8	10	1.6666...	56	11	12	1.6035...
17	6	6	1.4552...	37	9	10	1.6439...	57	11	12	1.5894...
18	6	7	1.6499...	38	9	10	1.6222...	58	11	13	1.7069...
19	6	6	1.3764...	39	9	10	1.6012...	59	11	12	1.5622...
20	6	7	1.5652...	40	9	10	1.5811...	60	11	13	1.6782...

Table 2: Basis sizes and characteristics of cyclic groups C_n for $n \le 60$.

The results of computer calculations suggest the following question.

Question 1. *Is $|r[C_{n+1}] - r[C_n]| \le 1$ for every $n \in \mathbb{N}$?*

The results of computer calculations given in Table 2 imply the following lower and upper bounds for the basis characteristic of the class Cyclic of finite cyclic groups.

Proposition 2. *The class* Cyclic *of finite cyclic groups has the basis characteristic*

$$1.7069 \cdots = \frac{13}{\sqrt{58}} = \delta[C_{58}] \le \delta[\mathsf{Cyclic}] < 2.$$

Proof. The estimate for an upper bound of the basis characteristic for the class Cyclic of finite cyclic groups is $\delta[\mathsf{Cyclic}] < 2$, see Theorem 3.

On the other hand, the basis size $r[C_{58}] = 13$ of the cyclic group C_{58} witnesses that

$$1.7069 \cdots = \delta[C_{58}] \leq \delta[\text{Cyclic}].$$

$\square$

Question 2. *Is $\delta[\text{Cyclic}] = \delta[C_{58}] = 1.7069\ldots$?*

Next we establish some new lower bounds for the basis size of a finite (Abelian) group.

Theorem 8. *Every finite Abelian group G has*

$$r[G] \geq \sqrt{2|G| - |G|/|G_2|} \quad \text{and} \quad \delta[G] \geq \sqrt{2 - 1/|G_2|},$$

where $G_2 = \{g \in G \,:\, g^{-1} = g\}$.

Proof. Fix a basis $A \subseteq G$ of cardinality $|A| = r[G]$. Observe that G_2 coincides with the kernel of the homomorphism $h : G \to G$, $h : x \mapsto x^2$, and hence the image $h[G]$ has cardinality $|G/G_2| = |G|/|G_2|$. For every $y \in G \setminus h[G]$ there are elements $a, b \in A$ with $y = ab = ba$. It follows from $y \notin h[G]$ that $a \neq b$ and hence

$$r[G]^2 = |A \times A| \geq 2 \cdot |G \setminus h[G]| + |h[G]| = 2 \cdot (|G| - |G|/|G_2|) + |G|/|G_2| = 2 \cdot |G| - |G|/|G_2|.$$

$\square$

The proof of the following theorem was suggested by the `Mathoverflow` users `lambda`, `user49822` and `Geoff Robinson`, who answered a question posed by the first author at `https://mathoverflow.net/q/384292/61536`.

Theorem 9. *Every non-trivial finite group G has $r[G] > \sqrt{|G|}$.*

Proof. To derive a contradiction, assume that $r[G] = \sqrt{|G|}$ for some finite group G. Choose a basis $B \subseteq G$ of cardinality $|B| = \sqrt{|G|}$. It follows that the multiplication map $\mu : B \times B \to G$, $\mu : (x, y) \mapsto xy$, is bijective. Consider the complex linear space $\mathbb{C}^G$ of functions from G to the complex plane. Assign to each $g \in G$ the characteristic function of the singleton $\{g\}$. This is a unique function $\chi_g : G \to \{0, 1\}$ such that $\chi_g^{-1}(1) = \{g\}$. It is clear that $\{\chi_g : g \in G\}$ is a basis of the linear space $\mathbb{C}^G$.

For every element $x \in G$, let $T_x : \mathbb{C}^G \to \mathbb{C}^G$ be the unique linear operator such that $T_x(\chi_g) = \chi_{xg}$ for every $g \in G$. It is easy to see that $T_x \circ T_y = T_{xy}$ for any elements $x, y \in G$. The operators T_g, $g \in G$, are elements of the algebra $\text{End}(\mathbb{C}^G)$ of linear endomorphisms of the linear space $\mathbb{C}^G$.

Consider the endomorphisms

$$\Phi = \frac{1}{|B|} \sum_{b \in B} T_b \quad \text{and} \quad E = \frac{1}{|G|} \sum_{g \in G} T_g.$$

Observe that for any $g \in G$ we have

$$E \circ T_g = \frac{1}{|G|} \sum_{x \in G} T_x \circ T_g = \frac{1}{|G|} \sum_{x \in G} T_{xg} = \frac{1}{|G|} \sum_{y \in G} T_y = E$$

and

$$T_g \circ E = \frac{1}{|G|} \sum_{x \in G} T_g \circ T_x = \frac{1}{|G|} \sum_{x \in G} T_{gx} = \frac{1}{|G|} \sum_{y \in G} T_y = E.$$

Also

$$E \circ E = \frac{1}{|G|} \sum_{x \in G} T_x \circ E = \frac{1}{|G|} \sum_{x \in G} E = E.$$

The bijectivity of the multiplication map $\mu : B \times B \to G$ implies

$$\Phi \circ \Phi = \frac{1}{|B|^2} \sum_{a,b \in B} T_a \circ T_b = \frac{1}{|G|} \sum_{a,b \in B} T_{ab} = \frac{1}{|G|} \sum_{g \in G} T_g = E.$$

It follows that

$$(\Phi - E) \circ (\Phi - E) = \Phi \circ \Phi - E \circ \Phi - \Phi \circ E + E \circ E = E - E - E + E = 0,$$

which means that the endomorphism $\Phi - E$ is nilpotent and hence has trace equal to zero. The bijectivity of the multiplication map μ implies that the basis B does not contain the identity e of the group G. Then for every $b \in B$ the matrix of the transformation T_b has zero trace and the endomorphism $\Phi = \frac{1}{|B|} \sum_{b \in B} T_b$ also has zero trace. On the other hand, the trace of the endomorphism $E = \frac{1}{|G|} \sum_{g \in G} T_g$ is equal to the trace of the operator $\frac{1}{|G|} T_e$ and hence it is equal to 1. Consequently the operator $\Phi - E$ has trace -1 and cannot be nilpotent. This contradiction shows that $r(G) > \sqrt{|G|}$. □

In the following Table 3 we present the results of computer calculations of basis sizes and characteristics of all non-cyclic Abelian groups G of order $|G| \le 60$. In this table

$$lb[G] := \max \left\{ \left\lceil \sqrt{2|G| + 0.25} - 0.5 \right\rceil, \left\lceil \sqrt{2|G| - |G|/|G_2|} \right\rceil \right\}$$

is the lower bound given in Theorems 2 and 8.

The results of computer calculations given in Table 2, Table 3 and Theorem 1 imply the following lower and upper bounds for the basis characteristic of the class Abelian of finite Abelian groups.

Proposition 3. *The class* Abelian *of finite Abelian groups has basis characteristic*

$$1.7677 \cdots = \frac{5}{\sqrt{8}} = \delta \left[(C_2)^3 \right] \le \delta[\text{Abelian}] \le \frac{4}{\sqrt{3}} \approx 2.3094.$$

Proof. The upper bound $\delta[\text{Abelian}] \le \frac{4}{\sqrt{3}}$ follows from Theorem 1 and the lower bound

$$\delta[\text{Abelian}] \ge \delta \left[(C_2)^3 \right] = \frac{5}{\sqrt{8}}$$

follows from the known value $r \left[(C_2)^3 \right] = 5$. □

Question 3. *Is $\delta[\text{Abelian}] = \delta \left[(C_2)^3 \right] = 1.7677\ldots$?*

G	$(C_2)^2$	$(C_2)^3$	$C_2 \times C_4$	$(C_3)^2$	$(C_2)^2 \times C_3$	$C_2 \times C_8$	$(C_4)^2$
$lb[G]$	3	4	4	4	5	6	6
$r[G]$	3	5	5	4	5	7	6
$\delta[C_n]$	1.5	1.7677...	1.7677...	1.3333...	1.4433...	1.75	1.5
G	$(C_2)^2 \times C_4$	$(C_2)^4$	$C_2 \times (C_3)^2$	$(C_2)^2 \times C_5$	$C_2 \times C_3 \times C_4$	$(C_2)^3 \times C_3$	$(C_5)^2$
$lb[G]$	6	6	6	6	7	7	7
$r[G]$	6	6	7	7	8	8	8
$\delta[C_n]$	1.5	1.5	1.6499...	1.5652...	1.6329...	1.6329...	1.6
G	$C_3 \times C_9$	$(C_3)^3$	$(C_2)^2 \times C_7$	$C_2 \times C_{16}$	$C_4 \times C_8$	$(C_2)^2 \times C_8$	$C_2 \times (C_4)^2$
$lb[G]$	7	7	7	8	8	8	8
$r[G]$	8	8	9	9	10	10	10
$\delta[C_n]$	1.5396...	1.5396...	1.7008...	1.5909...	1.7677...	1.7677...	1.7677...
G	$(C_2)^3 \times C_4$	$(C_2)^5$	$(C_6)^2$	$(C_2)^2 \times C_9$	$(C_3)^2 \times C_4$	$(C_2)^3 \times C_5$	$C_2 \times C_4 \times C_5$
$lb[G]$	8	8	8	8	8	9	9
$r[G]$	10	10	10	10	9	10	10
$\delta[C_n]$	1.7677...	1.7677...	1.6666...	1.6666...	1.5	1.5811...	1.5811...
G	$(C_2)^2 \times C_{11}$	$(C_3)^2 \times C_5$	$C_2 \times C_3 \times C_8$	$C_3 \times (C_4)^2$	$(C_2)^2 \times C_3 \times C_4$	$(C_2)^4 \times C_3$	$(C_7)^2$
$lb[G]$	9	9	10	10	10	10	10
$r[G]$	11	11	11	11	11	12	11
$\delta[C_n]$	1.6583...	1.6397...	1.5877...	1.5877...	1.5877...	1.7320...	1.5714...
G	$C_2 \times (C_5)^2$	$(C_2)^2 \times C_{13}$	$C_6 \times C_9$	$C_2 \times (C_3)^3$	$C_2 \times C_4 \times C_7$	$(C_2)^3 \times C_7$	$(C_2)^2 \times C_3 \times C_5$
$lb[G]$	10	10	10	10	11	11	11
$r[G]$	12	12	12	12	12	12	13
$\delta[C_n]$	1.6970...	1.6641...	1.6329...	1.6329...	1.6035...	1.6035...	1.6782...

Table 3: Basis sizes and characteristics of non-cyclic Abelian groups G of order $|G| \leq 60$.

In the following Table 4 we present the results of computer calculations of basis sizes and characteristics of all groups G of order $|G| \leq 40$. For calculating we use the system for computational discrete algebra GAP. The command SmallGroup(n, k) returns the k-th group of order n in the GAP catalogue. In Table 4, (n, k) denotes SmallGroup(n, k). In GAP the command StructureDescription(G) gives the description of the structure of a group G. For example, StructureDescription(SmallGroup(16, 12)) gives $C_2 \times Q_8$. With the red color we denote Abelian groups, blue are nilpotent non-Abelian groups. All other groups in Table 4 are solvable and non-nilpotent. They are denoted with black color.

The results of computer calculations given in Table 4 and Theorem 1 imply the following lower and upper bounds for the basis characteristic of the class NonAbelian of finite non-Abelian groups.

Proposition 4. *The class* NonAbelian *of finite non-Abelian groups has basis characteristic*

$$1.5 = \frac{6}{\sqrt{16}} = \delta\left[C_2 \times Q_8\right] \leq \delta\left[\text{NonAbelian}\right] \leq \frac{4}{\sqrt{3}} \approx 2.3094.$$

Proof. The upper bound $\delta\left[\text{NonAbelian}\right] \leq \frac{4}{\sqrt{3}}$ follows from Theorem 1 and the lower bound $\delta\left[\text{NonAbelian}\right] \geq \delta\left[C_2 \times Q_8\right] = \delta\left[\text{SmallGroup}(16, 12)\right] = 1.5$ follows from the known value $r\left[C_2 \times Q_8\right] = r\left[\text{SmallGroup}(16, 12)\right] = 6$. $\qquad\square$

G	r[G]	δ[G]	G	r[G]	δ[G]	G	r[G]	δ[G]	G	r[G]	δ[G]
(1,1)	1	1	(18,5)	7	1.6499...	(32,2)	8	1.4142...	(32,49)	7	1.2374...
(2,1)	2	1.4142...	(19,1)	6	1.3764...	(32,3)	10	1.7677...	(32,50)	7	1.2374...
(3,1)	2	1.1547...	(20,1)	6	1.3416...	(32,4)	7	1.2374...	(32,51)	10	1.7677...
(4,1)	3	1.5	(20,2)	7	1.5652...	(32,5)	7	1.2374...	(33,1)	9	1.5666...
(4,2)	3	1.5	(20,3)	5	1.1180...	(32,6)	6	1.0606...	(34,1)	7	1.2004...
(5,1)	3	1.3416...	(20,4)	5	1.1180...	(32,7)	7	1.2374...	(34,2)	9	1.5434...
(6,1)	3	1.2247...	(20,5)	7	1.5652...	(32,8)	7	1.2374...	(35,1)	9	1.5212...
(6,2)	4	1.6329...	(21,1)	6	1.3093...	(32,9)	7	1.2374...	(36,1)	8	1.3333...
(7,1)	4	1.5118...	(21,2)	7	1.5275...	(32,10)	7	1.2374...	(36,2)	10	1.6666...
(8,1)	4	1.4142...	(22,1)	6	1.2792...	(32,11)	6	1.0606...	(36,3)	8	1.3333...
(8,2)	5	1.7677...	(22,2)	8	1.7056...	(32,12)	7	1.2374...	(36,4)	7	1.1666...
(8,3)	3	1.0606...	(23,1)	8	1.6681...	(32,13)	7	1.2374...	(36,5)	10	1.6666...
(8,4)	4	1.4142...	(24,1)	6	1.2247...	(32,14)	7	1.2374...	(36,6)	8	1.3333...
(8,5)	5	1.7677...	(24,2)	8	1.6329...	(32,15)	7	1.2374...	(36,7)	8	1.3333...
(9,1)	4	1.3333...	(24,3)	6	1.2247...	(32,16)	9	1.5909...	(36,8)	9	1.5
(9,2)	4	1.3333...	(24,4)	7	1.4288...	(32,17)	7	1.2374...	(36,9)	7	1.1666...
(10,1)	4	1.2649...	(24,5)	6	1.2247...	(32,18)	7	1.2374...	(36,10)	7	1.1666...
(10,2)	5	1.5811...	(24,6)	6	1.2247...	(32,19)	7	1.2374...	(36,11)	7	1.1666...
(11,1)	5	1.5075...	(24,7)	7	1.4288...	(32,20)	7	1.2374...	(36,12)	8	1.3333...
(12,1)	5	1.4433...	(24,8)	6	1.2247...	(32,21)	10	1.7677...	(36,13)	8	1.3333...
(12,2)	5	1.4433...	(24,9)	8	1.6329...	(32,22)	8	1.4142...	(36,14)	10	1.6666...
(12,3)	4	1.1547...	(24,10)	6	1.2247...	(32,23)	8	1.4142...	(37,1)	10	1.6439...
(12,4)	4	1.1547...	(24,11)	6	1.2247...	(32,24)	7	1.2374...	(38,1)	8	1.2977...
(12,5)	5	1.4433...	(24,12)	5	1.0206...	(32,25)	8	1.4142...	(38,2)	10	1.6222...
(13,1)	5	1.3867...	(24,13)	6	1.2247...	(32,26)	8	1.4142...	(39,1)	8	1.2810...
(14,1)	5	1.3363...	(24,14)	6	1.2247...	(32,27)	6	1.0606...	(39,2)	10	1.6012...
(14,2)	6	1.6035...	(24,15)	8	1.6329...	(32,28)	7	1.2374...	(40,1)	8	1.2649...
(15,1)	6	1.5491...	(25,1)	8	1.6	(32,29)	7	1.2374...	(40,2)	10	1.5811...
(16,1)	6	1.5	(25,2)	8	1.6	(32,30)	6	1.0606...	(40,3)	8	1.2649...
(16,2)	6	1.5	(26,1)	6	1.1766...	(32,31)	6	1.0606...	(40,4)	8	1.2649...
(16,3)	5	1.25	(26,2)	8	1.5689...	(32,32)	7	1.2374...	(40,5)	8	1.2649...
(16,4)	5	1.25	(27,1)	8	1.5396...	(32,33)	7	1.2374...	(40,6)	8	1.2649...
(16,5)	7	1.75	(27,2)	8	1.5396...	(32,34)	7	1.2374...	(40,7)	9	1.4230...
(16,6)	5	1.25	(27,3)	6	1.1547...	(32,35)	7	1.2374...	(40,8)	8	1.2649...
(16,7)	5	1.25	(27,4)	6	1.1547...	(32,36)	10	1.7677...	(40,9)	10	1.5811...
(16,8)	5	1.25	(27,5)	8	1.5396...	(32,37)	7	1.2374...	(40,10)	8	1.2649...
(16,9)	5	1.25	(28,1)	7	1.3228...	(32,38)	8	1.4142...	(40,11)	8	1.2649...
(16,10)	6	1.5	(28,2)	9	1.7008...	(32,39)	7	1.2374...	(40,12)	8	1.2649...
(16,11)	5	1.25	(28,3)	6	1.1338...	(32,40)	7	1.2374...	(40,13)	8	1.2649...
(16,12)	6	1.5	(28,4)	9	1.7008...	(32,41)	7	1.2374...	(40,14)	10	1.5811...
(16,13)	5	1.25	(29,1)	8	1.4855...	(32,42)	7	1.2374...	(41,1)	10	1.5617...
(16,14)	6	1.5	(30,1)	7	1.2780...	(32,43)	6	1.0606...	(42,1)	7	1.0801...
(17,1)	6	1.4552...	(30,2)	7	1.2780...	(32,44)	6	1.0606...	(42,2)	8	1.2344...
(18,1)	5	1.1785...	(30,3)	7	1.2780...	(32,45)	10	1.7677...	(42,5)	8	1.2344...
(18,2)	7	1.6499...	(30,4)	8	1.4605...	(32,46)	8	1.4142...	(42,6)	10	1.5430...
(18,3)	5	1.1785...	(31,1)	9	1.6164...	(32,47)	8	1.4142...	(50,3)	8	1.1313...
(18,4)	5	1.1785...	(32,1)	9	1.5909...	(32,48)	8	1.4142...	(50,4)	8	1.1313...

Table 4: Basis sizes and characteristics of all groups G of order $|G| \le 40$.

Question 4. *Is* $\delta[\text{NonAbelian}] = \delta[C_2 \times Q_8] = 1.5$?

The results of computer calculations given in Table 2, Table 3, Table 4 and Theorem 1 imply

the following lower and upper bounds for the basis characteristic of the class Groups of all finite groups.

Proposition 5. *The class* Groups *of all finite groups has basis characteristic*

$$1.7677\cdots = \frac{5}{\sqrt{8}} = \delta\left[(C_2)^3\right] \leq \delta[\text{Groups}] \leq \frac{4}{\sqrt{3}} \approx 2.3094.$$

Proof. The upper bound $\delta[\text{Groups}] \leq \frac{4}{\sqrt{3}}$ follows from Theorem 1 and the lower bound

$$\delta[\text{Groups}] \geq \delta\left[(C_2)^3\right] = \frac{5}{\sqrt{8}}$$

follows from the known value $r[(C_2)^3] = 5$. □

The results of computer calculations suggest the following questions.

Question 5. *Is $\delta[\text{Groups}] = \delta[(C_2)^3] = 1.7677\ldots$?*

Question 6. *Is $\delta[\text{Groups}] = \delta[\text{Abelian}]$?*

Problem 1. *Is $r[S_n] = \left\lceil\sqrt{n!}\right\rceil$ for the symmetric group S_n of degree n?*

Remark 1. *Computer calculations show that $r[S_n] = \left\lceil\sqrt{n!}\right\rceil$ for $n \leq 5$.*

Problem 2. *Is $r[S_n] = (1 + o(1)) \cdot \sqrt{n!}$ as $n \to \infty$? Equivalently, is $\lim_{n\to\infty} \delta[S_n] = 1$?*

Problem 3. *Is $r[A_n] = \left\lceil\sqrt{n!/2}\right\rceil$ for alternating group A_n of degree n?*

Remark 2. *Computer calculations show that $r[A_n] = \left\lceil\sqrt{n!/2}\right\rceil$ for $n \leq 5$.*

Problem 4. *Is $r[A_n] = (1 + o(1)) \cdot \sqrt{\frac{n!}{2}}$ as $n \to \infty$? Equivalently, is $\lim_{n\to\infty} \delta[A_n] = 1$?*

Acknowledgments

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Підмножина B групи G називається *базою* групи G, якщо кожен елемент $g \in G$ можна подати у вигляді $g = ab$ для деяких елементів $a, b \in B$. Найменша потужність $|B|$ бази $B \subseteq G$ називається *базовим розміром* групи G та позначається через $r[G]$. Ми доводимо, що кожна скінченна група G має базовий розмір $r[G] > \sqrt{|G|}$. Якщо група G є абелевою, то має місце нерівність $r[G] \geq \sqrt{2|G| - |G|/|G_2|}$, де $G_2 = \{g \in G : g^{-1} = g\}$. Також ми обчислюємо базові розміри всіх абелевих груп порядку ≤ 60 та всіх неабелевих груп порядку ≤ 40.

Ключові слова і фрази: скінченна група, абелева група, база, базовий розмір, базова характеристика.