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Extreme points of $\mathcal{L}_s(^2l_\infty)$ and $\mathcal{P}(^2l_\infty)$

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For $n \geq 2$, we show that every extreme point of the unit ball of $\mathcal{L}_s(^2l_\infty^n)$ is extreme in $\mathcal{L}_s(^2l_\infty^{n+1})$, which answers the question in [Period. Math. Hungar. 2018, **77** (2), 274–290]. As a corollary we show that every extreme point of the unit ball of $\mathcal{L}_s(^2l_\infty^n)$ is extreme in $\mathcal{L}_s(^2l_\infty)$. We also show that every extreme point of the unit ball of $\mathcal{P}(^2l_\infty^2)$ is extreme in $\mathcal{P}(^2l_\infty^n)$. As a corollary we show that every extreme point of the unit ball of $\mathcal{P}(^2l_\infty^2)$ is extreme in $\mathcal{P}(^2l_\infty)$.

Key words and phrases: extreme point, symmetric bilinear form, 2-homogeneous polynomials on l_∞.

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Introduction

Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2$. We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is an $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $\operatorname{ext} B_E$ and $\exp B_E$ the set of extreme points and the set of exposed points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \ldots, x_n)|$. $\mathcal{L}_s(^nE)$ denotes the closed subspace of all continuous symmetric n-linear forms on E. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer the reader to [8].

Let us introduce the history of classification problems of the extreme points and the exposed points of the unit ball of continuous n-homogeneous polynomials on a Banach space. We let $l_p^n = \mathbb{R}^n$ for every $1 \leq p \leq \infty$ equipped with the l_p-norm. Y.S. Choi et al. [3–5] initiated and classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2$. Y.S. Choi and S.G. Kim [7] classified $\exp B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2, \infty$. B.C. Grecu [12] classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. In the paper [41], S.G. Kim et al. showed that if E is a separable real Hilbert space with $\dim(E) \geq 2$, then $\operatorname{ext} B_{\mathcal{P}(^2E)} = \exp B_{\mathcal{P}(^2E)}$. In [16], S.G. Kim classified $\exp B_{\mathcal{P}(^2l_p^2)}$ for $1 \leq p \leq \infty$, and, in [18, 20], he characterized $\operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with an octagonal norm

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$\|(x,y)\|_w = \max\left\{|x|, |y|, \frac{|x|+|y|}{1+w}\right\}$ for $0 < w < 1$. In [25], S.G. Kim classified $\exp B_{\mathcal{P}(^2 d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}(^2 d_*(1,w)^2)} \neq \operatorname{ext} B_{\mathcal{P}(^2 d_*(1,w)^2)}$. Recently, in [30, 33], he classified $\operatorname{ext} B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ and $\exp B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with a hexagonal norm $\|(x,y)\|_{h(\frac{1}{2})} = \max\left\{|y|, |x| + \frac{1}{2}|y|\right\}$.

Parallel to the classification problems of $\operatorname{ext} B_{\mathcal{P}(^n E)}$ and $\exp B_{\mathcal{P}(^n E)}$, it seems to be very natural to study the classification problems of the extreme points and the exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space.

In the works [17,19,21,22,24,28,29,32,34,36,37], S.G. Kim classified $\operatorname{ext} B_{\mathcal{L}_s(^2 l^2_\infty)}$, $\exp B_{\mathcal{L}_s(^2 l^2_\infty)}$, $\operatorname{ext} B_{\mathcal{L}_s(^2 d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}(^2 d_*(1,w)^2)}$, $\exp B_{\mathcal{L}_s(^2 d_*(1,w)^2)}$, $\exp B_{\mathcal{L}(^2 d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}_s(^2 l^3_\infty)}$, $\exp B_{\mathcal{L}_s(^3 l^2_\infty)}$, $\operatorname{ext} B_{\mathcal{L}(^2 l^n_\infty)}$, $\operatorname{ext} B_{\mathcal{L}_s(^2 l^n_\infty)}$, $\operatorname{ext} B_{\mathcal{L}(^n l^2_\infty)}$ and $\operatorname{ext} B_{\mathcal{L}_s(^n l^2_\infty)}$, characterized $\operatorname{ext} B_{\mathcal{L}(^2 l^3_\infty)}$, $\exp B_{\mathcal{L}(^2 l^3_\infty)}$ and studied $\operatorname{ext} B_{\mathcal{L}(^2 l_\infty)}$. He showed that $\operatorname{ext} B_{\mathcal{L}_s(^2 l^2_\infty)} = \exp B_{\mathcal{L}_s(^2 l^2_\infty)}$, $\operatorname{ext} B_{\mathcal{L}_s(^2 l^3_\infty)} = \exp B_{\mathcal{L}_s(^2 l^3_\infty)}$, $\operatorname{ext} B_{\mathcal{L}_s(^3 l^2_\infty)} = \exp B_{\mathcal{L}_s(^3 l^2_\infty)}$, $\exp B_{\mathcal{L}(^2 l^n_\infty)} = \operatorname{ext} B_{\mathcal{L}(^2 l^n_\infty)}$, $\exp B_{\mathcal{L}_s(^2 l^n_\infty)} = \operatorname{ext} B_{\mathcal{L}_s(^2 l^n_\infty)}$, $|\operatorname{ext} B_{\mathcal{L}(^n l^2_\infty)}| = 2^{(2^n)}$, $|\operatorname{ext} B_{\mathcal{L}_s(^n l^2_\infty)}| = 2^{n+1}$, $\exp B_{\mathcal{L}(^n l^2_\infty)} = \operatorname{ext} B_{\mathcal{L}(^n l^2_\infty)}$ and $\exp B_{\mathcal{L}_s(^n l^2_\infty)} = \operatorname{ext} B_{\mathcal{L}_s(^n l^2_\infty)}$. In [2], M. Cavalcante et al. characterized $\operatorname{ext} B_{\mathcal{L}(^n l^m_\infty)}$. In [38], S.G. Kim classified extreme points and exposed points of the unit ball of the space of bilinear symmetric forms on the real Banach space of bilinear symmetric forms on l^2_∞. It is shown that for this case, the set of extreme points is equal to the set of exposed points. In [39], he characterized $\operatorname{ext} B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}$ and $\operatorname{ext} B_{\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})}$, where $\mathbb{R}^m_{\|\cdot\|}$ is $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| = 2m$ for $m \geq 2$. Recently, S.G. Kim [40] characterized $\operatorname{ext} B_{\mathcal{L}(^n l_1)}$ and $\operatorname{ext} B_{\mathcal{L}_s(^n l_1)}$ for $n \geq 2$.

We refer the reader to [1, 6, 9–11, 13–15, 23, 26, 27, 31, 35–38, 42–49] and references therein for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces.

In this paper, for $n \geq 2$, we show that every extreme point of the unit ball of $\mathcal{L}_s(^2 l^n_\infty)$ is extreme in $\mathcal{L}_s(^2 l^{n+1}_\infty)$, which answers the question in [32]. As a corollary we show that every extreme point of the unit ball of $\mathcal{L}_s(^2 l^n_\infty)$ is extreme in $\mathcal{L}_s(^2 l_\infty)$. We also show that every extreme point of the unit ball of $\mathcal{P}(^2 l^2_\infty)$ is extreme in $\mathcal{P}(^2 l^n_\infty)$. As a corollary we show that every extreme point of the unit ball of $\mathcal{P}(^2 l^2_\infty)$ is extreme in $\mathcal{P}(^2 l_\infty)$.

1 Results

Let $n, m \geq 2$ and $l^m_\infty = \mathbb{R}^m$ with the l_1-norm. Set

$$\mathcal{W}_{n,m} := \left\{ \Big((1, w^{(1)}_2, \ldots, w^{(1)}_m), \ldots, (1, w^{(n)}_2, \ldots, w^{(n)}_m) \Big) : w^{(k)}_j = \pm 1 \ \text{ for } \ 1 \leq k \leq n, 2 \leq j \leq m \right\}.$$

Note that $\mathcal{W}_{n,m}$ has $2^{(m-1)n}$ elements in $S_{l^m_\infty} \times \cdots \times S_{l^m_\infty}$.

For $(a_{j_1}, \ldots, a_{j_n}), (a_{i_1}, \ldots, a_{i_n}) \in \mathcal{W}_{n,m}$, we define an equivalence relation $\sim$ by

$$(a_{j_1}, \ldots, a_{j_n}) \sim (a_{i_1}, \ldots, a_{i_n})$$

if and only if $j_1 = i_{\phi(1)}, \ldots, j_n = i_{\phi(n)}$ for some permutation ϕ on $\{1, \ldots, n\}$. In $\mathcal{W}_{n,m}$,

$$\Big[\Big((1, w^{(1)}_2, \ldots, w^{(1)}_m), \ldots, (1, w^{(n)}_2, \ldots, w^{(n)}_m) \Big) \Big]$$

denotes the equivalence class containing $\Big((1, w^{(1)}_2, \ldots, w^{(1)}_m), \ldots, (1, w^{(n)}_2, \ldots, w^{(n)}_m) \Big)$.

Let

$$\mathcal{U}_{n,m} := \Big\{ \Big[\Big((1, w_2^{(1)}, \ldots, w_m^{(1)}), \ldots, (1, w_2^{(n)}, \ldots, w_m^{(n)}) \Big) \Big] : \\ \Big((1, w_2^{(1)}, \ldots, w_m^{(1)}), \ldots, (1, w_2^{(n)}, \ldots, w_m^{(n)}) \Big) \in \mathcal{W}_{n,m} \Big\}.$$

Lemma 1. *Let $n \geq 2$ and $Z_1, \ldots, Z_{\frac{n(n+1)}{2}} \in \mathcal{U}_{2,n}$. Then there are $1 \leq i_1 < \cdots < i_n \leq \frac{n(n+1)}{2}$ and $Z_{i_k} = \Big[\Big(1, x_2^{(i_k)}, \ldots, x_n^{(i_k)}\Big), \Big(1, y_2^{(i_k)}, \ldots, y_n^{(i_k)}\Big) \Big]$ for $k = 1, \ldots, n$, such that*

$$\Big(1, x_2^{(i_1)}, \ldots, x_n^{(i_1)}\Big), \ldots, \Big(1, x_2^{(i_n)}, \ldots, x_n^{(i_n)}\Big)$$

are linearly independent in $\mathbb{R}^n$ or

$$\Big(1, y_2^{(i_1)}, \ldots, y_n^{(i_1)}\Big), \ldots, \Big(1, y_2^{(i_n)}, \ldots, y_n^{(i_n)}\Big)$$

are linearly independent in $\mathbb{R}^n$.

The following result answers the question in the front of Theorem 3.5 of [32].

Theorem 1. *The inclusion $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)} \subset \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^{n+1})}$ holds for every $n \geq 2$.*

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)}$. Suppose that $T = \frac{1}{2}(S_1 + S_2)$ for some $S_1, S_2 \in \mathcal{L}_s(^2l_\infty^{n+1})$ with $\|S_1\| = 1 = \|S_2\|$. Notice that, for $i = 1, 2$, we have $S_i\Big((x_1, \ldots, x_n, 0), (y_1, \ldots, y_n, 0) \Big) \in \mathcal{L}_s(^2l_\infty^n)$, $\Big\| S_i\Big((x_1, \ldots, x_n, 0), (y_1, \ldots, y_n, 0) \Big) \Big\| \leq 1$ and

$$T = \frac{1}{2}\Big(S_1\Big((x_1, \ldots, x_n, 0), (y_1, \ldots, y_n, 0) \Big) + S_2\Big((x_1, \ldots, x_n, 0), (y_1, \ldots, y_n, 0) \Big) \Big).$$

Since $T \in \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)}$, $T\Big((x_1, \ldots, x_n), (y_1, \ldots, y_n) \Big) = S_1\Big((x_1, \ldots, x_n, 0), (y_1, \ldots, y_n, 0) \Big)$. Hence,

$$\begin{aligned} S_1\Big((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}) \Big) &= T\Big((x_1, \ldots, x_n), (y_1, \ldots, y_n) \Big) \\ &\quad + b_1(x_1 y_{n+1} + x_{n+1} y_1) + b_2(x_2 y_{n+1} + x_{n+1} y_2) + \cdots \\ &\quad + b_n(x_n y_{n+1} + x_{n+1} y_n) + c x_{n+1} y_{n+1} \end{aligned}$$

and

$$\begin{aligned} S_2\Big((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}) \Big) &= T\Big((x_1, \ldots, x_n), (y_1, \ldots, y_n) \Big) \\ &\quad - b_1(x_1 y_{n+1} + x_{n+1} y_1) + b_2(x_2 y_{n+1} + x_{n+1} y_2) + \cdots \\ &\quad + b_n(x_n y_{n+1} + x_{n+1} y_n) + c x_{n+1} y_{n+1}. \end{aligned}$$

We claim that $0 = b_1 = \cdots = b_n = c$.

By [35, Theorem 3.4], there exist $Z_1, \ldots, Z_{\frac{n(n+1)}{2}} \in \mathcal{U}_{2,n}$ such that $|T(Z_k)| = 1$ for $1 \leq k \leq \frac{n(n+1)}{2}$. Write

$$Z_k = \Big[\Big(1, x_2^{(k)}, \ldots, x_n^{(k)}\Big), \Big(1, y_2^{(k)}, \ldots, y_n^{(k)}\Big) \Big] \quad \text{for } k = 1, \ldots, \frac{n(n+1)}{2}.$$

It follows that

$$\begin{aligned} 1 \ &\geq\ \max\Big\{\Big|S_i\Big(\big(1, x_2^{(k)}, \ldots, x_n^{(k)}, x_{n+1}\big), \big(1, y_2^{(k)}, \ldots, y_n^{(k)}, y_{n+1}\big)\Big)\Big| : \\ &\qquad |x_{n+1}| \leq 1, |y_{n+1}| \leq 1, 1 \leq k \leq \frac{n(n+1)}{2}, i = 1, 2\Big\} \\ &=\ \max\Big\{|T(Z_k)| + \Big|b_1(y_{n+1} + x_{n+1}) + b_2(x_2^{(k)}y_{n+1} + x_{n+1}y_2^{(k)}) + \cdots \\ &\qquad + b_n(x_n^{(k)}y_{n+1} + x_{n+1}y_n^{(k)}) + cx_{n+1}y_{n+1}\Big| : \\ &\qquad |x_{n+1}| \leq 1, |y_{n+1}| \leq 1, 1 \leq k \leq \frac{n(n+1)}{2}, i = 1, 2\Big\} \\ &=\ \max\Big\{1 + \Big|b_1(y_{n+1} + x_{n+1}) + b_2(x_2^{(k)}y_{n+1} + x_{n+1}y_2^{(k)}) + \cdots \\ &\qquad + b_n(x_n^{(k)}y_{n+1} + x_{n+1}y_n^{(k)}) + cx_{n+1}y_{n+1}\Big| : \\ &\qquad |x_{n+1}| \leq 1, |y_{n+1}| \leq 1, 1 \leq k \leq \frac{n(n+1)}{2}, i = 1, 2\Big\}, \end{aligned}$$

which imply that, for all $|x_{n+1}| \leq 1, |y_{n+1}| \leq 1, 1 \leq k \leq \frac{n(n+1)}{2}$,

$$0 = b_1(y_{n+1} + x_{n+1}) + b_2(x_2^{(k)}y_{n+1} + x_{n+1}y_2^{(k)}) + \cdots + b_n(x_n^{(k)}y_{n+1} + x_{n+1}y_n^{(k)}) + cx_{n+1}y_{n+1},$$

$$0 = b_1(-y_{n+1} - x_{n+1}) + b_2(-x_2^{(k)}y_{n+1} - x_{n+1}y_2^{(k)}) + \cdots + b_n(-x_n^{(k)}y_{n+1} - x_{n+1}y_n^{(k)}) + cx_{n+1}y_{n+1},$$

which shows that $c = 0$ and for $1 \leq k \leq \frac{n(n+1)}{2}$,

$$\begin{aligned} 0 &= b_1 + b_2x_2^{(k)} + \cdots + b_nx_n^{(k)}, \\ 0 &= b_1 + b_2y_2^{(k)} + \cdots + b_ny_n^{(k)}. \end{aligned} \tag{1}$$

By Lemma 1, there exist $Z_{i_1}, \ldots, Z_{i_n}$ for $1 \leq i_1 < \cdots < i_n \leq \frac{n(n+1)}{2}$ such that $\big(1, x_2^{(i_1)}, \ldots, x_n^{(i_1)}\big), \ldots, \big(1, x_2^{(i_n)}, \ldots, x_n^{(i_n)}\big)$ are linearly independent in $\mathbb{R}^n$ or $\big(1, y_2^{(i_1)}, \ldots, y_n^{(i_1)}\big), \ldots, \big(1, y_2^{(i_n)}, \ldots, y_n^{(i_n)}\big)$ are linearly independent in $\mathbb{R}^n$.

Let A, B be the $n \times n$ matrices such that

$$\Big[Row(A)\Big]_j = \Big(1, x_2^{(i_j)}, \ldots, x_n^{(i_j)}\Big), \qquad \Big[Row(B)\Big]_j = \Big(1, y_2^{(i_j)}, \ldots, y_n^{(i_j)}\Big)$$

for $1 \leq j \leq n$. By (1), we have

$$A(b_1, \ldots, b_n)^t = (0, \ldots, 0)^t, \qquad B(b_1, \ldots, b_n)^t = (0, \ldots, 0)^t.$$

If $\big(1, x_2^{(i_1)}, \ldots, x_n^{(i_1)}\big), \ldots, \big(1, x_2^{(i_n)}, \ldots, x_n^{(i_n)}\big)$ are linearly independent in $\mathbb{R}^n$, then A is invertible and so $0 = b_1 = \cdots = b_n$. Similarly, if $\big(1, y_2^{(i_1)}, \ldots, y_n^{(i_1)}\big), \ldots, \big(1, y_2^{(i_n)}, \ldots, y_n^{(i_n)}\big)$ are linearly independent in $\mathbb{R}^n$, then B is invertible and so $0 = b_1 = \cdots = b_n$.

Hence, $S_1 = T = S_2$. Therefore, $T \in \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^{n+1})}$. □

Theorem 2. *The inclusion* $\bigcup_{n=1}^{\infty} \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)} \subset \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty)}$ *holds.*

Proof. Let $T \in \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)}$ for some $n \geq 1$. Suppose that $T = \frac{1}{2}(S_1 + S_2)$ for some $S_1, S_2 \in \mathcal{L}_s(^2l_\infty)$ with $\|S_1\| = 1 = \|S_2\|$. Write

$$T\Big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\Big) = \sum_{1 \leq i,j \leq n} a_{ij}x_iy_j$$

and

$$S_k\Big((x_1,\ldots,x_n,\cdots),(y_1,\ldots,y_n,\cdots)\Big)=\sum_{i,j=1}^{\infty}b_{ij}^{(k)}x_iy_j\quad\text{for }k=1,2,$$

where $a_{ij}, b_{ij}^{(k)}\in\mathbb{R}$.

Claim. $b_{ij}^{(1)}=0=b_{ij}^{(2)}$ *if $i>n$ or $j>n$.*

Let $m=\max\{i,j\}>n$ and $T_k:=S_k|_{l_\infty^m\times l_\infty^m}\in\mathcal{L}_s(^2l_\infty^m)$ for $k=1,2$. By Theorem 1, $T\in\operatorname{ext}B_{\mathcal{L}_s(^2l_\infty^m)}$. Since $\|T_k\|\le 1$ for all $k=1,2$ and $T=\frac{1}{2}(T_1+T_2)$, $T=T_1=T_2$. Hence, $b_{ij}^{(1)}=0=b_{ij}^{(2)}$ and $a_{ls}=b_{ls}^{(1)}=b_{ls}^{(2)}$ for every $1\le l,s\le n$. Hence, $T=S_1=S_2$. Therefore, $T\in\operatorname{ext}B_{\mathcal{L}_s(^2l_\infty)}$. □

Question. *Is it true that* $\bigcup_{n=1}^{\infty}\operatorname{ext}B_{\mathcal{L}_s(^2l_\infty^n)}=\operatorname{ext}B_{\mathcal{L}_s(^2l_\infty)}$*?*

Let $n\in\mathbb{N}$ with $n\ge 2$ and $P\in\mathcal{P}(^2l_\infty^n)$. Write, for $(x_1,\ldots,x_n)\in l_\infty^n$,

$$P(x_1,\ldots,x_n)=\sum_{1\le i\le n}a_ix_i^2+\sum_{1\le j<k\le n}b_{jk}x_jx_k$$

for some $a_i, b_{jk}\in\mathbb{R}$.

In [5], it was shown that

$$\operatorname{ext}B_{\mathcal{P}(^2l_\infty^2)}=\Big\{\pm x_1^2,\ \pm x_2^2,\ \pm\Big(ax_1^2-ax_2^2\pm 2\sqrt{a(1-a)}x_1x_2\Big):\frac{1}{2}\le a\le 1\Big\}.$$

Theorem 3. *The inclusion* $\operatorname{ext}B_{\mathcal{P}(^2l_\infty^2)}\subset\operatorname{ext}B_{\mathcal{P}(^2l_\infty^n)}$ *holds for every* $n\ge 3$.

Proof. **Claim 1.** $P(x_1,\ldots,x_n)=ax_1^2-ax_2^2+2\sqrt{a(1-a)}x_1x_2\in\operatorname{ext}B_{\mathcal{P}(^2l_\infty^n)}$ *for* $\frac{1}{2}\le a\le 1$.

Use induction on n.

Let $n=3$. Suppose that $P=\frac{1}{2}(Q_1+Q_2)$ for some $Q_1,Q_2\in\mathcal{P}(^2l_\infty^3)$ with $\|Q_1\|=1=\|Q_2\|$. Write

$$Q_1(x,y,z)=(a+c_1)x^2-(a+c_2)y^2+\Big(2\sqrt{a(1-a)}+c_3\Big)xy+d_1xz+d_2yz+d_3z^2$$

and

$$Q_2(x,y,z)=(a-c_1)x^2-(a-c_2)y^2+\Big(2\sqrt{a(1-a)}-c_3\Big)xy-d_1xz-d_2yz-d_3z^2$$

for some $c_j,d_j\in\mathbb{R}$ for $j=1,2,3$.

Notice that $Q_j(x,y,0)\in\mathcal{P}(^2l_\infty^2)$, $\|Q_j(x,y,0)\|=1$, $P=\frac{1}{2}\big(Q_1(x,y,0)+Q_2(x,y,0)\big)$ for $j=1,2$. Since $P(x,y)\in\operatorname{ext}B_{\mathcal{P}(^2l_\infty^2)}$, $P(x,y)=Q_1(x,y,0)$, which shows that $c_j=0$ for $j=1,2,3$. Hence,

$$Q_1(x,y,z)=P(x,y)+d_1xz+d_2yz+d_3z^2\quad\text{and}\quad Q_2(x,y,z)=P(x,y)-d_1xz-d_2yz-d_3z^2.$$

It follows that

$$\begin{aligned}1\ &\ge\ \max\Big\{\Big|Q_j(1,\sqrt{\frac{1-a}{a}},z)\Big|,\ \Big|Q_j\Big(\sqrt{\frac{1-a}{a}},-1,z\Big)\Big|:|z|\le 1,j=1,2\Big\}\\ &=\ \max\Big\{1+\Big|d_1z+d_2\sqrt{\frac{1-a}{a}}z+d_3z^2\Big|,\ 1+\Big|d_1\sqrt{\frac{1-a}{a}}z-d_2z+d_3z^2\Big|\Big\},\end{aligned}$$

which imply that, for all $|z| \le 1$,

$$0 = d_1 z + d_2\sqrt{\frac{1-a}{a}}z + d_3 z^2, \qquad 0 = d_1\sqrt{\frac{1-a}{a}}z - d_2 z + d_3 z^2.$$

Therefore, $d_j = 0$ for $j = 1,2,3$. Hence, $P(x,y) \in \mathrm{ext}\, B_{\mathcal{P}(^2 l_\infty^3)}$ for $\frac{1}{2} \le a \le 1$.

Suppose that for $n = k$, $P(x,y) = ax^2 - ay^2 + 2\sqrt{a(1-a)}xy \in \mathrm{ext}\, B_{\mathcal{P}(^2 l_\infty^k)}$ for $\frac{1}{2} \le a \le 1$. We will show that $P(x,y) \in \mathrm{ext}\, B_{\mathcal{P}(^2 l_\infty^{k+1})}$ for $\frac{1}{2} \le a \le 1$. Suppose that $P = \frac{1}{2}(R_1 + R_2)$ for some $R_1, R_2 \in \mathcal{P}(^2 l_\infty^{k+1})$ with $\|R_1\| = 1 = \|R_2\|$. By the above argument, we may assume that

$$R_1(x,y,z_1,\ldots,z_{k-1}) = P(x,y) + Axz_{k-1} + Byz_{k-1} + Cz_{k-1}^2 + \sum_{1\le j\le k-2} D_j z_j z_{k-1}$$

and

$$R_2(x,y,z_1,\ldots,z_{k-1}) = P(x,y) - \Big(Axz_{k-1} + Byz_{k-1} + Cz_{k-1}^2 + \sum_{1\le j\le k-2} D_j z_j z_{k-1}\Big)$$

for some $A, B, C, D_j \in \mathbb{R}$, $1 \le j \le k-2$. It follows that

$$\begin{aligned}
1 \;\ge\; & \max\Big\{\Big|R_j\Big(1, \sqrt{\frac{1-a}{a}}, z_1,\ldots,z_{k-1}\Big)\Big|, \ \Big|Q_j\Big(\sqrt{\frac{1-a}{a}}, -1, z_1,\ldots,z_{k-1}\Big)\Big| : \\
& \quad |z_i| \le 1, i = 1,\ldots,k-1, j = 1,2\Big\} \\
=\; & \max\Big\{1 + \Big|Az_{k-1} + B\sqrt{\frac{1-a}{a}}z_{k-1} + Cz_{k-1}^2 + \sum_{1\le j\le k-2} D_j z_j z_{k-1}\Big|, \\
& \quad 1 + \Big|A\sqrt{\frac{1-a}{a}}z_{k-1} - Bz_{k-1} + Cz_{k-1}^2 + \sum_{1\le j\le k-2} D_j z_j z_{k-1}\Big|\Big\},
\end{aligned}$$

which imply that, for all $|z_i| \le 1$, $i = 1,\ldots,k-1$, we have

$$\begin{aligned}
0 &= Az_{k-1} + B\sqrt{\frac{1-a}{a}}z_{k-1} + Cz_{k-1}^2 + \sum_{1\le j\le k-2} D_j z_j z_{k-1}, \\
0 &= A\sqrt{\frac{1-a}{a}}z_{k-1} - Bz_{k-1} + Cz_{k-1}^2 + \sum_{1\le j\le k-2} D_j z_j z_{k-1}.
\end{aligned}$$

If $z_j = 0$ for $1 \le j \le k-2$, then

$$0 = Az_{k-1} + B\sqrt{\frac{1-a}{a}}z_{k-1} + Cz_{k-1}^2, \qquad 0 = A\sqrt{\frac{1-a}{a}}z_{k-1} - Bz_{k-1} + Cz_{k-1}^2,$$

which shows that $A = B = C = 0$. If $z_{k-1} = 1$, then

$$0 = \sum_{1\le j\le k-2} D_j z_j z_{k-1} = \sum_{1\le j\le k-2} D_j z_j \quad \text{for } |z_j| \le 1, j = 1,\ldots,k-2,$$

which implies that $D_j = 0$ for $j = 1,\ldots,k-2$. Hence, $P(x,y) \in \mathrm{ext}\, B_{\mathcal{P}(^2 l_\infty^{k+1})}$ for $\frac{1}{2} \le a \le 1$. We have shown Claim 1. Similarly, $-P \in \mathrm{ext}\, B_{\mathcal{P}(^2 l_\infty^n)}$.

Claim 2. $P(x_1,\ldots,x_n) = x_1^2 \in \mathrm{ext}\, B_{\mathcal{P}(^2 l_\infty^n)}$.

Suppose that $P = \frac{1}{2}(Q_1 + Q_2)$ for some $Q_1, Q_2 \in \mathcal{P}(^2 l_\infty^n)$ with $\|Q_1\| = 1 = \|Q_2\|$. Since $1 = P(e_1) = \frac{1}{2}(Q_1(e_1) + Q_2(e_1))$, we have $1 = Q_1(e_1) = Q_2(e_1)$. Hence,

$$Q_1(x_1,\ldots,x_n) = x_1^2 + \sum_{2\le i\le n} a_i x_i^2 + \sum_{1\le j<k\le n} b_{jk} x_j x_k$$

and

$$Q_2(x_1,\ldots,x_n) = x_1^2 - \sum_{2\le i\le n} a_i x_i^2 - \sum_{1\le j<k\le n} b_{jk}x_jx_k$$

for some $a_i, b_{jk} \in \mathbb{R}$.

We will show that $a_i = 0 = b_{jk}$ for every $2 \le i \le n, 1 \le j < k \le n$. Let $2 \le j \le n$ be fixed. It follows that, for $i = 1,2, |t| \le 1$,

$$1 \;\ge\; \Big|Q_i(e_1 + te_j)\Big| = 1 + \Big|a_j t^2 + b_{1j}t\Big|,$$

which shows that $a_j t^2 + b_{1j}t = 0$ for $|t| \le 1$, hence $a_j = b_{1j} = 0$ for $2 \le j \le n$. Let $2 \le k < j \le n$ be fixed. It follows that

$$1 \ge \Big|Q_i(e_1 + e_k + e_j)\Big| = 1 + \Big|b_{kj}\Big|,$$

which shows that $b_{kj} = 0$ for $2 \le k < j \le n$. We have shown Claim 2. Similarly, $-x_1^2$, $\pm x_2^2 \in \operatorname{ext} B_{\mathcal{P}(^2l_\infty^n)}$. Therefore, we complete the proof. □

Corollary 1. *The inclusion* $\operatorname{ext} B_{\mathcal{P}(^2l_\infty^2)} \subset \operatorname{ext} B_{\mathcal{P}(^2l_\infty)}$ *holds.*

Proof. Let $P \in \operatorname{ext} B_{\mathcal{P}(^2l_\infty^2)}$. Suppose that $P = \frac{1}{2}(Q_1 + Q_2)$ for some $Q_1, Q_2 \in \mathcal{P}(^2l_\infty)$ with $\|Q_1\| = 1 = \|Q_2\|$. Write

$$P\Big((x_1,x_2)\Big) = \sum_{i,j=1}^{2} a_{ij}x_ix_j \qquad \text{and} \qquad Q_k\Big((x_1,\ldots,x_n,\cdots)\Big) = \sum_{i,j=1}^{\infty} b_{ij}^{(k)} x_ix_j \quad \text{for } k = 1,2,$$

where $a_{ij}, b_{ij}^{(k)} \in \mathbb{R}$.

Claim. $b_{ij}^{(1)} = 0 = b_{ij}^{(2)}$ *if* $i > 2$ *or* $j > 2$.

Let $m = \max\{i,j\} > 2$ and $R_k := Q_k|_{l_\infty^m} \in \mathcal{P}(^2l_\infty^m)$ for $k = 1,2$. By Theorem 3, $P \in \operatorname{ext} B_{\mathcal{P}(^2l_\infty^m)}$. Since $\|R_k\| \le 1$ for all $k = 1,2$ and $P = \frac{1}{2}(R_1 + R_2)$, $P = R_1 = R_2$. Hence, $b_{ij}^{(1)} = 0 = b_{ij}^{(2)}$ and $a_{ls} = b_{ls}^{(1)} = b_{ls}^{(2)}$ for every $1 \le l,s \le n$. Hence, $P = Q_1 = Q_2$. Therefore, $P \in \operatorname{ext} B_{\mathcal{P}(^2l_\infty)}$. □

References

[1] Aron R.M., Klimek M. *Supremum norms for quadratic polynomials*. Arch. Math. (Basel) 2001, **76** (1), 73–80. doi:10.1007/s000130050544

[2] Cavalcante M., Pellegrino D., Teixeira E. *Geometry of multilinear forms.* Commun. Contemp. Math. 2020, **22** (2), article ID 1950011. doi:10.1142/S0219199719500111

[3] Choi Y.S., Kim S.G., Ki H. *Extreme polynomials and multilinear forms on* l_1. J. Math. Anal. Appl. 1998, **228** (2), 467–482. doi:10.1006/jmaa.1998.6161

[4] Choi Y.S., Kim S.G. *The unit ball of* $\mathcal{P}(^2l_2^2)$. Arch. Math. (Basel) 1998, **71** (6), 472–480. doi:10.1007/s000130050292

[5] Choi Y.S., Kim S.G. *Extreme polynomials on* c_0. Indian J. Pure Appl. Math. 1998, **29**, 983–989.

[6] Choi Y.S., Kim S.G. *Smooth points of the unit ball of the space* $\mathcal{P}(^2l_1^2)$. Results Math. 1999, **36** (1–2), 26–33. doi:10.1007/BF03322099

[7] Choi Y.S., Kim S.G. *Exposed points of the unit balls of the spaces* $\mathcal{P}(^2l_p^2)$ $(p = 1,2,\infty)$. Indian J. Pure Appl. Math. 2004, **35**, 37–41.

[8] Dineen S. Complex Analysis on Infinite Dimensional Spaces. Springer-Verlag, London, 1999.

[9] Gámez-Merino J., Muñoz-Fernández G., Sánchez V., Seoane-Sepúlveda J. *Inequalities for polynomials on the unit square via the Krein-Milman Theorem.* J. Convex Anal. 2013, **20** (1), 125–142.

[10] Grecu B.C. *Geometry of three-homogeneous polynomials on real Hilbert spaces.* J. Math. Anal. Appl. 2000, **246** (1), 217–229. doi:10.1006/jmaa.2000.6783

[11] Grecu B.C. *Smooth 2-homogeneous polynomials on Hilbert spaces.* Arch. Math. (Basel) 2001, **76** (6), 445–454. doi:10.1007/PL00000456

[12] Grecu B.C. *Geometry of 2-homogeneous polynomials on l_p spaces,* $1 < p < \infty$. J. Math. Anal. Appl. 2002, **273** (2), 262–282. doi:10.1016/S0022-247X(02)00217-2

[13] Grecu B.C. *Extreme 2-homogeneous polynomials on Hilbert spaces.* Quaest. Math. 2002, **25** (4), 421–435. doi:10.2989/16073600209486027

[14] Grecu B.C. *Geometry of homogeneous polynomials on two- dimensional real Hilbert spaces.* J. Math. Anal. Appl. 2004, **293** (2), 578–588. doi:10.1016/j.jmaa.2004.01.020

[15] Grecu B.C., Muñoz-Fernández G., Seoane-Sepúlveda J. *The unit ball of the complex $P(^3H)$.* Math. Z. 2009, **263**, 775–785. doi:10.1007/s00209-008-0438-y

[16] Kim S.G. *Exposed 2-homogeneous polynomials on L_P^2,* $1 \leq P \leq \infty$. Math. Proc. R. Ir. Acad. 2007, **107A** (2), 123–129.

[17] Kim S.G. *The unit ball of $\mathcal{L}_s(^2l_\infty^2)$.* Extracta Math. 2009, **24** (1), 17–29.

[18] Kim S.G. *The unit ball of $\mathcal{P}(^2d_*(1,w)^2)$.* Math. Proc. R. Ir. Acad. 2011, **111A** (2), 77–92. doi:10.3318/pria.2011.111.1.9

[19] Kim S.G. *The unit ball of $\mathcal{L}_s(^2d_*(1,w)^2)$.* Kyungpook Math. J. 2013, **53** (2), 295–306.

[20] Kim S.G. *Smooth polynomials of $\mathcal{P}(^2d_*(1,w)^2)$.* Math. Proc. R. Ir. Acad. 2013, **113A** (1), 45–58.

[21] Kim S.G. *Extreme bilinear forms of $\mathcal{L}(^2d_*(1,w)^2)$.* Kyungpook Math. J. 2013, **53** (4), 625–638.

[22] Kim S.G. *Exposed symmetric bilinear forms of $\mathcal{L}_s(^2d_*(1,w)^2)$.* Kyungpook Math. J. 2014, **54** (3), 341–347.

[23] Kim S.G. *Polarization and unconditional constants of $\mathcal{P}(^2d_*(1,w)^2)$.* Commun. Korean Math. Soc. 2014, **29** (3), 421–428. doi:10.4134/ckms.2014.29.3.421

[24] Kim S.G. *Exposed bilinear forms of $\mathcal{L}(^2d_*(1,w)^2)$.* Kyungpook Math. J. 2015, **55** (1), 119–126.

[25] Kim S.G. *Exposed 2-homogeneous polynomials on the two-dimensional real predual of Lorentz sequence space.* Mediterr. J. Math. 2016, **13** (5), 2827–2839. doi:10.1007/s00009-015-0658-4

[26] Kim S.G. *The unit ball of $\mathcal{L}(^2\mathbb{R}^2_{h(w)})$.* Bull. Korean Math. Soc. 2017, **54** (2), 417–428. doi:10.4134/bkms.b150851

[27] Kim S.G. *Extremal problems for $\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})$.* Kyungpook Math. J. 2017, **57** (2), 223–232.

[28] Kim S.G. *The unit ball of $\mathcal{L}_s(^2l_\infty^3)$.* Comment. Math. 2017, **57** (1), 1–7. doi:10.14708/cm.v57i1.1230

[29] Kim S.G. *The geometry of $\mathcal{L}_s(^3l_\infty^2)$.* Commun. Korean Math. Soc. 2017, **32** (4), 991–997. doi:10.4134/ckms.c170016

[30] Kim S.G. *Extreme* 2*-homogeneous polynomials on the plane with a hexagonal norm and applications to the polarization and unconditional constants.* Studia Sci. Math. Hungar. 2017, **54** (3), 362–393. doi:10.1556/012.2017.54.3.1371

[31] Kim S.G. *The geometry of $\mathcal{L}(^3l_\infty^2)$ and optimal constants in the Bohnenblust-Hille inequality for multilinear forms and polynomials.* Extracta Math. 2018, **33** (1), 51–66. doi:10.17398/2605-5686.33.1.51

[32] Kim S.G. *Extreme bilinear forms on $\mathbb{R}^n$ with the supremum norm*. Period. Math. Hungar. 2018, **77** (2), 274–290. doi:10.1007/s10998-018-0246-z

[33] Kim S.G. *Exposed polynomials of $\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})$.* Extracta Math. 2018, **33** (2), 127–143. doi:10.17398/2605-5686.33.2.127

[34] Kim S.G. *The unit ball of the space of bilinear forms on $\mathbb{R}^3$ with the supremum norm.* Commun. Korean Math. Soc. 2019, **34** (2), 487–494. doi:10.4134/ckms.c180111

[35] Kim S.G. *Smooth points of $\mathcal{L}_s({}^n l_\infty^2)$*. Bull. Korean Math. Soc. 2020, **57** (2), 443–447. doi:10.4134/bkms.b190311

[36] Kim S.G. *Extreme points of the space $\mathcal{L}({}^2 l_\infty)$*. Commun. Korean Math. Soc. 2020, **35** (3), 799–807. doi: 10.4134/CKMS.c190300

[37] Kim S.G. *Extreme and exposed points of $\mathcal{L}({}^n l_\infty^2)$ and $\mathcal{L}_s({}^n l_\infty^2)$*. Extracta Math. 2020, **35** (2), 127–135. doi: 10.17398/2605-5686.35.2.127

[38] Kim S.G. *Extreme and exposed symmetric bilinear forms on the space $\mathcal{L}_s({}^2 l_\infty^2)$*. Carpathian Math. Publ. 2020, **12** (2), 340–352. doi:10.15330/cmp.12.2.340-352

[39] Kim S.G. *Geometry of multilinear forms on $\mathbb{R}^m$ with a certain norm.* Acta Sci. Math. (Szeged) 2021, **87** (1–2), 233–245. doi:10.14232/actasm-020-824-2

[40] Kim S.G. *Geometry of multilinear forms on l_1*. Acta Comment. Univ. Tartu. Math. 2021, **25** (1), 57–66. In print.

[41] Kim S.G., Lee S.H. *Exposed 2-homogeneous polynomials on Hilbert spaces.* Proc. Amer. Math. Soc. 2003, **131** (2), 449–453. doi:10.1090/S0002-9939-02-06544-9

[42] Krein M.G., Milman D.P. *On extreme points of regular convex sets.* Studia Math. 1940, **9**, 133–137.

[43] Milev L., Naidenov N. *Strictly definite extreme points of the unit ball in a polynomial space.* C. R. Acad. Bulgare Sci. 2008, **61** (11), 1393–1400.

[44] Milev L., Naidenov N. *Semidefinite extreme points of the unit ball in a polynomial space.* J. Math. Anal. Appl. 2013, **405** (2), 631–641. doi:10.1016/j.jmaa.2013.04.026

[45] Muñoz-Fernández G., Pellegrino D., Seoane-Sepúlveda J., Weber A. *Supremum norms for 2-homogeneous polynomials on circle sectors.* J. Convex Anal. 2014, **21** (3), 745–764.

[46] Muñoz-Fernández G., Révész S.G., Seoane-Sepúlveda J. *Geometry of homogeneous polynomials on non symmetric convex bodies.* Math. Scand. 2009, **105** (1), 147–160. doi:10.7146/math.scand.a-15111

[47] Muñoz-Fernández G., Seoane-Sepúlveda J. *Geometry of Banach spaces of trinomials.* J. Math. Anal. Appl. 2008, **340** (2), 1069–1087. doi:10.1016/j.jmaa.2007.09.010

[48] Neuwirth S. *The maximum modulus of a trigonometric trinomial*. J. Anal. Math. 2008, **104** (1), 371–396. doi:10.1007/s11854-008-0028-2

[49] Ryan R.A., Turett B. *Geometry of spaces of polynomials.* J. Math. Anal. Appl. 1998, **221** (2), 698–711. doi:10.1006/jmaa.1998.5942

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Кім С.Ґ. *Екстремальні точки просторів $\mathcal{L}_s({}^2 l_\infty)$ та $\mathcal{P}({}^2 l_\infty)$* // Карпатські матем. публ. — 2021. — Т.13, №2. — C. 289–297.

Ми доводимо, що для $n \geq 2$ кожна екстремальна точка одиничної кулі простору $\mathcal{L}_s({}^2 l_\infty^n)$ є екстремальною в $\mathcal{L}_s({}^2 l_\infty^{n+1})$. Це дає відповідь на питання, поставлене в [Period. Math. Hungar. 2018, **77** (2), 274–290]. Як наслідок, ми показуємо, що кожна екстремальна точка одиничної кулі простору $\mathcal{L}_s({}^2 l_\infty^n)$ є екстремальною в $\mathcal{L}_s({}^2 l_\infty)$. Також ми показуємо, що кожна екстремальна точка одиничної кулі простору $\mathcal{P}({}^2 l_\infty^2)$ є екстремальною в $\mathcal{P}({}^2 l_\infty^n)$. Як наслідок, ми показуємо, що кожна екстремальна точка одиничної кулі простору $\mathcal{P}({}^2 l_\infty^2)$ є екстремальною в $\mathcal{P}({}^2 l_\infty)$.

Ключові слова і фрази: екстремальна точка, симетрична білінійна форма, двоходнорідні поліноми на просторі l_∞.