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Error bounds of a function related to generalized Lipschitz class via the pseudo-Chebyshev wavelet and its applications in the approximation of functions

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In this paper, a new computation method derived to solve the problems of approximation theory. This method is based upon pseudo-Chebyshev wavelet approximations. The pseudo-Chebyshev wavelet is being presented for the first time. The pseudo-Chebyshev wavelet is constructed by the pseudo-Chebyshev functions. The method is described and after that the error bounds of a function is analyzed. We have illustrated an example to demonstrate the accuracy and efficiency of the pseudo-Chebyshev wavelet approximation method and the main results. Four new error bounds of the function related to generalized Lipschitz class via the pseudo-Chebyshev wavelet are obtained. These estimators are the new fastest and best possible in theory of wavelet analysis.

Key words and phrases: $\mathrm{Lip}_{[0,1)}\alpha$ class of functions, $\mathrm{Lip}_{[0,1)}\xi$ class of functions, wavelet, multiresolution analysis, pseudo-Chebyshev function, pseudo-Chebyshev wavelet.

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Introduction

Orthogonal functions play an important role while solving the various problems in differential equations, integral equations, approximation theory, dynamical systems, and many more. Our approach in using the orthogonal function is to transform the underlying problems into a simpler approximating truncated orthogonal function. One of the orthogonal polynomials is the Chebyshev polynomial $T_m(t)$, $m \geq 0$, where $0 \leq t \leq 1$. This is numerically more effective as in [5, 6, 23, 31, 33, 34]. The pseudo-Chebyshev functions of fractional degree are recently introduced by P.E. Ricci [32] and some of its important properties like orthogonality and many more were studied by C. Cesarano and P.E. Ricci [7].

Wavelets, which are relatively recent introduced in 1980's, have considerably expanded their domain and thus attracted a large number of researchers like J. Morlet et al. [26, 27], I. Daubechies [10], C.K. Chui [8, 9], G. Strang and T. Nguyen [36], I.P. Natanson [28], Y. Meyer [24], I. Daubechies and J.C. Lagarias [11], G.G. Walter [39, 40], M.R. Islam et al. [12], F. Mohammadi [25], S. Lal et al. [14, 19, 21, 22], Y.V. Venkatesh [38], E. Keshavarz et al. [13] and others from the area of pure and applied mathematics. Along with Harmonic theory and Fourier analysis, wavelets are rapidly growing under the influence of fractals and approximation theory. Working in this field, a great many researchers like S. Rehman and A.H. Siddiqi [30], G. Strang [35], S. Lal et al. [15–18, 20], F. Bastin [3], J. Biazar et al. [4], E. Babolian and F. Fattahzadeh [1, 2]

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developed the application of wavelet theory. They established the utility of wavelet theory as wonderful tools for science and technology. The fractals are continuous but nowhere differentiable functions. The Brownian trajectories, fractional Brownian motion, typical Feynmann path, complex Bernoulli spiral, and turbulent fluid motion are related to irregular fractals. The irregular fractals are further specified at every point by a locally Lipschitz condition between specific finite intervals. This is the fact that inspired us for considering the approximation of functions belonging to the Lipschitz class and its generalized class via the pseudo-Chebyshev wavelet and its applications. But no satisfactory work seems to have been done so far, to obtain the error bounds of functions related to $\mathrm{Lip}_{[0,1)}\alpha$ and $\mathrm{Lip}_{[0,1)}\xi$ for the different values of $\alpha \geq 0$ and $0 < \alpha < 1$, using the orthogonal projection operator via the pseudo-Chebyshev wavelet.

In the present research article, a new approximation method has been proposed for the error bounds of a function related to Lipschitz class and its generalized classes. The method is based upon pseudo-Chebyshev wavelet approximation. This wavelet is utilized to determine the approximation of functions.

1 Definitions and preliminaries

1.1 Function of Lipα class and function of Lipξ class

A function $f \in \mathrm{Lip}\alpha$ if

$$|f(x+t) - f(x)| = O(|t|^{\alpha}) \quad \text{for } 0 < \alpha \leq 1,$$

and $f \in \mathrm{lip}\alpha$ if

$$|f(x+t) - f(x)| = o(|t|^{\alpha}) \quad \text{for } 0 < \alpha \leq 1.$$

Let ξ be a monotonic increasing function of t. Then a function $f \in \mathrm{Lip}\xi$ if

$$|f(x+t) - f(x)| = O\left(|\xi\,(t)|^{\alpha}\right) \quad \text{for } 0 < \alpha \leq 1,$$

and $f \in \mathrm{lip}\xi$ if

$$|f(x+t) - f(x)| = o\left(|\xi\,(t)|^{\alpha}\right) \quad \text{for } 0 < \alpha \leq 1.$$

1.2 Wavelets and Multiresolution Analysis

Wavelets. A function $\psi \in L^2(\mathbb{R})$ is said to be basic wavelet, if it satisfies the "admissibility" condition

$$C_{\psi} = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty.$$

Wavelets are a set $\{\psi_{m,n} \colon m, n \in \mathbb{Z}\}$ of functions constructed from translation and dilation of a single basic wavelet ψ also called the mother wavelet. If the dilation parameter a and the translation parameter b vary continuously, then the following family of continuous wavelets are

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\, \psi\left(\frac{t-b}{a}\right), \quad a \neq 0, b \in \mathbb{R}.$$

Multiresolution Analysis. A sequence of closed subspaces V_n of $L^2(\mathbb{R})$, $n \in \mathbb{Z}$, are said to be Multiresolution Analysis, if it satisfies the following properties:

(i) V_n is a subset of V_{n+1},

(ii) $f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$,

(iii) $f(x) \in V_0 \Longleftrightarrow f(x+1) \in V_0$,

(iv) $\overline{\bigcup_{n=-\infty}^{\infty} V_n} = L^2(\mathbb{R})$ and $\bigcap_{n=-\infty}^{\infty} V_n = \{0\}$,

(v) there exists a function ϕ such that the set $\{\phi(x-m) \colon m \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Since $\psi \in L^2(\mathbb{R}), \psi_{n,m} := 2^{\frac{n}{2}}\psi(2^n - m)$ with

$$W_n := clos\,\langle \psi_{n,m} : m \in \mathbb{Z} \rangle\,,$$

and this family of subspaces of $L^2(\mathbb{R})$ gives a direct sum decomposition of $L^2(\mathbb{R})$ is the same that every $f \in L^2(\mathbb{R})$ has a unique decomposition

$$g(x) = \sum_{n\in\mathbb{Z}} g_n(x) = \cdots + g_{-2}(x) + g_{-1}(x) + g_0(x) + g_1(x) + g_2(x) + \cdots\,,$$

where $g_n \in W_n$ for all $n \in \mathbb{Z}$ and we describe this by writing

$$L^2(\mathbb{R}) = V_n \bigoplus_{i=n}^{\infty} W_i,$$

where

$$V_n := \bigoplus_{m=-\infty}^{n-1} W_m,$$

$\{\psi_{n,m} \colon m \in \mathbb{Z}\}$ is a Riesz basis of W_n. Therefore

$$g(x) = \sum_{n=-\infty}^{\infty} \langle g, \phi_{n,m}\rangle\, \phi_{n,m}(x) + \sum_{k=m}^{\infty} \sum_{n=-\infty}^{\infty} \langle g, \psi_{n,m}\rangle\, \psi_{n,m}(x).$$

1.3 Pseudo Chebyshev wavelets

In recent research article of P.E. Ricci [32], the sets of classical Chebyshev polynomials of the first and second kind have been extended to the case of fractional indices for the studies of the complex Bernoulli spirals. The resulting functions are said to be pseudo-Chebyshev functions.

The pseudo-Chebyshev functions of first kind $T_{n+1/2}(x)$ and second kind $U_{n+1/2}$ are defined by C. Cesarano and P.E. Ricci [7,32] as

$$T_{m+1/2}(x) = \cos(m + 1/2(\arccos x)) \quad \text{and} \quad U_{m+1/2}\sin(m + 1/2(\arccos x)),$$

where $m \in \mathbb{N} \cup \{0\}$. The recurrence relation of the first kind pseudo-Chebyshev polynomials and its orthogonal properties are given (see [7]) by

$$T_{m+1/2}(x) = 2xT_{m-1/2}(x) - T_{m-3/2}(x), \text{ where } T_{1/2}(x) = T_{-1/2}(x) = \sqrt{\frac{1+x}{2}},\ m \in \mathbb{N}.$$

The first few pseudo-Chebyshev polynomials of first kind are

$$T_{-1/2} = \sqrt{\frac{1+x}{2}}, \quad T_{1/2} = \sqrt{\frac{1+x}{2}}, \quad T_{3/2} = (2x-1)\sqrt{\frac{1+x}{2}}.$$

These polynomials satisfy the following conditions

$$\int\limits_{-1}^{1} \frac{T_{m+1/2}(x)T_{n+1/2}(x)}{\sqrt{1-x^2}} = \begin{cases} \frac{\pi}{2}, & \text{for } m=n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the set $\{T_{m+1/2}(x)\colon m \geq 0\}$ is an orthogonal subset of $L^2(-1,1)$ with respect to weight functions $\omega_{k,n}(x) = \omega(2^k x - 2n + 1)$ where $\omega(x) = \frac{1}{\sqrt{1-x^2}}$.

In the special case of half-integer indices, the pseudo-Chebyshev functions satisfy the analogous properties of the classical counterparts including differential equations, recurrence relations, orthogonality properties and so fourth, as mentioned in [7,32].

Pseudo Chebyshev Wavelets. Let $T_{m+1/2}(x)$ be the pseudo-Chebyshev functions of indices $m+1/2$. Then the pseudo-Chebyshev wavelets are defined by

$$\psi_{n,m}(x) := \psi_{(k,n,m)}(x) = \begin{cases} \sqrt{\frac{2^{k+1}}{\pi}}T_{m+1/2}(2^k x - 2n + 1), & \text{for } \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $m \geq 0$, $n = 1, 2, 3, \ldots, 2^{k-1}$ and $k \in \mathbb{N}$.

1.4 Orthogonal projection operators $P_n\,(f)$

An orthogonal projection operator $P_n(f)$ of a function $f \in L^2[0,1)$ onto V_n is defined (see [37]) as

$$P_n\,(f)(\,t) \;=\; \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) = \sum_{m=0}^{\infty} \langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \psi_{n,m}(t),$$

where $c_{n,m} = \int\limits_{0}^{1} f(t)\psi_{n,m}(t)\omega_{k,n}(t)\,dt$ and $n = 1, 2, 3, \ldots, 2^{k-1}$, $k \in \mathbb{N}$.

1.5 Wavelet series

A function $f \in L^2[0,1)$ is expanded by wavelet series (see [29]) as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \psi_{n,m}(t), \tag{1}$$

where $c_{n,m} = \int\limits_{-\infty}^{\infty} f(t)\psi_{n,m}(t)\omega_{k,n}(t)\,dt$.

If it is truncated by $P_{2^{k-1},M}\,(f)$ then $f = \lim\limits_{M\to\infty}\lim\limits_{k\to\infty} P_{2^{k-1},M}\,(f)$.

1.6 Wavelet approximation

The error bounds of wavelet approximation $E_{2^{k-1},M}\,(f)$ of a function $f \in L^2[0,1)$ by the orthogonal projection operators $P_{2^{k-1},M}\,(f)$ is defined (see [41]) as

$$E_{2^{k-1},M}(f) \;=\; \inf_{P_{2^{k-1},M}(f)} \|P_{2^{k-1},M}\,(f) - f\|_2,$$

where $M, k \in \mathbb{N}$. If $E_{2^{k-1},M}\,(f) \to 0$ as $k \to \infty$ or $M \to \infty$, then $P_{2^{k-1},M}\,(f)$ is called the best wavelet approximation of a function $f \in L^2[0,1)$ (see [41]).

1.7 Auxiliary lemmas

For the proof of main results, the following lemmas are required.

Lemma 1. *Let* $f_n : \left[\frac{n-1/2}{2^{k-1}}, \frac{n}{2^{k-1}}\right] \longrightarrow [0,1]$ *be a real valued monotonic function defined by* $f_n(t) = 2^k t - 2n + 1$, *where* $n = 1, 2, 3, \ldots, 2^{k-1}$, *and* $k = 1, 2, 3, \ldots$. *Then*

$$(f_n(t))^\alpha \leq \begin{cases} 1 + \alpha(2^k t - 2n), & \text{for } 0 < \alpha < 1 \text{ and } t \in \left[\frac{n-1/2}{2^{k-1}}, \frac{n}{2^{k-1}}\right], \\ 1 + 2^k t - 2n, & \text{for } \alpha \geq 1, \text{ and } t \in \left[\frac{n-1/2}{2^{k-1}}, \frac{n}{2^{k-1}}\right]. \end{cases}$$

Proof. We have $f_n'(t) = 2^k > 0$ with $f_n\left(\frac{n-1/2}{2^{k-1}}\right) = 0$ and $f_n\left(\frac{n}{2^{k-1}}\right) = 1$. □

Lemma 2. *Let f be a bounded real valued measurable function on the non-negative countably additive finite measurable space (X, ζ, μ) and E be a measurable subset of X. Then there exists $M_0 > 0$ such that*

$$|f(t_0)| \leq M_0 \mu(X)\mu(E) \text{ a.e., where } t_0 \in E.$$

In particular, if $X = [0,1)$ and $E = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$, where $n = 1, 2, 3, \ldots, 2^{k-1}$, and $k = 1, 2, 3, \ldots$, then

$$f\left(\frac{2n-1}{2^k}\right) \leq \frac{M_0}{2^{k-1}}.$$

Proof. Since $\mu(E) \leq \mu(X)$ and f is bounded real valued function, there exists $M_0\,(\mu(E))^2$ such that $|f(t)| \leq M_0\,(\mu(E))^2$ for all $t \in E$. □

2 Main results

Theorem 1. *Let $f \in Lip_{[0,1)}\alpha$ and its pseudo-Chebyshev wavelet series is given by equation* (1) *with M^{th} partial sums of an orthogonal projection operators*

$$P_{2^{k-1},M}(f)(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} P_{n,m}(f)(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \langle f, \psi_{n,m}\rangle_{\omega_{k,n}} \psi_{n,m}(x).$$

Then the pseudo Chebyshev wavelet error bounds $E_{2^{k-1},M}(f)$ of a function f by $P_{2^{k-1},M}(f)$ satisfy

$$E_{2^{k-1},M}(f) = \inf_{P_{2^{k-1},M}(f)} \|f - P_{2^{k-1},M}(f)\|_2 = \begin{cases} O\left(\frac{1}{2^{k+1}\,(M-1/2)}\right), & \text{for } \alpha \geq 1, \\ O\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right), & \text{for } 0 < \alpha < 1. \end{cases}$$

Theorem 2. *Let $f \in Lip_{[0,1)}\xi$, i.e. $|f(x) - f(y)| = O(\xi(x-y))$, where ξ is a non-negative monotonic increasing function. Then the pseudo-Chebyshev wavelet error bounds $E_{2^{k-1},M}(f)$ of a function f by $P_{2^{k-1},M}(f)$ satisfy*

$$E_{2^{k-1},M}(f) = \inf_{P_{2^{k-1},M}(f)} \|f - P_{2^{k-1},M}(f)\|_2 = \begin{cases} O\left(\xi\left(\frac{1}{2^{k+1}\,(M-1/2)}\right)\right), & \text{for } \alpha \geq 1, \\ O\left(\xi\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right)\right), & \text{for } 0 < \alpha < 1. \end{cases}$$

Theorem 3. *If $f \in lip_{[0,1)}\alpha$, then the pseudo-Chebyshev wavelet error bounds $E_{2^{k-1},M}(f)$ of a function f by $P_{2^{k-1},M}(f)$ satisfy*

$$E_{2^{k-1},M}(f) = \inf_{P_{2^{k-1},M}(f)} \|f - P_{2^{k-1},M}(f)\|_2 = \begin{cases} o\left(\frac{1}{2^{k+1}(M-1/2)}\right), & \text{for } \alpha \geq 1, \\ o\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right), & \text{for } 0 < \alpha < 1. \end{cases}$$

Theorem 4. *Let $f \in lip_{[0,1)}\xi$, i.e. $|f(x) - f(y)| = o(\xi(x-y))$, where ξ is a non-negative monotonic increasing function. Then the pseudo-Chebyshev wavelet error bounds $E_{2^{k-1},M}(f)$ of a function f by $P_{2^{k-1},M}(f)$ satisfy*

$$E_{2^{k-1},M}(f) = \inf_{P_{2^{k-1},M}(f)} \|f - P_{2^{k-1},M}(f)\|_2 = \begin{cases} o\left(\xi\left(\frac{1}{2^{k+1}(M-1/2)}\right)\right), & \text{for } \alpha \geq 1, \\ o\left(\xi\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right)\right), & \text{for } 0 < \alpha < 1. \end{cases}$$

2.1 Proof of theorems

Proof of Theorem 1. Since

$$\begin{aligned}
c_{n,m} = \langle f, \psi_{n,m}\rangle_{\omega_{k,n}} &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\psi_{n,m}(t)\omega_{k,n}(t)\,dt \\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(t) - f\left(\tfrac{2n-1}{2^k}\right) + f\left(\tfrac{2n-1}{2^k}\right)\right)\psi_{n,m}(t)\omega_{k,n}(t)\,dt \\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(\left|t - \left(\tfrac{2n-1}{2^k}\right)\right|^\alpha + f\left(\tfrac{2n-1}{2^k}\right)\right)\psi_{n,m}(t)\omega_{k,n}(t)\,dt, \\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(\left|\tfrac{2^k t - 2n+1}{2^k}\right|^\alpha + f\left(\tfrac{2n-1}{2^k}\right)\right)\psi_{n,m}(t)\omega_{k,n}(t)\,dt \\
&= \frac{1}{2^{k\alpha}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left|2^k t - 2n + 1\right|^\alpha \psi_{n,m}(t)\omega_{k,n}(t)\,dt \\
&\quad + f\left(\tfrac{2n-1}{2^k}\right)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \psi_{n,m}(t)\omega_{k,n}(t)\,dt \\
&= \sqrt{\frac{2^{k+1}}{\pi}}\left(\frac{1}{2^{k\alpha}}I_1 + f\left(\tfrac{2n-1}{2^k}\right)I_2\right),
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left|2^k t - 2n + 1\right|^\alpha T_{m+\frac{1}{2}}(2^k t - 2n + 1)\omega_{k,n}(t)\,dt \\
&= \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (\bar{t})^\alpha T_{m+1/2}(\bar{t})\omega_{k,n}(t)\,dt + \int_{\frac{n-1}{2^{k-1}}}^{\frac{n-1/2}{2^{k-1}}} (-\bar{t})^\alpha T_{m+1/2}(\bar{t})\omega_{k,n}(t)\,dt = I_{1,1} + I_{1,2}, \\
I_2 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_{m+1/2}(\bar{t})\omega_{k,n}(t)\,dt.
\end{aligned}$$

Above we set $\bar{t} = 2^k t - 2n + 1$, $\bar{t} \geq 0$ for $t \in \left[\frac{n-1/2}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ and $\bar{t} \leq 0$ for $t \in \left[\frac{n-1}{2^{k-1}}, \frac{n-1/2}{2^{k-1}}\right]$.

Therefore

$$c_{n,m} = \sqrt{\frac{2^{k+1}}{\pi}}\left(\frac{1}{2^{k\alpha}}(I_{1,1} + I_{1,2}) + f\left(\tfrac{2n-1}{2^k}\right)I_2\right). \tag{2}$$

If $\alpha \geq 1$, then by the Lemma 1 we have

$$
\begin{aligned}
I_{1,1} &= \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (2^k t - 2n + 1)^{\alpha} T_{m+1/2}(\bar{t})\omega_{k,n}(t)\, dt \\
&\leq \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (2^k t - 2n + 1) T_{m+1/2}(2^k t - 2n + 1)\omega_{k,n}(t)\, dt \\
&\leq \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (2^k t - 2n + 1) T_{m+1/2}(2^k t - 2n + 1)\omega(2^k t - 2n + 1)\, dt \\
&= \frac{1}{2^k} \int_0^{\frac{\pi}{2}} \cos\theta T_{m+1/2}(\cos\theta)\, d\theta \\
&= \frac{1}{2^k} \int_0^{\frac{\pi}{2}} \cos\theta \cos\left(m + 1/2\right)\theta\, d\theta,
\end{aligned}
$$

where we put $\cos\theta = 2^k t - 2n + 1$. Similarly,

$$
I_{1,2} \leq \frac{1}{2^k} \int_{\frac{\pi}{2}}^{\pi} \cos\theta \cos\left(m + 1/2\right)\theta\, d\theta.
$$

Therefore,

$$
\begin{aligned}
I_1 = \frac{1}{2^k}(I_{1,1} + I_{1,2}) &\leq \frac{1}{2^k} \int_0^{\pi} \cos\theta \cos\left(m + 1/2\right)\theta\, d\theta \\
&= \frac{1}{2^k}\frac{1}{2} \int_0^{\pi} \left(\cos\left(m + 3/2\right)\theta + \cos\left(m - 1/2\right)\theta\right) d\theta \\
&= (-1)^m \frac{1}{2^{k+1}} \left(\frac{1}{m - 1/2} - \frac{1}{m + 3/2} \right) \\
&\leq (-1)^m \frac{1}{2^{k+1}} \frac{1}{m - 1/2}.
\end{aligned}
$$

Next

$$
\begin{aligned}
I_2 = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_{m+1/2}(\bar{t})\omega_{k,n}(t)\, dt &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_{m+1/2}(2^k t - 2n + 1)\omega_{k,n}(t)\, dt \\
&= \frac{1}{2^k} \int_0^{\pi} \cos\left(m + 1/2\right)\theta\, d\theta = (-1)^m \frac{1}{2^k} \frac{1}{m + 1/2}.
\end{aligned}
$$

Therefore, by the equation (2) and Lemma 2 we obtain

$$
\begin{aligned}
c_{n,m} &\leq \sqrt{\frac{2^{k+1}}{\pi}} \left((-1)^m \frac{1}{2^{k\alpha}} \frac{1}{2^{k+1}} \frac{1}{m - 1/2} + (-1)^m f\left(\frac{2n-1}{2^k}\right) \frac{1}{2^k} \frac{1}{m + 1/2} \right) \\
&\leq \frac{(-1)^m}{2^k} \sqrt{\frac{2^{k+1}}{\pi}} \left(\frac{1}{2^{k\alpha+1}} \frac{1}{m - 1/2} + \frac{4M_0}{2^{k+1}} \frac{1}{m + 1/2} \right) \\
&\leq \frac{(-1)^m}{2^{k+1}2^k} \sqrt{\frac{2^{k+1}}{\pi}} \frac{1}{m - 1/2} \left(1 + 4M_0\right) \\
&= (-1)^m \left(\frac{1 + 4M_0}{2} \right) \sqrt{\frac{2}{\pi}} \frac{1}{2^{3k/2}} \frac{1}{m - 1/2}.
\end{aligned}
$$

If $0 < \alpha < 1$, then by the Lemma 1 we have

$$\begin{aligned}
I_{1,1} &= \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (2^k t - 2n + 1)^{\alpha} T_{m+1/2}(\bar{t})\omega_{k,n}(t)\, dt \\
&\leq \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (2^k \alpha t - 2\alpha n + 1) T_{m+1/2}(2^k t - 2n + 1)\omega_{k,n}(t)\, dt \\
&\leq \alpha \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} (2^k t - 2n + 1) T_{m+1/2}(2^k t - 2n + 1)\omega(2^k t - 2n + 1)\, dt \\
&\quad + (1-\alpha) \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_{m+1/2}(2^k t - 2n + 1)\omega(2^k t - 2n + 1)\, dt \\
&= \frac{\alpha}{2^k} \int_0^{\frac{\pi}{2}} \cos\theta T_{m+1/2}(\cos\theta)\, d\theta + \frac{(1-\alpha)}{2^k} \int_0^{\frac{\pi}{2}} T_{m+1/2}(\cos\theta)\, d\theta \\
&= \frac{\alpha}{2^k} \int_0^{\frac{\pi}{2}} \cos\theta \cos\left(m + 1/2\right)\theta\, d\theta + \frac{(1-\alpha)}{2^k} \int_0^{\frac{\pi}{2}} \cos\left(m + 1/2\right)\theta\, d\theta,
\end{aligned}$$

where we put $\cos\theta = 2^k t - 2n + 1$. Similarly

$$I_{1,2} \;\leq\; \frac{\alpha}{2^k} \int_{\frac{\pi}{2}}^{\pi} \cos\theta \cos\left(m + 1/2\right)\theta\, d\theta + \frac{(\alpha-1)}{2^k} \int_{\frac{\pi}{2}}^{\pi} \cos\left(m + 1/2\right)\theta\, d\theta.$$

Therefore,

$$\begin{aligned}
I_1 = \frac{1}{2^k}(I_{1,1} + I_{1,2}) &\leq \frac{\alpha}{2^k} \int_0^{\pi} \cos\theta \cos\left(m + 1/2\right)\theta\, d\theta + \frac{(1-\alpha)}{2^k} \int_0^{\frac{\pi}{2}} T_{m+1/2}(\cos\theta)\, d\theta \\
&\quad + \frac{(\alpha-1)}{2^k} \int_{\frac{\pi}{2}}^{\pi} T_{m+1/2}(\cos\theta)\, d\theta \\
&= \frac{\alpha}{2^k}\frac{1}{2} \int_0^{\pi} \left(\cos\left(m + 3/2\right)\theta + \cos\left(m - 1/2\right)\theta\right)\, d\theta + 0 \\
&= (-1)^m \frac{\alpha}{2^{k+1}} \left(\frac{1}{m - 1/2} - \frac{1}{m + 3/2}\right) \leq (-1)^m \frac{\alpha}{2^{k+1}} \frac{1}{m - 1/2}.
\end{aligned}$$

Therefore, by the equation (2) and Lemma 2 we obtain

$$\begin{aligned}
c_{n,m} &\leq \sqrt{\frac{2^{k+1}}{\pi}} \left((-1)^m \frac{1}{2^{k\alpha}} \frac{\alpha}{2^{k+1}} \frac{1}{m - 1/2} + (-1)^m f\left(\frac{2n-1}{2^k}\right) \frac{1}{2^k} \frac{1}{m + 1/2}\right) \\
&\leq \frac{(-1)^m}{2^{k\alpha}} \sqrt{\frac{2^{k+1}}{\pi}} \left(\frac{\alpha}{2^{k+1}} \frac{1}{m - 1/2} + \frac{4M_0}{2^{k+1}} \frac{1}{m + 1/2}\right) \\
&= \frac{(-1)^m}{2^{k\alpha}} \sqrt{\frac{2^{k+1}}{\pi}} \frac{1}{2^{k+1}} \left(\frac{\alpha}{m - 1/2} + \frac{4M_0}{m + 1/2}\right) \\
&\leq \frac{(-1)^m}{2^{k\alpha}} \frac{1}{2^{k/2}} \frac{1}{m - 1/2} \frac{\alpha + 4M_0}{2} \sqrt{\frac{2}{\pi}}.
\end{aligned}$$

Hence

$$c_{n,m} \leq \begin{cases} (-1)^m \left(\frac{1+4M_0}{2}\right) \sqrt{\frac{2}{\pi}}\, \frac{1}{2^{3k/2}}\, \frac{1}{m-1/2}, & \text{for } \alpha \geq 1, \\ (-1)^m \left(\frac{\alpha+4M_0}{2}\right) \sqrt{\frac{2}{\pi}}\, \frac{1}{2^{k\alpha}} \frac{1}{2^{k/2}} \frac{1}{m-1/2}, & \text{for } 0 < \alpha < 1. \end{cases}$$

Write

$$
\begin{aligned}
f(t)-P_{2^{k-1},M}(f)(t) &= \left(\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)c_{n,m}\psi_{n,m}(t)\\
&= \left(\left(\sum_{n=1}^{2^{k-1}}+\sum_{n=2^k}^{\infty}\right)\sum_{m=0}^{\infty}-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)c_{n,m}\psi_{n,m}(t)\\
&= \left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty}+\sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty}-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)c_{n,m}\psi_{n,m}(t)\\
&= \left(\sum_{n=1}^{2^{k-1}}\left(\sum_{m=0}^{M-1}+\sum_{m=M}^{\infty}\right)+\sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty}-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)c_{n,m}\psi_{n,m}(t)\\
&= \left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}+\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}+\sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty}-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\right)c_{n,m}\psi_{n,m}(t)\\
&= \left(\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}+\sum_{n=2^k}^{\infty}\sum_{m=0}^{\infty}\right)c_{n,m}\psi_{n,m}(t).
\end{aligned}
$$

Now we use the property of the o.n. wavelets $\{\psi_{n,m}(t)\}$ in the disjoint intervals $\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right]$. Then

$$
\begin{aligned}
\left(f(t)-P_{2^{k-1},M}(f)(t)\right)^2 =& \left(\left(\sum_{n_1=1}^{2^{k-1}}\sum_{m_1=M}^{\infty}+\sum_{n_1=2^k}^{\infty}\sum_{m_1=0}^{\infty}\right)c_{n_1,m_1}\psi_{n_1,m_1}(t)\right)\\
&\times\left(\left(\sum_{n_2=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}+\sum_{n_2=2^k}^{\infty}\sum_{m_2=0}^{\infty}\right)c_{n_2,m_2}\psi_{n_2,m_2}(t)\right)=\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}|^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(E_{2^{k-1},M}(f))^2=\|E_{n,m}(f)\|_2^2 &= \int_0^1|E_{n,m}(f)(t)|^2\,dt=\inf_{P_{2^{k-1},M}(f)}\int_0^1\left|f(t)-P_{2^{k-1},M}(f)(t)\right|^2\,dt\\
&= \inf_M\int_0^1\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}|^2\,dt=\inf_M\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}|^2\\
&\leq \inf_M\begin{cases}\sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=M}^{\infty}\frac{(1+4M_0)^2}{2\pi}\left(\frac{1}{2^{3k/2}}\frac{1}{m-1/2}\right)^2, & \text{for } \alpha\geq 1,\\ \sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=M}^{\infty}\frac{(\alpha+4M_0)^2}{2\pi}\left(\frac{1}{2^{k\alpha}2^{k/2}}\frac{1}{m-1/2}\right)^2, & \text{for } 0<\alpha<1,\end{cases}\\
&\leq \begin{cases}\frac{(1+4M_0)^2}{\pi}\left(\frac{1}{2^{k+1}}\frac{1}{M-1/2}\right)^2, & \text{for } \alpha\geq 1,\\ \frac{(\alpha+4M_0)^2}{\pi}\left(\frac{1}{2^{k\alpha+1}}\frac{1}{M-1/2}\right)^2, & \text{for } 0<\alpha<1.\end{cases}
\end{aligned}
$$

Therefore,

$$
E_{2^{k-1},M}(f) = \begin{cases}O\left(\frac{1}{2^{k+1}(M-1/2)}\right), & \text{for } \alpha\geq 1,\\ O\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right), & \text{for } 0<\alpha<1.\end{cases}
$$

Thus the Theorem 1 is completely established. □

Proof of Theorem 2. Following the proof of Theorem 1, we write

$$\left|E_{2^{k-1},M}(f)(t)\right| = \left|f(t) - P_{2^{k-1},M-1}(f)(t)\right| = \begin{cases} O\Big(\xi\left(\frac{1}{2^{k+1}(M-1/2)}\right)\Big), & \text{for } \alpha \geq 1, \\ O\Big(\xi\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right)\Big), & \text{for } 0 < \alpha < 1, \end{cases}$$

and

$$\begin{aligned} E_{2^{k-1},M}(f) &= \left(\inf_{P_{2^{k-1},M-1}(f)} \int_0^1 \left|f(t) - P_{2^{k-1},M-1}(f)(t)\right|^2 dt\right)^{1/2} \\ &= \begin{cases} \int_0^1 O\Big(\xi\left(\frac{1}{2^{k+1}(M-1/2)}\right)\Big)\, dt, & \text{for } \alpha \geq 1, \\ \int_0^1 O\Big(\xi\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right)\Big)\, dt, & \text{for } 0 < \alpha < 1, \end{cases} \\ &= \begin{cases} O\Big(\xi\left(\frac{1}{2^{k+1}(M-1/2)}\right)\Big) \int_0^1 dt, & \text{for } \alpha \geq 1, \\ O\Big(\xi\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right)\Big) \int_0^1 dt, & \text{for } 0 < \alpha < 1, \end{cases} \\ &= \begin{cases} O\Big(\xi\left(\frac{1}{2^{k+1}(M-1/2)}\right)\Big), & \text{for } \alpha \geq 1, \\ O\Big(\xi\left(\frac{1}{2^{k\alpha+1}(M-1/2)}\right)\Big), & \text{for } 0 < \alpha < 1. \end{cases} \end{aligned}$$

Thus the Theorem 2 is completely established. □

Proofs of Theorems 3 and 4 can be developed on the lines of proofs of Theorems 1 and 2 considering $f \in lip_{[0,1)}\alpha$ and $f \in lip_{[0,1)}\xi$, respectively.

2.2 Corollaries

In this section, four new corollaries related to Theorems 1, 2, 3 and 4 have been established.

Corollary 1. *Let $f \in Lip_{[0,1)}\alpha$ and its pseudo-Chebyshev wavelet series for $m = 0$ is given by*

$$f(t) = \sum_{n=1}^{\infty} \langle f, \psi_{n,0}\rangle_{\omega_{k,n}} \psi_{n,0}(t)$$

with the 2^{k-1}th order orthogonal projection operator

$$P_{2^{k-1},0}(f)(x) = \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}(x) = \sum_{n=1}^{2^{k-1}} \langle f, \psi_{n,0}\rangle_{\omega_{k,n}} \psi_{n,0}(x).$$

Then the pseudo-Chebyshev wavelet error bounds $E_{2^{k-1},0}(f)$ of a function f by $P_{2^{k-1},0}(f)$ satisfy

$$E_{2^{k-1},0}(f) = \inf_{P_{2^{k-1},0}(f)} \|f - P_{2^{k-1},0}(f)\| = \begin{cases} O\Big(\frac{1}{2^{k+1}}\Big), & \text{for } \alpha \geq 1, \\ O\Big(\frac{1}{2^{k\alpha+1}}\Big), & \text{for } 0 < \alpha < 1. \end{cases}$$

Corollary 2. *Let $f \in Lip_{[0,1)}\xi$, i.e. $|f(x) - f(y)| = O(\xi(x-y))$, where ξ is a non-negative monotonic increasing function. Then the pseudo-Chebyshev wavelet error bounds $E_{2^{k-1},0}(f)$ of a function f by $P_{2^{k-1},0}(f)$ satisfy*

$$E_{2^{k-1},0}(f) = \min_{P_{2^{k-1},0}(f)} \|f - P_{2^{k-1},0}(f)\| = \begin{cases} O\Big(\xi\left(\frac{1}{2^{k+1}}\right)\Big), & \text{for } \alpha \geq 1, \\ O\Big(\xi\left(\frac{1}{2^{k\alpha+1}}\right)\Big), & \text{for } 0 < \alpha < 1. \end{cases}$$

Corollary 3. *If $f \in lip_{[0,1)}\alpha$, then the pseudo-Chebyshev wavelet error bounds $E_{2^{k-1},0}(f)$ of a function f by $P_{2^{k-1},0}(f)$ satisfy*

$$E_{2^{k-1},0}(f) = \min_{P_{2^{k-1},0}(f)} \|f - P_{2^{k-1},0}(f)\| = \begin{cases} o\left(\frac{1}{2^{k+1}}\right), & \text{for } \alpha \geq 1, \\ o\left(\frac{1}{2^{k\alpha+1}}\right), & \text{for } 0 < \alpha < 1. \end{cases}$$

Corollary 4. *Let $f \in lip_{[0,1)}\xi$, i.e. $|f(x) - f(y)| = o(\xi(x-y))$, where ξ is a non-negative monotonic increasing function. Then the pseudo-Chebyshev wavelet error bounds $E_{2^{k-1},0}(f)$ of a function f by $P_{2^{k-1},0}(f)$ satisfy*

$$E_{2^{k-1},0}(f) = \min_{P_{2^{k-1},0}(f)} \|f - P_{2^{k-1},0}(f)\| = \begin{cases} o\left(\xi\left(\frac{1}{2^{k+1}}\right)\right), & \text{for } \alpha \geq 1, \\ o\left(\xi\left(\frac{1}{2^{k\alpha+1}}\right)\right), & \text{for } 0 < \alpha < 1. \end{cases}$$

2.3 Proof of corollaries

Proof of Corollary 1. Consider,

$$\begin{aligned} c_{n,0} = \langle f, \psi_{n,0}\rangle_{\omega_{k,n}} &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\psi_{n,0}(t)\omega_{k,n}(t)\,dt \\ &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(t) - f\left(\tfrac{2n-1}{2^k}\right)\right)\psi_{n,0}(t)\omega_{k,n}(t)\,dt + \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f\left(\tfrac{2n-1}{2^k}\right)\psi_{n,0}(t)\omega_{k,n}(t)\,dt \\ &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left|t - \tfrac{2n-1}{2^k}\right|^\alpha \psi_{n,0}(t)\omega_{k,n}(t)\,dt + f\left(\tfrac{2n-1}{2^k}\right)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \psi_{n,0}(t)\omega_{k,n}(t)\,dt \\ &= \frac{1}{2^{k\alpha}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left|2^k t - 2n + 1\right|^\alpha \psi_{n,0}(t)\omega_{k,n}(t)\,dt + f\left(\tfrac{2n-1}{2^k}\right)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \psi_{n,0}(t)\omega_{k,n}(t)\,dt \\ &= \frac{1}{2^{k\alpha}}I_1 + f\left(\tfrac{2n-1}{2^k}\right)I_2. \end{aligned}$$

Next,

$$\begin{aligned} I_2 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \psi_{n,0}(t)\omega_{k,n}(t)\,dt = \sqrt{\frac{2^{k+1}}{\pi}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_{1/2}(2^k t - 2n + 1)\omega(2^k t - 2n + 1)\,dt \\ &= \sqrt{\frac{2^{k+1}}{\pi}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \sqrt{\frac{2^k t - 2n + 2}{2}}\sqrt{\frac{1}{1 - (2^k t - 2n + 1)^2}}\,dt \\ &= \sqrt{\frac{2^k}{\pi}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \sqrt{\frac{1 + (2^k t - 2n + 1)}{1 - (2^k t - 2n + 1)^2}}\,dt = \sqrt{\frac{2^k}{\pi}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{1}{\sqrt{2n - 2^k t}}\,dt \\ &= \sqrt{\frac{1}{\pi}}\frac{1}{2^{k/2}}\int_0^2 \frac{1}{\sqrt{u}}\,du = \sqrt{\frac{8}{\pi}}\frac{1}{2^{k/2}} \end{aligned}$$

and

$$\begin{aligned} I_1 &= \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(2^k t - 2n + 1\right)^\alpha \psi_{n,0}(t)\omega_{k,n}(t)\,dt \\ &\quad + \int_{\frac{n-1}{2^{k-1}}}^{\frac{n-1/2}{2^{k-1}}} \left(-(2^k t - 2n + 1)\right)^\alpha \psi_{n,0}(t)\omega_{k,n}(t)\,dt \\ &= I_{1,1} + I_{1,2}. \end{aligned}$$

If $\alpha \geq 1$, then by the Lemma 1

$$\begin{aligned}
I_{1,1} &= \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(2^k t - 2n + 1\right)^{\alpha} \psi_{n,0}(t)\omega_{k,n}(t)\, dt \\
&\leq \sqrt{\frac{2^{k+1}}{\pi}} \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(2^k t - 2n + 1\right) \sqrt{\frac{1 + 2^k t - 2n + 1}{2}} \sqrt{\frac{1}{1-(2^k t - 2n+1)^2}}\, dt \\
&\leq \sqrt{\frac{2^k}{\pi}} \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{\left(1-(2n-2^k t)\right)}{\sqrt{2n-2^k t}}\, dt = \frac{1}{2^{k/2}\sqrt{\pi}} \int_0^1 \frac{1-u}{\sqrt{u}}\, du = \frac{4}{3}\frac{1}{2^{k/2}}\sqrt{\frac{1}{\pi}}.
\end{aligned}$$

Similarly,

$$I_{1,2} \leq \frac{4}{3}\frac{1}{2^{k/2}}\sqrt{\frac{1}{\pi}}.$$

Therefore,

$$I_1 = I_{1,1} + I_{1,2} \leq \frac{4\sqrt{2}}{3\, 2^{k/2}}\sqrt{\frac{2}{\pi}}.$$

So, by the Lemma 2

$$\begin{aligned}
c_{n,0} &\leq \frac{1}{2^{k\alpha}}\frac{4\sqrt{2}}{3\, 2^{k/2}}\sqrt{\frac{2}{\pi}} + f\left(\frac{2n-1}{2^k}\right)\sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}} \leq \sqrt{\frac{2}{\pi}}\frac{1}{2^{k/2}}\left(\frac{4\sqrt{2}}{3\, 2^{k\alpha}} + \left(\frac{2M_0}{2^k}\right)\right) \\
&\leq \sqrt{\frac{8}{\pi}}\frac{1}{2^{k/2}}\left(\frac{2\sqrt{2}}{3\, 2^k} + \left(\frac{M_0}{2^k}\right)\right) = \sqrt{\frac{8}{\pi}}\,\frac{2\sqrt{2}+3M_0}{3}\,\frac{1}{2^{3k/2}}.
\end{aligned} \tag{3}$$

Again, if $0 < \alpha < 1$, then by the Lemma 1

$$\begin{aligned}
I_{1,1} &= \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(2^k t - 2n + 1\right)^{\alpha} \psi_{n,0}(t)\omega_{k,n}(t)\, dt \\
&\leq \sqrt{\frac{2^{k+1}}{\pi}} \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(1-\alpha(2n-2^k t)\right) \sqrt{\frac{1 + 2^k t - 2n + 1}{2}} \sqrt{\frac{1}{1-(2^k t - 2n+1)^2}}\, dt \\
&= \sqrt{\frac{2^k}{\pi}} \int_{\frac{n-1/2}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{\left(1-\alpha(2n-2^k t)\right)}{\sqrt{2n-2^k t}}\, dt = \frac{1}{2^{k/2}\sqrt{\pi}} \int_0^1 \frac{1-\alpha u}{\sqrt{u}}\, du = \frac{2(3-\alpha)}{3\,\sqrt{\pi}}\,\frac{1}{2^{k/2}}.
\end{aligned}$$

Similarly,

$$I_{1,2} \leq \frac{2(3-\alpha)}{3\,\sqrt{\pi}}\,\frac{1}{2^{k/2}}.$$

Therefore,

$$I_1 = I_{1,1} + I_{1,2} \leq \frac{(3-\alpha)\sqrt{2}}{3}\sqrt{\frac{8}{\pi}}\,\frac{1}{2^{k/2}}.$$

So, by the Lemma 2

$$\begin{aligned}
c_{n,0} &\leq \frac{1}{2^{k\alpha}}\left(\frac{(3-\alpha)\sqrt{2}}{3}\sqrt{\frac{8}{\pi}}\,\frac{1}{2^{k/2}}\right) + f\left(\frac{2n-1}{2^k}\right)\sqrt{\frac{8}{\pi}}\frac{1}{2^{k/2}} \\
&\leq \sqrt{\frac{8}{\pi}}\,\frac{1}{2^{k/2}}\left(\frac{(3-\alpha)\sqrt{2}}{3\, 2^{k\alpha}} + \frac{2M_0}{2^k}\right) \leq \sqrt{\frac{8}{\pi}}\,\frac{1}{2^{k/2}}\left(\frac{(3-\alpha)\sqrt{2}}{3\, 2^{k\alpha}} + \frac{2M_0}{2^{k\alpha}}\right) \\
&= \sqrt{\frac{8}{\pi}}\,\frac{1}{2^{k/2}}\left(\frac{(3-\alpha)\sqrt{2}+6M_0}{3}\right)\frac{1}{2^{k\alpha}}.
\end{aligned} \tag{4}$$

By equations (3) and (4)

$$c_{n,0} \le \begin{cases} \sqrt{\frac{8}{\pi}}\left(\frac{2\sqrt{2}+3M_0}{3}\right)\frac{1}{2^{3k/2}}, & \text{for } \alpha \ge 1, \\ \sqrt{\frac{8}{\pi}}\left(\frac{(3-\alpha)\sqrt{2}+6M_0}{3}\right)\frac{1}{2^{k\alpha}}\frac{1}{2^{k/2}}, & \text{for } 0<\alpha<1. \end{cases}$$

Since

$$\begin{aligned}
(E_{2^{k-1},0}(f))^2 &= \|E_{2^{k-1},0}(f)(t)\|_2^2 = \int_0^1 \left|E_{2^{k-1},0}(f)(t)\right|^2 dt \\
&= \inf_{P_{2^{k-1},0}(f)} \int_0^1 \left|f(t) - P_{2^{k-1},0}(f)(t)\right|^2 dt = \int_0^1 \sum_{n=1}^{2^{k-1}} |c_{n,0}|^2 \, dt = \sum_{n=1}^{2^{k-1}} |c_{n,0}|^2 \\
&\le \begin{cases} \sum_{n=1}^{2^{k-1}} \left|\sqrt{\frac{8}{\pi}}\left(\frac{2\sqrt{2}+3M_0}{3}\right)\frac{1}{2^{3k/2}}\right|^2, & \text{for } \alpha \ge 1, \\ \sum_{n=1}^{2^{k-1}} \left|\sqrt{\frac{8}{\pi}}\left(\frac{(3-\alpha)\sqrt{2}+6M_0}{3}\right)\frac{1}{2^{k\alpha}}\frac{1}{2^{k/2}}\right|^2, & \text{for } 0<\alpha<1, \end{cases} \\
&\le \begin{cases} \left(\sqrt{\frac{8}{\pi}}\left(\frac{2\sqrt{2}+3M_0}{3}\right)\frac{1}{2^{k+1}}\right)^2, & \text{for } \alpha \ge 1, \\ \left(\sqrt{\frac{8}{\pi}}\left(\frac{(3-\alpha)\sqrt{2}+6M_0}{3}\right)\frac{1}{2^{k\alpha+1}}\right)^2, & \text{for } 0<\alpha<1, \end{cases}
\end{aligned}$$

therefore

$$E_{2^{k-1},0}(f) = \begin{cases} O\left(\frac{1}{2^{k+1}}\right), & \text{for } \alpha \ge 1, \\ O\left(\frac{1}{2^{k\alpha+1}}\right), & \text{for } 0<\alpha<1. \end{cases}$$

Thus the Corollary 1 is completely established. □

Proof of Corollary 2. Following the proof of Corollary 1,

$$\begin{aligned}
E_{2^{k-1},0}(f) &= \left(\min_{P_{2^{k-1},0}(f)} \int_0^1 \left|f(t) - P_{2^{k-1},0}(f)(t)\right|^2\right)^{1/2} \\
&= \begin{cases} \int_0^1 O\left(\xi\left(\frac{1}{2^{k+1}}\right)\right) dt, & \text{for } \alpha \ge 1, \\ \int_0^1 O\left(\xi\left(\frac{1}{2^{k\alpha+1}}\right)\right) dt, & \text{for } 0<\alpha<1, \end{cases} \\
&= \begin{cases} O\left(\xi\left(\frac{1}{2^{k+1}}\right)\right) \int_0^1 dt, & \text{for } \alpha \ge 1, \\ O\left(\xi\left(\frac{1}{2^{k\alpha+1}}\right)\right) \int_0^1 dt, & \text{for } 0<\alpha<1, \end{cases} \\
&= \begin{cases} O\left(\xi\left(\frac{1}{2^{k+1}}\right)\right), & \text{for } \alpha \ge 1, \\ O\left(\xi\left(\frac{1}{2^{k\alpha+1}}\right)\right), & \text{for } 0<\alpha<1. \end{cases}
\end{aligned}$$

Thus the Corollary 2 is completely established. □

Proofs of Corollaries 3 and 4 can be developed on the lines of proofs of Corollaries 1 and 2 considering $f \in lip_{[0,1)}\alpha$ and $f \in lip_{[0,1)}\xi$, respectively.

3 Approximation of function by pseudo-Chebyshev wavelet

3.1 Function approximation

Suppose that

$$\left\{\psi_{n,m}(t)\colon n=1,2,3,\dots,2^{k-1}, m=0,1,2,\dots,M-1, k,M\in\mathbb{N}\right\}\subset L^2([0,1))$$

is the set of pseudo-Chebyshev wavelets and

$$\begin{aligned} W &= \text{cl span}\left\{\psi_{1,0}(t),\psi_{1,1}(t),\cdots,\psi_{1,M-1},\cdots,\psi_{2^{k-1},0},\psi_{2^{k-1},1},\cdots,\psi_{2^{k-1},M-1}\right\}\\ &= \overline{\text{span}\left\{\psi_{n,m}(t)\right\}}\end{aligned}$$

be a finite dimensional closed subspace of the vector space $L^2([0,1))$ of dimension $2^{k-1}M$. Let $f\in L^2([0,1))$ be an arbitrary element. Then a function $f_0\in W$ is said to be the best approximation out of W, if for any $g\in W$ the following condition

$$\|f-f_0\|_2^2 \;=\; \int_0^1 |f(t)-f_0(t)|^2\,dt \le \int_0^1 |f(t)-g(t)|^2\,dt = \|f-g\|_2^2$$

is satisfied, or equivalently,

$$|f(t)-f_0(t)|\le|f(t)-g(t)| \quad \forall\, 0\le t<1.$$

Since $f_0(t)\in W$, therefore there exist the unique coefficients $a_{i,j}$'s, $i=1,2,3,\dots,2^{k-1}$ and $j=0,1,2,\dots,M-1$ such that

$$f(x)\approx f_0(x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}a_{n,m}\psi_{n,m}(x)=\langle A,\Psi(x)\rangle=A^\tau\Psi(x), \tag{5}$$

where A^τ indicates transpose of a matrix A. Here A and $\Psi(x)$ are $2^{k-1}M\times 1$ matrices given by

$$A=\left(a_{1,0},a_{1,1},a_{1,2},\cdots,a_{1,M-1},a_{2,0},a_{2,1},\cdots,a_{2,M-1},\cdots,a_{2^{k-1},0},a_{2^{k-1},1},\cdots,a_{2^{k-1},M-1}\right)^\tau,$$

$$\Psi=\left(\psi_{1,0},\psi_{1,1},\psi_{1,2},\cdots,\psi_{1,M-1},\psi_{2,0},\psi_{2,1},\cdots,\psi_{2,M-1},\cdots,\psi_{2^{k-1},0},\psi_{2^{k-1},1},\cdots,\psi_{2^{k-1},M-1}\right)^\tau,$$

and $\langle A,\Psi(t)\rangle$ is an inner product of column vectors of A and Ψ. To compute the column vector of A, let

$$f_{i,j}=\int_0^1 f(t)\psi_{i,j}(t)\,dt.$$

By equation (5)

$$f_{i,j}=\int_0^1\langle A,\Psi(t)\rangle\,\psi_{i,j}(t)\,dt=\int_0^1\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}a_{n,m}\psi_{n,m}(t)\right)\psi_{i,j}(t)\,dt=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}a_{n,m}b_{i,j}^{n,m},$$

where $i=1,2,3,\dots,2^{k-1}$, $j=0,1,2,\dots,M-1$, and

$$b_{i,j}^{n,m}=\int_0^1\psi_{n,m}(t)\psi_{i,j}(t)\,dt.$$

Therefore, $f_{i,j} = \langle A, B_{i,j} \rangle = A^\tau B_{i,j}$, and

$$B_{i,j} = \left(b_{i,j}^{1,0}, b_{i,j}^{1,1}, b_{i,j}^{1,2}, \cdots, b_{i,j}^{1,M-1}, b_{i,j}^{2,0}, \cdots, b_{i,j}^{2,M-1}, \cdots, b_{i,j}^{2^{k-1},0}, b_{i,j}^{2^{k-1},1}, \cdots, b_{i,j}^{2^{k-1},M-1}\right)^\tau,$$

where $i = 1, 2, 3, \ldots, 2^{k-1}$, $j = 0, 1, 2, \cdots, M-1$. If

$$F = \left(f_{1,0}, f_{1,1}, \cdots, f_{1,M-1}, f_{2,0}, \cdots, f_{2,M-1}, \cdots, f_{2^{k-1},0}, \cdots, f_{2^{k-1},M-1}\right)^\tau,$$

then $F^\tau = A^\tau B$, where B is a square matrix of order $2^{k-1}M \times 2^{k-1}M$ and it is given by

$$B = \left(b_{i,j}^{n,m}\right)_{1 \le i,n \le 2^{k-1}, 0 \le j,m \le M-1} = \int_0^1 (\Psi(x) \otimes \Psi(x))\, dx = \int_0^1 \Psi(x)\Psi^\tau(x)\, dx,$$

where $\Psi \otimes \Psi$ denotes the tensor product of vectors Ψ. Since F^τ and B is known because of it is fully determined with the help of pseudo-Chebyshev wavelets $\psi_{n,m}$ and function f. The best approximation $f_0(t)$ can be determined with the help of the equation (5) and $A^\tau = F^\tau B^{-1}$. The existence of B^{-1} is insured by the orthogonality property of the pseudo-Chebyshev wavelets $\psi_{n,m}$.

Therefore, we obtained

$$f(x) \approx f_0(x) = \left\langle \left(B^{-1}\right)^\tau F, \Psi(x) \right\rangle = F^\tau B^{-1}\Psi(x) \quad \forall\, 0 \le x < 1.$$

In particular, if $B = I$ then $f(x) \approx f_0(x) = \langle A, \Psi(x) \rangle = A^\tau \Psi(x)$ for all $0 \le x < 1$, $A_i^\tau = (a_{i,0}, a_{i,1}, a_{i,2}, \cdots, a_{i,M-1})$ for fixed i, and

$$a_{i,j} = \frac{\int_0^1 f(x)\psi_{i,j}(x)\omega(x)\, dx}{\int_0^1 \psi_{i,j}(x)\psi_{i,j}(x)\omega(x)\, dx}$$

for fixed j with $i = 1, 2, 3, \cdots 2^{k-1}$.

3.2 Illustrative example

In this section, we calculate the approximation of the function

$$f(x) = \begin{cases} x^{1/2} - 3x^{3/2} + 7x^{5/2} + 9x^{7/2}, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases}$$

by the pseudo-Chebyshev wavelet approximation method.

In the Theorem 1, if $k = 1$, then $n = 1$ and

$$P_{1,M}(f)(x) = \sum_{m=0}^{M-1} \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,m}(x) = \sum_{m=0}^{M-1} a_{1,m}\psi_{1,m}(x), \quad a_{1,m} = \frac{\int_0^1 f(x)\psi_{1,m}(x)\omega(x)dx}{\int_0^1 \psi_{1,m}(x)\psi_{1,m}(x)\omega(x)dx}.$$

Next, we evaluate $P_{1,1}(f)(x)$, $P_{1,2}(f)(x)$, $P_{1,3}(f)(x)$, $P_{1,4}(f)(x)$, $E_{1,1}(f)(x)$, $E_{1,2}(f)(x)$, $E_{1,3}(f)(x)$, $E_{1,4}(f)(x)$ and $P_{1,M}(f)(x)$, $E_{1,M}(f)(x)$.

x	f(x)	$P_{1,1}(f)(x)$	$E_{1,1}(f)(x)$	$P_{1,2}(f)(x)$	$E_{1,2}(f)(x)$	$P_{1,3}(f)(x)$	$E_{1,3}(f)(x)$	$P_{1,4}(f)(x)$	$E_{1,4}(f)(x)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.2463	2.5446	2.2983	-1.0653	1.3116	0.3556	0.1092	0.2463	0.0000
0.2000	0.3363	3.5987	3.2624	-0.7211	1.0574	0.3217	0.0146	0.3363	0.0000
0.3000	0.5329	4.4075	3.8745	0.0787	0.4542	0.4214	0.115	0.5329	0.0000
0.4000	0.9462	5.0893	4.1431	1.2017	0.2555	0.8060	0.1402	0.9462	0.0000
0.5000	1.6794	5.6900	4.0106	2.5854	0.9060	1.5799	0.0994	1.6794	0.0000
0.6000	2.8381	6.2331	3.3950	4.1925	1.3544	2.8268	0.0113	2.8381	0.0000
0.7000	4.5322	6.7325	2.2003	5.9978	1.4656	4.6178	0.0857	4.5322	0.0000
0.8000	6.8764	7.1973	0.3210	7.9828	1.1064	7.0162	0.1399	6.8764	0.0000
0.9000	9.9906	7.6339	2.3566	10.1331	0.1425	10.0792	0.0886	9.9906	0.0000
1.0000	14.0000	8.0469	5.9531	12.4375	1.5625	13.8594	0.1406	14.0000	0.0000

Table 1. Comparison between pseudo-Chebyshev wavelet approximate functions $P_{1,M}(f)(x)$ and exact function $f(x)$ for different values of $M = 1,2,3,4$, and $k = 1$.

If

$$A_1^M = (a_{1,0}, a_{1,1}, a_{1,2}, \dots, a_{1,M-1})^\tau \qquad \text{and} \qquad \Psi_1^M = (\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, \dots, \psi_{1,M-1})^\tau,$$

then

$$\begin{aligned} P_1(f) &= \sum_{m=0}^{\infty} a_{1,m}\psi_{1,m} = \sum_{m=0}^{\infty} \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} \psi_{1,m} = \lim_{M\to\infty} \sum_{m=0}^{M-1} a_{1,m}\psi_{1,m}, \\ &= \lim_{M\to\infty} \left\langle A_1^M, \Psi_1^M \right\rangle = \lim_{M\to\infty} \left(\left(A_1^M\right)^\tau \Psi_1^M \right) = \lim_{M\to\infty} P_{1,M}(f), \end{aligned}$$

where

$$a_{1,m} = \langle f, \psi_{1,m}\rangle_{\omega_{1,1}} = \int_0^1 f(t)\psi_{1,m}(t)\omega_{1,1}(t)\,dt.$$

Next, we calculate $a_{1,m}$ for $m \geq 0$, namely

$$\begin{aligned} a_{1,0} &= \int_0^1 f(t)\psi_{1,0}(t)\omega_{1,1}(t)\,dt \approx 7.1314, & a_{1,1} &= \int_0^1 f(t)\psi_{1,1}(t)\omega_{1,1}(t)\,dt \approx 3.8911, \\ a_{1,2} &= \int_0^1 f(t)\psi_{1,2}(t)\omega_{1,1}(t)\,dt \approx 1.2601, & a_{1,3} &= \int_0^1 f(t)\psi_{1,3}(t)\omega_{1,1}(t)dt \approx 0.1246, \\ a_{1,4} &= \int_0^1 f(t)\psi_{1,4}(t)\omega_{1,1}(t)\,dt = 0, & a_{1,5} &= \cdots = a_{1,M-1} = 0 \ \text{ for } \ M \geq 5, \end{aligned}$$

i.e. $A_1^M = (7.1314, 3.8911, 1.2601, 0.1246, 0, 0, \dots, 0)^\tau$.

Since $P_{1,M}(f) = \sum\limits_{m=0}^{M-1} a_{1,m}\psi_{1,m} = \left(A_1^M\right)^\tau \Psi_1^M$ and $E_{1,M}(f) = \sum\limits_{m=M}^{\infty} a_{1,m}\psi_{1,m}$, therefore

$$\begin{aligned} P_{1,M}(f)(x) &\approx 7.1314\ \psi_{1,0}(x) + 3.8911\ \psi_{1,1}(x) + 1.2601\ \psi_{1,2}(x) + 0.1246\ \psi_{1,3}(x) + 0 + \cdots + 0, \\ &= P_{1,4}(f)(x) = P_1(f)(x) \approx f(x), \end{aligned}$$

and $E_{1,M}(f)(x) \approx 0$ for $M \geq 4$.

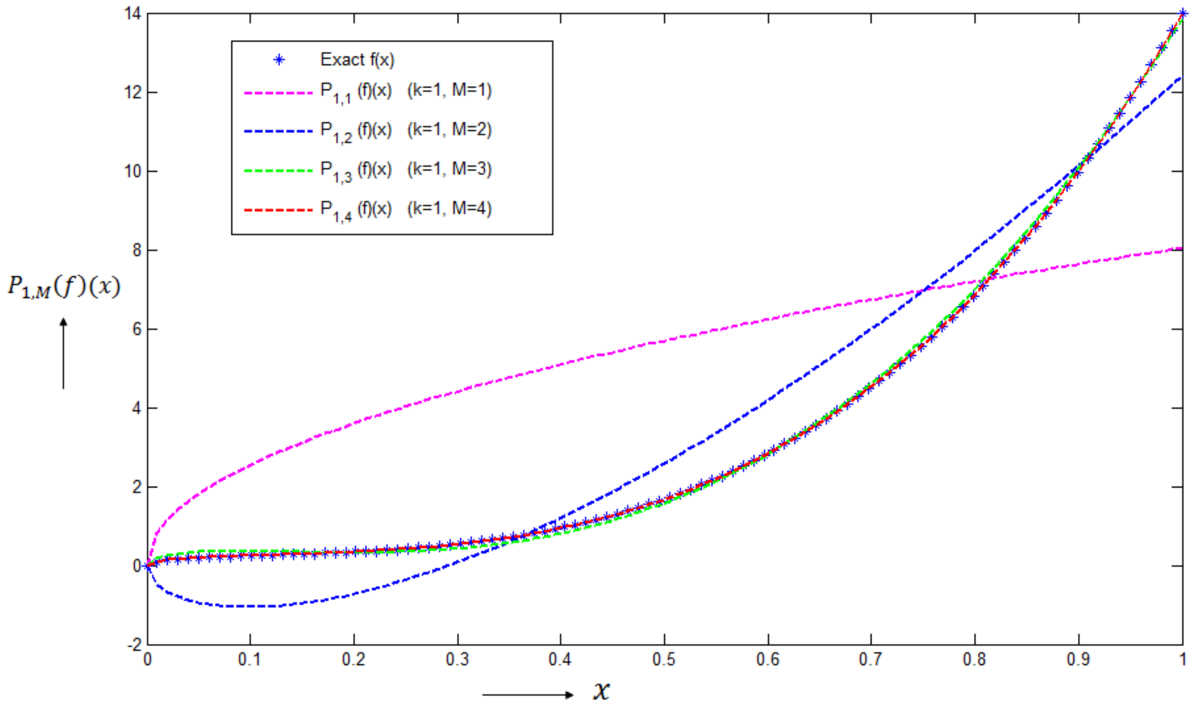

Figure 1. Graph of exact function $f(x)$ and approximate functions $P_{1,M}(f)(x)$ for different values of $M = 1, 2, 3, 4$, and $k = 1$.

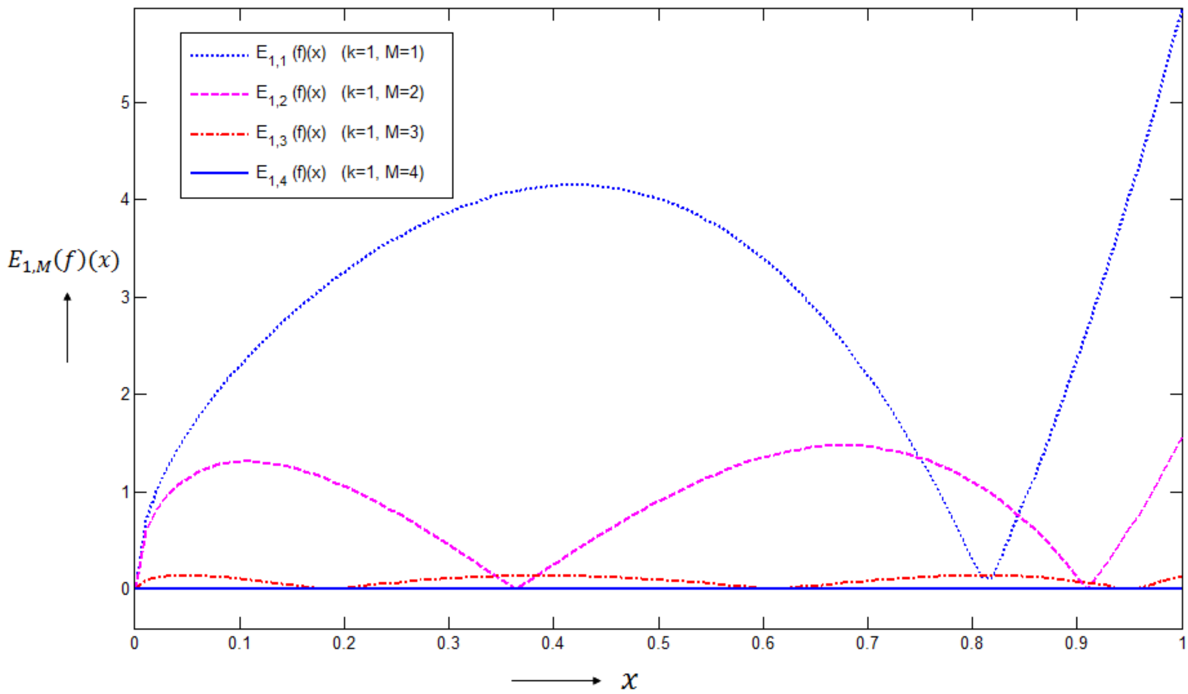

Figure 2. Graph of error bounds of function $E_{1,M}(f)(x)$ for different values of $M = 1, 2, 3, 4$, and $k = 1$.

4 Conclusions

(i) Since $E_{2^{k-1},M}f \to 0$ as $k \to \infty$ or $M \to \infty$ in above results, therefore the wavelet approximations determined in this results are best possible in the wavelet analysis (see [41]).

(ii) Four very important Corollaries 1, 2, 3 and 4 have been derived from our main Theorems 1, 2, 3 and 4 respectively.

(iii) Independent proofs of these Corollaries 1, 2, 3 and 4 can be developed for specific contributions of these estimates in wavelet analysis.

(iv) Our numerical findings are compared with exact values in the Tabel 1 and Figure 1, which shows that this approach can solve the problem effectively.

(v) The absolute error of the numerical results is shown in the Table 1 and Figure 2, which indicates that $E_{1,M} << 0$ as $M >> 1$.

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Лал С., Кумар С., Мішра С.К., Авасті А.К. *Оцінки похибок функцій з узагальненого класу Ліпшиця, отримані за допомогою псевдочебишевського вейвлету, та їх застосування у апроксимації функцій* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 29–48.

У цій статті виведено новий обчислювальний метод для розв'язання задач теорії апроксимації. Цей метод ґрунтується на псевдочебишевських вейвлет наближеннях. Псевдочебишевський вейвлет представлено вперше, він збудований за допомогою псевдочебишевських функцій. Ми описали цей метод і проаналізували оцінки похибок функцій. Ми навели приклад, щоб продемострувати точність та ефективність методу псевдочебишевського вейвлет наближення. За допомогою псевдочебишевського вейвлету отримано чотири нові оцінки похибки функції з узагальненого класу Ліпшиця. Ці оцінки є новими, найшвидшими і найкращими в теорії вейвлет аналізу.

Ключові слова і фрази: $\mathrm{Lip}_{[0,1)}\alpha$ клас функцій, $\mathrm{Lip}_{[0,1)}\xi$ клас функцій, вейвлет, мультирозділький аналіз, псевдочебишевська функція, псевдочебишевський вейвлет.