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ON THE POSITIVE DIMANT STRONGLY p-SUMMING MULTILINEAR OPERATORS

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In 2003, Dimant V. has defined and studied the interesting class of strongly p-summing multilinear operators. In this paper, we introduce and study a new class of operators between two Banach lattices, where we extend the previous notion to the positive framework, and prove, among other results, the domination, inclusion and composition theorems. As consequences, we investigate some connections between our class and other classes of operators, such as duality and linearization.

Key words and phrases: Banach lattice, Pietsch domination theorem, positive strongly p-summing multilinear operator, positive p-summing operator, tensor norm.

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INTRODUCTION AND PRELIMINARIES

The theory of summing linear operators returns to A. Grothendieck in the 1950*s* after that, it was developed by A. Pietsch in 1967. Many authors have extended differents summability concepts to multilinear operators. For example, V. Dimant in [6] defined the concept of strongly p-summing multilinear operators. Next, D. Achour and L. Mezrag in [2] introduced and studied the new notion called Cohen strongly p-summing multilinear operators, this last notion was extended by D. Achour and A. Belacel to the positive linear case (see [1]) and, by A. Bougoutaia and A. Belacel to the positive multilinear case (see [3]). In this way, our objective is to extend the concept of Dimant strongly p-summing multilinear operators to positive framework, also to study its ties with other known classes of summability. On the other hand, this work presents a continuation of the article [3].

We begin this section by recalling briefly some basic notations and terminology. Throughout this paper, n and m are positive integers, $X, X_1, \ldots, X_m, Y$ will be Banach spaces and $E, E_1, \ldots, E_m, F, G$ will be Banach lattices over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $1 \leq p \leq \infty$, and p^* is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. We write X^* to denote the topological dual of X. The closed unit ball of X is represented by B_X. For a Banach lattice E, E^+ will denote its positive cone, let $x^+ := \sup\{x, 0\}$, $x^- := \sup\{-x, 0\}$ be the positive part, the negative part of x, respectively, and we have $x = x^+ - x^-$and $|x| = x^+ + x^-$. The Banach space of all continuous m-linear operators from $X_1 \times \cdots \times X_m$ into Y endowed with the sup norm will be denoted by $\mathcal{L}(X_1, \ldots, X_m; Y)$. If $Y = \mathbb{K}$, we write $\mathcal{L}(X_1, \ldots, X_m)$. In the case $X_1 = \cdots = X_m = X$, we will simply write $\mathcal{L}(^m X; Y)$ and an m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is called positive if $T(x^1, \ldots, x^m) \in F^+$, whenever $(x^1, \ldots, x^m) \in E_1^+ \times \cdots \times E_m^+$.

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Q. Bu and G. Buskes in [4], and D.H. Fremlin in [7] introduced a lattice tensor product, called the positive projective tensor product. The projective cone on the tensor product $E_1 \otimes \cdots \otimes E_m$ is defined as

$$E_1^+ \otimes \cdots \otimes E_m^+ = \left\{ \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m : x_i^j \in E_j^+,\ j = 1, \ldots, m,\ n \in \mathbb{N} \right\}.$$

The positive projective tensor norm on $E_1 \otimes \cdots \otimes E_m$ is defined as

$$\|u\|_{|\pi|} = \inf \left\{ \sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\| : x_i^j \in E_j^+, n \in \mathbb{N}, |u| \leq \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m \right\}$$

for every $u \in E_1 \otimes \cdots \otimes E_m$. By $E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$ we will denote the complete positive m-fold projective tensor product of $E_1, \ldots, E_m$ (Fremlin projective tensor product). Then $E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$ with the above norm is a Banach lattice; we will use the next notation $\widehat{E} = E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$.

Every $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ has an associated linear operator $T_L \in \mathcal{L}(\widehat{E}; F)$ defined by $\forall x^j \in E_j^+,\ j = 1, \ldots, m,\ T_L\left(x^1 \otimes \cdots \otimes x^m\right) = T(x^1, \ldots, x^m)$ (see [4] for details). Let $\delta_m : E_1 \times \cdots \times E_m \longrightarrow \widehat{E}$ denote the canonical m-linear operator, that is $\delta_m\left(x^1, \ldots, x^m\right) = x^1 \otimes \cdots \otimes x^m$. Then $T = T_L \circ \delta_m$.

The adjoint of an m-linear operator $T \in \mathcal{L}\left(X_1, \ldots, X_m; Y\right)$ is defined as follows

$$T^* : Y^* \longrightarrow \mathcal{L}\left(X_1, \ldots, X_m\right), y^* \longrightarrow T^*(y^*) : X_1 \times \cdots \times X_m \longrightarrow \mathbb{K}$$

with $T^*(y^*)\left(x^1, \ldots, x^m\right) = y^*\left(T\left(x^1, \ldots, x^m\right)\right).$

We denote by $\ell_p^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X equipped with the norm

$$\|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}},$$

and by $\ell_{p,weak}^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X equipped with the norm

$$w_p\left((x_i)_{i=1}^n\right) = \|(x_i)_{i=1}^n\|_{\ell_{p,weak}^n(X)} = \sup_{\varphi \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, \varphi \rangle|^p \right)^{\frac{1}{p}}.$$

It is well known that for $1 \leq p \leq \infty$ and $(\varphi_i)_{i=1}^n \in \ell_{p^*,weak}^n(Y^*)$ we have

$$\left\|(\varphi_i)_{i=1}^n\right\|_{\ell_{p^*,weak}^n(Y^*)} = \sup_{\varphi \in B_{Y^{**}}} \left(\sum_{i=1}^{n} |\varphi(\varphi_i)|^{p^*} \right)^{\frac{1}{p^*}} = \sup_{y \in B_Y} \|\varphi_i(y)\|_{p^*}.$$

Consider the case where X is replaced by a Banach lattice E, and define

$$\ell_{p,|weak|}^n(E) := \left\{ (x_i)_{i=1}^n : (|x_i|)_{i=1}^n \in \ell_{p,weak}^n(E) \right\}.$$

Let $B_{E^*}^{+} = B_{E^*} \cap E^{*+}$. If $x_1, \ldots, x_n \geq 0$, we have that

$$\left\|(x_i)_{i=1}^{n}\right\|_{\ell_{p,|weak|}^{n}(E)} = \sup_{\xi \in B_{E^*}^{+}} \left(\sum_{i=1}^{n} \langle x_i, \xi \rangle^{p}\right)^{\frac{1}{p}}.$$

The notion of strongly p-summing multilinear operators was initiated by V. Dimant in [6], such that an m-linear operator $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$ is strongly p-summing, $1 \leq p \leq \infty$, if there exists a positive constant C such that for every $x_1^j, \ldots, x_n^j \in X_j$, $1 \leq j \leq m$, we have

$$\left(\sum_{i=1}^{n} \left\|T\left(x_i^1, \ldots, x_i^m\right)\right\|^{p}\right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{\mathcal{L}(X_1,\ldots,X_m)}} \left(\sum_{i=1}^{n} \left|\varphi\left(x_i^1, \ldots, x_i^m\right)\right|^{p}\right)^{\frac{1}{p}}. \tag{1}$$

The class of all strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into Y, which is denoted by $\mathcal{L}_{ss,p}^{m}(X_1, \ldots, X_m; Y)$, is a Banach space with the norm $d_{s,p}^{m}(.)$, where $d_{s,p}^{m}(T)$ is the smallest constant C such that the inequality (1) holds. Furthermore, L. Mezrag and K. Saadi in [8] showed that, T^* belongs to the class of Cohen strongly p^*-summing operators, then T belongs to the class of strongly p-summing multilinear operators.

We recall some definitions that we need in the sequel.

Definition 1 ([1]). *Let $1 \leq p < \infty$. An operator $u : E \longrightarrow X$ is said to be positive p-summing if there exists a constant $C > 0$ such that the inequality*

$$\left\|u\,(x_i)_{i=1}^{n}\right\|_{p} \leq C \left\|(x_i)_{i=1}^{n}\right\|_{\ell_{p,|weak|}^{n}(E)} = C \sup_{x^* \in B_{E^*}^{+}} \left(\sum_{i=1}^{n} \langle x_i, x^* \rangle^{p}\right)^{1/p} \tag{2}$$

holds for every positive integer n and for all $x_1, \ldots, x_n \in E$.

For $p = \infty$

$$\sup_{1 \leq i \leq n} \left\|u\,(x_i)\right\| \leq C \left\|(x_i)_{i=1}^{n}\right\|_{\ell_{\infty,|weak|}^{n}(E)}.$$

We denote by $\Pi_p^{+}(E, X)$ the space of positive p-summing operators from E into X. $\Pi_p^{+}(E, X)$ becomes a Banach space with norm $\pi_p^{+}(.)$, $\pi_p^{+}(u)$ is giving by the infimum of the constant $C > 0$ that verify the inequality (2). We have $\Pi_\infty^{+}(E, X) = \mathcal{L}(E, X)$.

Definition 2 ([5]). *Let $1 \leq p \leq \infty$. A continuous linear operator $u : X \longrightarrow Y$ is strongly p-summing if there is a positive constant C such that for all $(x_i)_{i=1}^{n} \subset X$ and $\left(y_i^*\right)_{i=1}^{n} \subset Y^*$ we have*

$$\sum_{i=1}^{n} \left|\langle u(x_i), y_i^* \rangle\right| \leq C \left\|(x_i)_{i=1}^{n}\right\|_{p} \left\|(y_i^*)_{i=1}^{n}\right\|_{\ell_{p^*,weak}^{n}(Y^*)}.$$

The class of all strongly p-summing operators between X and Y is denoted by $\mathcal{D}_p(X, Y)$. The infimum of all the constants C in the inequality defines the norm $d_p(.)$ on $\mathcal{D}_p(X, Y)$. We have $\mathcal{D}_1(X, Y) = \mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X into Y.

Definition 3 ([1]). *Let $1 \leq p \leq \infty$. An operator $u : X \longrightarrow F$ is positive strongly p-summing if there exists a constant $C > 0$ such that for all finite sets $n \in \mathbb{N}$, $(x_i)_{i=1}^{n} \subset X$ and $\left(y_i^*\right)_{i=1}^{n} \subset F^*$ we have*

$$\sum_{i=1}^{n} \left|\langle u(x_i), y_i^* \rangle\right| \leq C \left\|(x_i)_{i=1}^{n}\right\|_{p} \left\|(y_i^*)_{i=1}^{n}\right\|_{\ell_{p^*,|weak|}^{n}(F^*)}. \tag{3}$$

The class of all positive strongly p-summing operators between X and F is denoted by $\mathcal{D}_p^+(X,F)$. The infimum of all the constants C in the inequality (3) defines the norm $d_p^+(.)$ on $\mathcal{D}_p^+(X,F)$.

Definition 4 ([2])**.** *An m-linear operator $T: X_1 \times \cdots \times X_m \longrightarrow Y$ is Cohen strongly p-summing if and only if there is a constant $C > 0$ such that for any $x_1^j, \ldots, x_n^j \in X_j$, $1 \leq j \leq m$, and any $y_1^*, y_2^*, \ldots, y_n^* \in Y^*$*

$$\left\| \left\langle T(x_i^1, \ldots, x_i^m), y_i^* \right\rangle \right\|_{\ell_1^n} \leq C \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \left\| x_i^j \right\|_{X_j}^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \| y_i^*(y) \|_{\ell_{p^*}^n}. \tag{4}$$

The smallest constant C, which is noted by $d_p^m(T)$, such that the inequality (4) holds, defines a norm on the space $\mathcal{D}_p^m(X_1, \ldots, X_m; Y)$ of all Cohen strongly p-summing operators from $X_1 \times \cdots \times X_m$ into Y which is a Banach space.

Definition 5 ([3])**.** *Let $1 \leq p \leq +\infty$. An m-linear operator $T: X_1 \times \cdots \times X_m \longrightarrow F$, $m \in \mathbb{N}^*$, is Cohen positive strongly p-summing multilinear operator if there is a constant $C > 0$ such that for any $x_1^j, \ldots, x_n^j \in X_j$, $1 \leq j \leq m$, and any $y_1^*, \ldots, y_n^* \in F^*$*

$$\left\| \left\langle T(x_i^1, \ldots, x_i^m), y_i^* \right\rangle \right\|_{\ell_1^n} \leq C \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \left\| x_i^j \right\|_{X_j}^p \right)^{\frac{1}{p}} \| (y_i^*)_{i=1}^n \|_{\ell_{p^*,|weak|}^n(F^*)}. \tag{5}$$

Moreover, the class of all Cohen positive strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into F is denoted by $\mathcal{D}_p^{m+}(X_1, \ldots, X_m; F)$. Our space is a Banach space with the norm $d_p^{m+}(.)$, which defined by the smallest constant C such that the inequality (5) holds.

1 POSITIVE STRONGLY p-SUMMING MULTILINEAR OPERATORS

In this section we introduce and study a new class of operators between two Banach lattices, where we extend a notion defined by V. Dimant. We prove a natural analog to the Pietsch domination theorem for this class.

Definition 6. *Let $1 \leq p \leq \infty$. An m-linear operator $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ is said to be positive strongly p-summing if there exists a constant $C > 0$ such that for every $x_1^j, \ldots, x_n^j \in E_j^+$, $j = 1, \ldots, m$, we have*

$$\left(\sum_{i=1}^{n} \left\| T\left(x_i^1, \ldots, x_i^m\right) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B^+_{\mathcal{L}(E_1, \ldots, E_m)}} \left(\sum_{i=1}^{n} \left(\varphi\left(x_i^1, \ldots, x_i^m\right) \right)^p \right)^{\frac{1}{p}}. \tag{6}$$

Moreover, the class of all positive strongly p-summing m-linear operators from $E_1 \times \cdots \times E_m$ into F is denoted by $\mathcal{L}_{ss,p}^{m+}(E_1, \ldots, E_m; F)$. Our space is a Banach space with the norm $d_{s,p}^{m+}(.)$, which defined by the smallest constant C such that the inequality (6) holds.

Proposition 1 (Ideal property)**.** *Let $T \in \mathcal{L}(E_1, \ldots, E_m; F)$. Let R be a positive operator in $\mathcal{L}(F, G)$ and S_j be a positive operator in $\mathcal{L}(F_j, E_j)$, $1 \leq j \leq m$.*

1) If T is positive strongly p-summing, $R \circ T$ is positive strongly p-summing and

$$d_{s,p}^{m+}(R \circ T) \leq \|R\| \, d_{s,p}^{m+}(T).$$

2) If T is positive strongly p-summing, $T \circ (S_1, \ldots, S_m)$ is positive strongly p-summing and

$$d_{s,p}^{m+}(T \circ (S_1, \ldots, S_m)) \leq d_{s,p}^{m+}(T) \prod_{j=1}^{m} \|S_j\|.$$

Proof. 1) Let $T \in \mathcal{L}_{ss,p}^{m+}(E_1, \ldots, E_m; F)$. Then for every $x_1^j, \ldots, x_n^j \in E_j^+$, $j = 1, \ldots, m$, we have

$$\left(\sum_{i=1}^{n} \left\| R \circ T\left(x_i^1, \ldots, x_i^m\right) \right\|^p\right)^{\frac{1}{p}} \leq \|R\| \left(\sum_{i=1}^{n} \left\| T\left(x_i^1, \ldots, x_i^m\right) \right\|^p\right)^{\frac{1}{p}}$$

$$\leq \|R\| \, d_{s,p}^{m+}(T) \sup_{\varphi \in B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\sum_{i=1}^{n} \left(\varphi\left(x_i^1, \ldots, x_i^m\right)\right)^p\right)^{\frac{1}{p}}.$$

So $R \circ T \in \mathcal{L}_{ss,p}^{m+}(E_1, \ldots, E_m; G)$ and $d_{s,p}^{m+}(R \circ T) \leq \|R\| \, d_{s,p}^{m+}(T)$.

2) Let $T \in \mathcal{L}_{ss,p}^{m+}(E_1, \ldots, E_m; F)$. Then

$$\left(\sum_{i=1}^{n} \| T \circ (S_1, \ldots, S_m)(x_i^1, \ldots, x_i^m) \|^p\right)^{\frac{1}{p}}$$

$$\leq d_{s,p}^{m+}(T) \sup_{\varphi \in B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\sum_{i=1}^{n} \left(\varphi(S_1(x_i^1), \ldots, S_m(x_i^m))\right)^p\right)^{\frac{1}{p}}$$

$$\leq d_{s,p}^{m+}(T) \prod_{j=1}^{m} \|S_j\| \sup_{\varphi \in B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\sum_{i=1}^{n} \left(\frac{1}{\prod_{j=1}^{m} \|S_j\|} \varphi(S_1, \ldots, S_m)(x_i^1, \ldots, x_i^m)\right)^p\right)^{\frac{1}{p}}$$

$$\leq d_{s,p}^{m+}(T) \prod_{j=1}^{m} \|S_j\| \sup_{\psi \in B^+_{\mathcal{L}(F_1,\ldots,F_m)}} \left(\sum_{i=1}^{n} \left(\psi(x_i^1, \ldots, x_i^m)\right)^p\right)^{\frac{1}{p}}.$$

We have $T \circ (S_1, \ldots, S_m)$ is positive strongly p-summing multilinear operator and

$$d_{s,p}^{m+}(T \circ (S_1, \ldots, S_m)) \leq d_{s,p}^{m+}(T) \prod_{j=1}^{m} \|S_j\|.$$

□

For the proof of the next theorem, we will use the full general Pietsch domination theorem recently presented by D. Pellegrino et al. in [9], where, in [6], V. Dimant used Hahn-Banach theorem.

Theorem 1. *An m-linear operator $T \in \mathcal{L}(E_1, \ldots, E_m; F)$ is positive strongly p-summing if and only if there exist a regular probability measure μ on $B^+_{\mathcal{L}(E_1,\ldots,E_m)}$ with the weak star topology, and a positive constant C such that*

$$\left\| T(x^1, \ldots, x^m) \right\| \leq C \left(\int_{B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\varphi(x^1, \ldots, x^m)\right)^p d\mu(\varphi) \right)^{\frac{1}{p}} \tag{7}$$

for every $(x^1, \ldots, x^m) \in E_1^+ \times \cdots \times E_m^+$. Moreover, in this case

$$d_{s,p}^{m+}(T) = \inf\{C \text{ which satisfies the inequality } (7)\}.$$

Proof. Suppose that T is positive strongly p-summing. We will verify the hypotheses of the general Pietsch domination theorem. We put

$$\begin{cases} G = \mathbb{K}, \\ \mathcal{H} = \mathcal{L}\left(E_1, \ldots, E_m; F\right), \\ Z = E_1^+ \times \cdots \times E_m^+, \\ K = B^+_{\left(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_m\right)^*}, \\ x_0 = (0, \ldots, 0). \end{cases}$$

It is not difficult to see that the notion of positive strongly p-summing multilinear operator is precisely the concept of $R-S$-abstract p-summing, where

$$\begin{aligned} R : B^+_{(E_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_m)^*} \times E_1^+ \times \cdots \times E_m^+ \times \mathbb{K} \longrightarrow [0, \infty), \\ R\left(\varphi, \left(x^1, \ldots, x^m\right), \lambda\right) = |\lambda| \left(\varphi\left(x^1, \ldots, x^m\right)\right), \\ S : \mathcal{L}(E_1, \ldots, E_m; F) \times E_1^+ \times \cdots \times E_m^+ \times \mathbb{K} \longrightarrow [0, \infty), \\ S\left(T, \left(x^1, \ldots, x^m\right), \lambda\right) = |\lambda| \left\| T\left(x^1, \ldots, x^m\right)\right\|, \end{aligned}$$

here R and S satisfy the conditions of full general Pietsch domination theorem. Then the result follows immediately. □

We need a small review about $R-S$-abstract p-summing.

Let X, Y and Z be (arbitrary) sets, $\mathcal{H}$ be a family of mappings from X to Y, G be a Banach space and K be a compact Hausdorff topological space. Suppose that the maps

$$\begin{aligned} R : K \times Z \times G \longrightarrow [0, \infty), \\ S : \mathcal{H} \times Z \times G \longrightarrow [0, \infty) \end{aligned}$$

be such that:

a) for each $f \in \mathcal{H}$, there is $x_0 \in Z$ such that $R\left(\varphi, x_0, b\right) = S\left(f, x_0, b\right)$ for every $\varphi \in K$ and $b \in G$;

b) the mapping $R_{x,b} : K \longrightarrow [0, \infty)$ defined by $R_{x,b}\left(\varphi\right) = R\left(\varphi, x, b\right)$ is continuous for every $x \in Z$ and $b \in G$;

c) it holds that $R\left(\varphi, x, \eta b\right) \leq \eta R\left(\varphi, x, b\right)$ and $\eta S\left(f, x, b\right) \leq S\left(f, x, \eta b\right)$ for every $\varphi \in K$, $x \in Z$, $0 \leq \eta \leq 1$, $b \in G$ and $f \in \mathcal{H}$.

Definition 7 ([9]). *Let R and S be as above and $0 < p < \infty$. A mapping $f \in \mathcal{H}$ is said to be $R-S$-abstract p-summing if there is a constant $C_1 > 0$ so that*

$$\left(\sum_{j=1}^{m} S\left(f, x^j, b^j\right)^p\right)^{\frac{1}{p}} \leq C_1 \sup_{\varphi \in K} \left(\sum_{j=1}^{m} R\left(\varphi, x^j, b^j\right)^p\right)^{\frac{1}{p}}$$

for all $x^1, \ldots, x^m \in Z$ and $b^1, \ldots, b^m \in G$. The infimum of such constants C_1 is denoted by $\pi_{RS,p}\left(f\right)$.

Theorem 2 ([9]). *Let R and S be as above, $0 < p < \infty$ and $f \in \mathcal{H}$. Then f is $R-S$-abstract p-summing if and only if there exist a constant $C > 0$ and a regular Borel probability measure μ on K such that for all $x \in Z$ and $b \in G$,*

$$S(f,x,b) \leq C\left(\int_K (R(\varphi,x,b))^p d\mu(\varphi)\right)^{\frac{1}{p}}.$$

Moreover, the infimum of such constants C equals to $\pi_{RS,p}(f)$.

Proposition 2. *$T \in \mathcal{L}(E_1,\ldots,E_m;F)$ is positive strongly p-summing if and only if there exist a regular probability measure μ on $B^+_{\mathcal{L}(E_1,\ldots,E_m)}$ with the weak star topology, and a constant $K > 0$, such that the inequality*

$$\left\|T\left(x^1,\ldots,x^m\right)\right\| \leq K\left(\int_{B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\varphi(|x^1|,\ldots,|x^m|)\right)^p d\mu(\varphi)\right)^{\frac{1}{p}},$$

holds for every $\left(x^1,\ldots,x^m\right) \in E_1 \times \cdots \times E_m$.

Proof. For convenience, we prove the inequality for $m = 2$ only. Let $(x,y) \in E_1 \times E_2$. By Theorem 1, we have

$$\begin{aligned}
\|T(x,y)\| &= \|T(x^+ - x^-, y^+ - y^-)\| = \left\|T\left(x^+,y^+\right) - T\left(x^+,y^-\right) - T\left(x^-,y^+\right) + T\left(x^-,y^-\right)\right\| \\
&\leq \left\|T\left(x^+,y^+\right)\right\| + \left\|T\left(x^+,y^-\right)\right\| + \left\|T\left(x^-,y^+\right)\right\| + \left\|T\left(x^-,y^-\right)\right\| \\
&\leq C_1\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\varphi(x^+,y^+))^p d\mu(\varphi)\right)^{\frac{1}{p}} + C_2\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\varphi(x^+,y^-))^p d\mu(\varphi)\right)^{\frac{1}{p}} \\
&\leq C_3\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\varphi(x^-,y^+))^p d\mu(\varphi)\right)^{\frac{1}{p}} + C_4\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\varphi(x^-,y^-))^p d\mu(\varphi)\right)^{\frac{1}{p}} \\
&\leq K\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\varphi(|x|,|y|))^p d\mu(\varphi)\right)^{\frac{1}{p}}.
\end{aligned}$$

To prove the converse, just apply Theorem 1 for $\left(x^1,\ldots,x^m\right) \in E_1^+ \times \cdots \times E_m^+$. □

Proposition 3. *If $1 \leq p < q < \infty$, then*

$$\mathcal{L}^{m+}_{ss,p}(E_1,\ldots,E_m;F) \subset \mathcal{L}^{m+}_{ss,q}(E_1,\ldots,E_m;F) \quad \text{and} \quad d^{m+}_{s,q}(T) \leq d^{m+}_{s,p}(T).$$

Proof. Let $T \in \mathcal{L}^{m+}_{ss,p}(E_1 \times \cdots \times E_m;F)$. Then by (7)

$$\begin{aligned}
\left\|T(x^1,\ldots,x^m)\right\| &\leq d^{m+}_{s,p}(T)\left(\int_{B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\varphi(x^1,\ldots,x^m)\right)^p d\mu(\varphi)\right)^{\frac{1}{p}} \\
&\leq d^{m+}_{s,p}(T)\left(\int_{B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\varphi(x^1,\ldots,x^m)\right)^q d\mu(\varphi)\right)^{\frac{1}{q}},
\end{aligned}$$

for every $\left(x^1,\ldots,x^m\right) \in E_1^+ \times \cdots \times E_m^+$. This implies that $T \in \mathcal{L}^{m+}_{ss,q}(E_1 \times \cdots \times E_m;F)$ and $d^{m+}_{s,q}(T) \leq d^{m+}_{s,p}(T)$. □

Here is another important result.

Proposition 4. *Let $1 \leq p < \infty$. Let $A_1 : E_1 \times \cdots \times E_{m_1} \to F_1, \ldots, A_k : E_{1+m_{k-1}} \times \cdots \times E_{m_k} \to F_k$ and $T : F_1 \times \cdots \times F_k \to F$ be non null continuous multilinear operators. If all $A_1, \ldots, A_k$ are positive strongly p-summing, then $T \circ (A_1, \ldots, A_k)$ is positive strongly p-summing and*

$$d_{s,p}^{m+}\left(T \circ (A_1, \ldots, A_k)\right) \leq \|T\| \prod_{j=1}^{k} d_{s,p}^{m+}\left(A_j\right) \quad \text{for} \quad 1 \leq j \leq k.$$

Proof. From the domination theorem, there exist regular Borel probabilities μ_1 on $B_1 = B^+_{\mathcal{L}\left(E_1,\ldots,E_{m_1}\right)}, \ldots, \mu_k$ on $B_k = B^+_{\mathcal{L}\left(E_{1+m_{k-1}},\ldots,E_{m_k}\right)}$ such that for each $\left(x^1, \ldots, x^{m_1}\right) \in E_1^+ \times \cdots \times E_{m_1}^+, \ldots, (x^{1+m_{k-1}}, \ldots, x^{m_k}) \in E_{1+m_{k-1}}^+ \times \cdots \times E_{m_k}^+$ we have

$$\|A_1(x^1, \ldots, x^{m_1})\| \leq d_{s,p}^{m+}(A_1) \left(\int_{B_1} \left(\varphi_1\left(x^1, \ldots, x^{m_1}\right)\right)^p d\mu_1\left(\varphi_1\right) \right)^{\frac{1}{p}},$$

$$\cdots$$

$$\|A_k(x^{1+m_{k-1}}, \ldots, x^{m_k})\| \leq d_{s,p}^{m+}(A_k) \left(\int_{B_k} \left(\varphi_k\left(x^{1+m_{k-1}}, \ldots, x^{m_k}\right)\right)^p d\mu_k\left(\varphi_k\right) \right)^{\frac{1}{p}}.$$

Fubini's theorem gives us that for each $\left(x^1, \ldots, x^{m_k}\right) \in E_1^+ \times \cdots \times E_{m_k}^+$ we have

$$\begin{aligned} &\|A_1(x^1, \ldots, x^{m_1})\|^p \cdots \|A_k(x^{1+m_{k-1}}, \ldots, x^{m_k})\|^p \leq \left[\prod_{j=1}^{k} d_{s,p}^{m+}\left(A_j\right)\right]^p \\ &\times \int \cdots \int_{B_1 \times \cdots \times B_k} \left(\varphi_1\left(x^1, \ldots, x^{m_1}\right)\right)^p \cdots \left(\varphi_k\left(x^{1+m_{k-1}}, \ldots, x^{m_k}\right)\right)^p d\left(\mu_1 \times \cdots \times \mu_k\right)\left(\varphi_1, \ldots, \varphi_k\right). \end{aligned} \tag{8}$$

Let $1 \leq i \leq n$ and $\left(x_i^1, \ldots, x_i^{m_k}\right) \in E_1^+ \times \cdots \times E_{m_k}^+$. Since

$$\|T \circ (A_1, \ldots, A_k)\left(x_i^1, \ldots, x_i^{m_k}\right)\| \leq \|T\| \, \|A_1\left(x_i^1, \ldots, x_i^{m_1}\right)\| \cdots \|A_k\left(x_i^{1+m_{k-1}}, \ldots, x_i^{m_k}\right)\|,$$

by (8), we find

$$\begin{aligned} &\left(\sum_{i=1}^{n} \left\|T \circ (A_1, \ldots, A_k)\left(x_i^1, \ldots, x_i^{m_k}\right)\right\|^p\right)^{1/p} \leq \|T\| \prod_{j=1}^{k} d_{s,p}^{m+}(A_j) \\ &\times \left(\sum_{i=1}^{n} \int \cdots \int_{B_1 \times \cdots \times B_k} \varphi_1\left(x_i^1, \ldots, x_i^m\right)^p \cdots \varphi_k\left(x_i^{1+m_{k-1}}, \ldots, x_i^{m_k}\right)^p d\left(\mu_1 \times \cdots \times \mu_k\right)\left(\varphi_1, \ldots, \varphi_k\right)\right)^{1/p}. \end{aligned} \tag{9}$$

For $(\varphi_1, \ldots, \varphi_k) \in B_1 \times \cdots \times B_k$ we define $\varphi_1 \otimes \cdots \otimes \varphi_k : E_1 \times \cdots \times E_{m_k} \longrightarrow \mathbb{K}$ by

$$\left(\varphi_1 \otimes \cdots \otimes \varphi_k\right)\left(x^1, \ldots, x^{m_1}, \ldots, x^{1+m_{k-1}}, \ldots, x^{m_k}\right) = \varphi_1\left(x^1, \ldots, x^{m_1}\right) \cdots \varphi_k\left(x^{1+m_{k-1}}, \ldots, x^{m_k}\right).$$

Then

$$\varphi_1 \otimes \cdots \otimes \varphi_k \in \mathcal{L}\left(E_1 \times \cdots \times E_{m_k}\right), \quad \|\varphi_1 \otimes \cdots \otimes \varphi_k\| \leq \|\varphi_1\| \cdots \|\varphi_k\|$$

and

$$\begin{aligned}\left(\sum_{i=1}^{n}\varphi_1\left(x_i^1,\ldots,x_i^{m_1}\right)^p\cdots\varphi_k\left(x_i^{1+m_{k-1}},\ldots,x_i^{m_k}\right)^p\right)^{1/p}\\ &= \left(\sum_{i=1}^{n}\left(\varphi_1\otimes\cdots\otimes\varphi_k\right)\left(x_i^1,\ldots,x_i^{m_k}\right)^p\right)^{1/p} \\ &\leq \sup_{\varphi_1\otimes\cdots\otimes\varphi_k\in B^+_{\mathcal{L}\left(E_1,\ldots,E_{m_k}\right)}}\left(\sum_{i=1}^{n}\Big(\left(\varphi_1\otimes\cdots\otimes\varphi_k\right)\left(x_i^1,\ldots,x_i^{m_k}\right)\Big)^p\right)^{\frac{1}{p}}.\end{aligned} \tag{10}$$

Using (10) and the fact that $\mu_1,\ldots,\mu_k$ are probability measures, by (9), we get

$$\begin{aligned}&\left(\sum_{i=1}^{n}\left\|T\circ\left(A_1,\ldots,A_k\right)\left(x_i^1,\ldots,x_i^{m_k}\right)\right\|^p\right)^{1/p}\\ &\leq \|T\|\prod_{j=1}^{k}d_{s,p}^{m+}\left(A_j\right)\sup_{\varphi_1\otimes\cdots\otimes\varphi_k\in B^+_{\mathcal{L}\left(E_1,\ldots,E_{m_k}\right)}}\left(\sum_{i=1}^{n}\Big(\left(\varphi_1\otimes\cdots\otimes\varphi_k\right)\left(x_i^1,\ldots,x_i^{m_k}\right)\Big)^p\right)^{\frac{1}{p}}.\end{aligned}$$

Then from Definition 6, $T\circ\left(A_1,\ldots,A_k\right)$ is a positive strongly p-summing operator. □

2 TIES WITH KNOWN OTHER CLASSES OF SUMMABILITY

Our main results of this section is to analyse some connections between the different classes investigated in this paper.

Theorem 3. *Let $1<p\leq\infty$. If $T\in\mathcal{L}\left(E_1,\ldots,E_m;F\right)$ is such that T^* is a Cohen positive strongly p^*-summing linear operator, then T is positive strongly p-summing multilinear operator.*

Proof. Suppose that $T^*\in\mathcal{L}\left(F^*;\mathcal{L}\left(E_1,\ldots,E_m\right)\right)$ is Cohen positive strongly p^*-summing linear operator. By (3), we have

$$\sum_{i=1}^{n}\left|\langle T^*(y_i^*),z_i^*\rangle\right|\leq d_{p^*}^+\left(T^*\right)\left(\sum_{i=1}^{n}\|y_i^*\|^{p^*}\right)^{\frac{1}{p^*}}\sup_{\varphi\in B^+_{\mathcal{L}(E_1,\ldots,E_m)}}\left(\sum_{i=1}^{n}\left(z_i^*\left(\varphi\right)\right)^p\right)^{\frac{1}{p}}.$$

Let $\left(x_i^j\right)_{i=1}^n\subset E_j^+$, $1\leq j\leq m$. We consider the linear form $T_{\left(x_i^1,\ldots,x_i^m\right)}:\mathcal{L}\left(E_1,\ldots,E_m\right)\longrightarrow\mathbb{K}$ defined by $T_{\left(x_i^1,\ldots,x_i^m\right)}\left(\varphi\right)=\varphi\left(x_i^1,\ldots,x_i^m\right)$. We have

$$\begin{aligned}\sum_{i=1}^{n}\left|\left\langle T\left(x_i^1,\ldots,x_i^m\right),y_i^*\right\rangle\right| &= \sum_{i=1}^{n}\left|\left\langle T^*(y_i^*),T_{\left(x_i^1,\ldots,x_i^m\right)}\right\rangle\right|\\ &\leq d_{p^*}^+\left(T^*\right)\left(\sum_{i=1}^{n}\|y_i^*\|^{p^*}\right)^{\frac{1}{p^*}}\sup_{\varphi\in B^+_{\mathcal{L}(E_1,\ldots,E_m)}}\left(\sum_{i=1}^{n}\Big(T_{\left(x_i^1,\ldots,x_i^m\right)}(\varphi)\Big)^p\right)^{\frac{1}{p}}\\ &\leq d_{p^*}^+\left(T^*\right)\left(\sum_{i=1}^{n}\|y_i^*\|^{p^*}\right)^{\frac{1}{p^*}}\sup_{\varphi\in B^+_{\mathcal{L}(E_1,\ldots,E_m)}}\left(\sum_{i=1}^{n}\varphi\Big(x_i^1,\ldots,x_i^m\Big)^p\right)^{\frac{1}{p}}.\end{aligned}$$

Taking the supremum over all sequences $(y_i^*)_{i=1}^n$ with $\left(\sum\limits_{i=1}^n \|y_i^*\|^{p^*}\right)^{\frac{1}{p^*}} \leq 1$, we obtain

$$\left(\sum_{i=1}^n \|T(x_i^1,\ldots,x_i^m)\|^p\right)^{\frac{1}{p}} = \sup_{\left(\sum\limits_{i=1}^n \|y_i^*\|^{p^*}\right)^{\frac{1}{p^*}} \leq 1} \left(\sum_{i=1}^n \left|\left\langle T\left(x_i^1,\ldots,x_i^m\right), y_i^*\right\rangle\right|\right)$$
$$\leq d_{p^*}^+(T^*) \sup_{\varphi \in B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\sum_{i=1}^n \varphi(x_i^1,\ldots,x_i^m)^p\right)^{\frac{1}{p}}.$$

Then, T is positive strongly p-summing and $d_{s,p}^{m+}(T) \leq d_{p^*}^+(T^*)$. □

Open problem. *Is the inverse true?*

Proposition 5. *Let $T \in \mathcal{L}(E_1,\ldots,E_m;F)$ and R be a positive operator in $\mathcal{L}(F,G)$. If R is positive p-summing, then $R \circ T$ is positive strongly p-summing and*

$$d_{s,p}^{m+}(R \circ T) \leq \pi_p^+(R)\|T\|.$$

Proof. Let R be an operator in $\Pi_p^+(F,G)$. From Definition 1, we have

$$\left\|R \circ T(x_i^1,\ldots,x_i^m)\right\|_p \leq \pi_p^+(R) \sup_{y^* \in B^+_{F^*}} \left(\sum_{i=1}^n \left\langle T(x_i^1,\ldots,x_i^m), y^*\right\rangle^p\right)^{1/p}$$
$$\leq \pi_p^+(R) \sup_{y^* \in B^+_{F^*}} \left(\sum_{i=1}^n \left\langle T_{(x_i^1,\ldots,x_i^m)}, T^*(y^*)\right\rangle^p\right)^{1/p}$$
$$\leq \pi_p^+(R)\|T\| \sup_{y^* \in B^+_{F^*}} \left(\sum_{i=1}^n \left\langle T_{(x_i^1,\ldots,x_i^m)}, \frac{T^*(y^*)}{\|T\|}\right\rangle^p\right)^{1/p}$$
$$\leq \pi_p^+(R)\|T\| \sup_{\varphi \in B^+_{\mathcal{L}(E_1,\ldots,E_m)}} \left(\sum_{i=1}^n \left(\varphi(x_i^1,\ldots,x_i^m)\right)^p\right)^{\frac{1}{p}}.$$

So $R \circ T \in \mathcal{L}_{ss,p}^{m+}(E_1,\ldots,E_m;G)$ and $d_{s,p}^{m+}(R \circ T) \leq \pi_p^+(R)\|T\|$. □

Corollary 1. *Let $1 \leq p \leq \infty$. If $T_L \in \Pi_p^+(\widehat{E};F)$, then $T \in \mathcal{L}_{ss,p}^{m+}(E_1,\ldots,E_m;F)$.*

Proof. Let $T_L \in \Pi_p^+(\widehat{E};F)$. Since $T = T_L \circ \delta_m$, by the above Proposition we have

$$T \in \mathcal{L}_{ss,p}^{m+}(E_1,\ldots,E_m;F) \quad \text{and} \quad d_{s,p}^{m+}(T) \leq \pi_p^+(T_L).$$

□

Open Problem. *Is the opposite implication in the last corollary true?*

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REFERENCES

[1] Achour D., Belacel A. *Domination and factorization theorems for positive strongly p-summing operators*. Positivity 2014, **18**, 785–804. doi:10.1007/s11117-014-0276-6

[2] Achour D., Mezrag L. *On the Cohen strongly p-summing multilinear operators*. J. Math. Anal. Appl. 2007, **327**, 550–563. doi:10.1016/j.jmaa.2006.04.065

[3] Bougoutaia A., Belacel A. *Cohen positive strongly p-summing and p-convex multilinear operators*. Positivity 2019, **23**, 379–395. doi:10.1007/s11117-018-0612-3

[4] Bu Q., Buskes G. *Polynomials on Banach lattices and positive tensor products*. J. Math. Anal. Appl. 2012, **388**, 845–862. doi:10.1016/j.jmaa.2011.10.001

[5] Cohen J.S. *Absolutely p-summing, p-nuclear operators and thier conjugates*. Math. Ann. 1973, **201**, 177–200. doi:10.1007/BF01427941

[6] Dimant V. *Strongly p-summing multilinear operators*. J. Math. Anal. Appl. 2003, **278**, 182–193. doi:10.1016/S0022-247X(02)00710-2

[7] Fremlin D.H. *Tensor products of Banach lattices*. Math. Ann. 1974, **211**, 87–106. doi:10.1007/BF01344164

[8] Mezrag L., Saadi K. *Inclusion theorems for Cohen strongly summing multilinear operators*. Bull. Belg. Math. Soc. Simon Stevin. 2009, **16**, 1–11. doi:10.36045/bbms/1235574187

[9] Pellegrino D., Santos J., Seoane-Sepúlveda J. *Some techniques on nonlinear analysis and applications*. Adv. Math. 2012, **229**, 1235–1265. doi:10.1016/j.aim.2011.09.014

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У 2003 році В. Дімант визначила і вивчала цікавий клас строго p-сумовних мультилінійних операторів. У цій роботі ми вводимо та вивчаємо новий клас операторів між двома банаховими гратками, де ми поширюємо попереднє поняття на позитивні рамки та доводимо, серед інших результатів, теореми про домінування, включення та композицію. Як наслідок, ми встановлюємо деякі зв'язки між нашим класом та іншими класами операторів, такі як двоїстість та лінеаризація.

Ключові слова і фрази: банахова гратка, теорема Пітча про домінування, додатний строго p-сумовний мультилінійний оператор, додатний p-сумовний оператор, тензорна норма.