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TOPOLOGICAL MONOIDS OF ALMOST MONOTONE INJECTIVE CO-FINITE PARTIAL SELFMAPS OF POSITIVE INTEGERS

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In this paper we study the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ of partial co-finite almost monotone bijective transformations of the set of positive integers $\mathbb{N}$. We show that the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. Also we prove that every Baire topology τ on $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ such that $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau)$ is a semitopological semigroup is discrete, describe the closure of $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau)$ in a topological semigroup and construct non-discrete Hausdorff semigroup topologies on $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$.

Introduction and preliminaries

In this paper all spaces are assumed to be Hausdorff. Furthermore we shall follow the terminology of [6, 7, 10, 27]. By ω we shall denote the first infinite cardinal.

An algebraic semigroup S is called *inverse* if for any element $x\in S$ there exists the unique $x^{-1}\in S$ such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$. The element x^{-1} is called the *inverse of* $x\in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S\to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion*.

If S is a semigroup, then by $E(S)$ we shall denote the *band* (i. e. the subset of idempotents) of S. If the band $E(S)$ is a non-empty subset of S, then the semigroup operation on S determines the partial order $\leqslant$ on $E(S)$: $e\leqslant f$ if and only if $ef=fe=e$. This order is called *natural*. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or *chain* if the semilattice operation admits a linear natural order on E. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [25, Definition II.5.12] chain L is called ω-chain if L is isomorphic to $\{0,-1,-2,-3,\ldots\}$ with the usual order $\leqslant$. Let E be a semilattice and $e\in E$. We denote

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$\downarrow e = \{f \in E \mid f \leqslant e\}$ and $\uparrow e = \{f \in E \mid e \leqslant f\}$. By $(\mathscr{P}_{<\omega}(\mathbb{N}), \subseteq)$ we shall denote the free semilattice with identity over the set of positive integers $\mathbb{N}$.

If S is a semigroup, then by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$ and $\mathscr{H}$ the Green relations on S (see [7]):

$$a\mathscr{R}b \text{ if and only if } aS^1 = bS^1;$$
$$a\mathscr{L}b \text{ if and only if } S^1a = S^1b;$$
$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L};$$
$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}.$$

A semigroup S is called *simple* if S does not contain proper two-sided ideals and *bisimple* if all elements of S are $\mathscr{D}$-equivalent.

A *semitopological* (resp. *topological*) *semigroup* is a topological space together with a separately (resp. jointly) continuous semigroup operation.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of a set X of cardinality λ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\}$, for $\alpha, \beta \in \mathscr{I}_\lambda$. The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [7]). The symmetric inverse semigroup was introduced by Wagner [29] and it plays a major role in the theory of semigroups.

We denote $\mathscr{I}_\lambda^n = \{\alpha \in \mathscr{I}_\lambda \mid \operatorname{rank}\alpha \leqslant n\}$, for $n = 1, 2, 3, \ldots$. Obviously, $\mathscr{I}_\lambda^n$ ($n = 1, 2, 3, \ldots$) is an inverse semigroup, $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}_\lambda$ for each $n = 1, 2, 3, \ldots$. Further, we shall call the semigroup $\mathscr{I}_\lambda^n$ the *symmetric inverse semigroup of finite transformations of the rank n.*

Let $\mathbb{N}$ be the set of all positive integers. By $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ we shall denote the semigroup of monotone, non-decreasing, injective partial transformations of $\mathbb{N}$ such that the sets $\mathbb{N} \setminus \operatorname{dom}\varphi$ and $\mathbb{N} \setminus \operatorname{rank}\varphi$ are finite for all $\varphi \in \mathscr{I}_\infty^{\nearrow}(\mathbb{N})$. Obviously, $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathscr{I}_\omega$. The semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is called *the semigroup of co-finite monotone partial bijections* of $\mathbb{N}$ [19].

We shall denote every element α of the semigroup $\mathscr{I}_\omega$ by $\begin{pmatrix} n_1 & n_2 & n_3 & n_4 & \ldots \\ m_1 & m_2 & m_3 & m_4 & \ldots \end{pmatrix}$ and this means that α maps the positive integer n_i into m_i for all $i = 1, 2, 3, \ldots$. We observe that an element α of the semigroup $\mathscr{I}_\omega$ is an element of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ if and only if it satisfies the following conditions:

(i) the sets $\mathbb{N} \setminus \{n_1, n_2, n_3, n_4, \ldots\}$ and $\mathbb{N} \setminus \{m_1, m_2, m_3, m_4, \ldots\}$ are finite;

(ii) $n_1 < n_2 < n_3 < n_4 < \ldots$ and $m_1 < m_2 < m_3 < m_4 < \ldots$.

A partial map $\alpha \colon \mathbb{N} \rightharpoonup \mathbb{N}$ is called *almost monotone* if there exists a finite subset A of $\mathbb{N}$ such that the restriction $\alpha|_{\mathbb{N}\setminus A} \colon \mathbb{N} \setminus A \rightharpoonup \mathbb{N}$ is a monotone partial map.

By $\mathscr{I}_\infty^{\,\nearrow}(\mathbb{N})$ we shall denote the semigroup of monotone, almost non-decreasing, injective partial transformations of $\mathbb{N}$ such that the sets $\mathbb{N} \setminus \operatorname{dom}\varphi$ and $\mathbb{N} \setminus \operatorname{rank}\varphi$ are finite for all $\varphi \in \mathscr{I}_\infty^{\,\nearrow}(\mathbb{N})$. Obviously, $\mathscr{I}_\infty^{\,\nearrow}(\mathbb{N})$ is an inverse subsemigroup of the semigroup $\mathscr{I}_\omega$ and the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ is an inverse subsemigroup of $\mathscr{I}_\infty^{\,\nearrow}(\mathbb{N})$ too. The semigroup $\mathscr{I}_\infty^{\,\nearrow}(\mathbb{N})$ is called *the semigroup of co-finite almost monotone partial bijections* of $\mathbb{N}$. We observe that an element α of the semigroup $\mathscr{I}_\omega$ is an element of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ if and only if it satisfies conditions (i) and (iii):

(iii) there exists a positive integer i such that $n_i < n_{i+1} < n_{i+2} < n_{i+3} < \ldots$ and $m_i < m_{i+1} < m_{i+2} < m_{i+3} < \ldots$.

Further by $\mathbb{I}$ we shall denote the identity of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$.

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by elements p and q subject only to the condition $pq = 1$. The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathscr{C}(p,q)$ under h is a cyclic group (see [7, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known result of Andersen [1] states that a (0–)simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete topology and a topological semigroup S can contain $\mathscr{C}(p,q)$ only as an open subset [9]. Neither stable nor Γ-compact topological semigroups can contain a copy of the bicyclic semigroup [2, 21]. Also, the bicyclic semigroup does not embed into a countably compact topological inverse semigroup [18]. Moreover, in [3] and [4] the conditions were given when a countable compact or pseudocompact topological semigroup does not contain the bicyclic semigroup. However, Banakh, Dimitrova and Gutik constructed with set-theoretic assumptions (Continuum Hypothesis or Martin Axiom) an example of a Tychonoff countable compact topological semigroup which contains the bicyclic semigroup [4].

Many semigroup theorists have considered a topological semigroup of (continuous) transformations of an arbitrary topological space. Beĭda [5], Orlov [23, 24], and Subbiah [28] have considered semigroup and inverse semigroup topologies of semigroups of partial homeomorphisms of some classes of topological spaces.

Gutik and Pavlyk [14] considered the special case of the semigroup $\mathscr{I}_\lambda^n$: an infinite topological semigroup of $\lambda\times\lambda$-matrix units B_λ. They showed that an infinite topological semigroup of $\lambda\times\lambda$-matrix units B_λ does not embed into a compact topological semigroup and that B_λ is algebraically h-closed in the class of topological inverse semigroups. They also described the Bohr compactification of B_λ, minimal semigroup and minimal semigroup inverse topologies on B_λ.

Gutik, Lawson and Repovš [13] introduced the notion of a semigroup with a tight ideal series and investigated their closures in semitopological semigroups, particularly inverse semigroups with continuous inversion. As a corollary they showed that the symmetric inverse semigroup of finite transformations $\mathscr{I}_\lambda^n$ of infinite cardinal λ is algebraically closed in the class of (semi)topological inverse semigroups with continuous inversion. They also derived related results about the nonexistence of (partial) compactifications of classes of considered semigroups.

Gutik and Reiter [16] showed that the topological inverse semigroup $\mathscr{I}_\lambda^n$ is algebraically h-closed in the class of topological inverse semigroups. They also proved that a topological semigroup S with countably compact square $S\times S$ does not contain the semigroup $\mathscr{I}_\lambda^n$

for infinite cardinals λ and showed that the Bohr compactification of an infinite topological semigroup $\mathscr{I}_\lambda^n$ is the trivial semigroup.

In [17] Gutik and Reiter showed that the symmetric inverse semigroup of finite transformations $\mathscr{I}_\lambda^n$ of infinite cardinal λ is algebraically closed in the class of semitopological inverse semigroups with continuous inversion. There they described all congruences on the semigroup $\mathscr{I}_\lambda^n$ and all compact and countably compact topologies τ on $\mathscr{I}_\lambda^n$ such that $(\mathscr{I}_\lambda^n, \tau)$ is a semitopological semigroup.

Gutik, Pavlyk and Reiter [15] showed that a topological semigroup of finite partial bijections $\mathscr{I}_\lambda^n$ of infinite set with a compact subsemigroup of idempotents is absolutely H-closed. They proved that no Hausdorff countably compact topological semigroup and no Tychonoff topological semigroup with pseudocompact square contain $\mathscr{I}_\lambda^n$ as a subsemigroup. They proved that every continuous homomorphism from topological semigroup $\mathscr{I}_\lambda^n$ into a Hausdorff countably compact topological semigroup or Tychonoff topological semigroup with pseudocompact square is annihilating. Also they gave sufficient conditions for a topological semigroup $\mathscr{I}_\lambda^1$ to be non-H-closed and showed that the topological inverse semigroup $\mathscr{I}_\lambda^1$ is absolutely H-closed if and only if the band $E(\mathscr{I}_\lambda^1)$ is compact [15].

In [19] Gutik and Repovš studied the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ of partial cofinite monotone bijective transformations of the set of positive integers $\mathbb{N}$. They showed that the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. They proved that every locally compact topology τ on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}), \tau)$ is a topological inverse semigroup, is discrete and describe the closure of $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}), \tau)$ in a topological semigroup.

We remark that the bicyclic semigroup is isomorphic to the semigroup $\mathscr{C}_{\mathbb{N}}(\pi, \sigma)$ which is generated by partial transformations π and σ of the set of positive integers $\mathbb{N}$, defined as follows:

$$(n)\pi = n+1 \quad \text{if } n \geqslant 1, \qquad \text{and} \qquad (n)\sigma = n-1 \quad \text{if } n > 1.$$

Therefore the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ contains an isomorphic copy of the bicyclic semigroup $\mathscr{C}(p, q)$.

In the present paper we study the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ of partial co-finite almost monotone bijective transformations of the set of positive integers $\mathbb{N}$. We show that the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ has algebraic properties similar to the bicyclic semigroup: it is bisimple and all of its non-trivial group homomorphisms are either isomorphisms or group homomorphisms. Also we prove that every Baire topology τ on $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ such that $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}), \tau)$ is a semitopological semigroup is discrete, describe the closure of $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}), \tau)$ in a topological semigroup and construct non-discrete Hausdorff semigroup topologies on $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$.

1 Algebraic properties of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$

Proposition 1.1. *(i) An element α of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is an idempotent if and only if $(x)\alpha = x$ for every $x \in \operatorname{dom}\alpha$, and hence $E(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})) = E(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}))$.*

(ii) If $\varepsilon, \iota \in E(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}))$, then $\varepsilon \leqslant \iota$ if and only if $\operatorname{dom}\varepsilon \subseteq \operatorname{dom}\iota$.

(*iii*) *The semilattice* $E(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}))$ *is isomorphic to* $(\mathscr{P}_{<\omega}(\mathbb{N}), \subseteq)$ *under the mapping* $(\varepsilon)h = \mathbb{N} \setminus \operatorname{dom} \varepsilon$.

(*iv*) *Every maximal chain in* $E(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}))$ *is an* ω*-chain.*

(*v*) *For every* $\varepsilon, \iota \in E(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}))$ *there exists* $\alpha \in \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ *such that* $\alpha\alpha^{-1} = \varepsilon$ *and* $\alpha^{-1}\alpha = \iota$.

(*vi*) $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ *is a simple semigroup.*

(*vii*) $\alpha\mathscr{R}\beta$ *in* $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

(*viii*) $\alpha\mathscr{L}\beta$ *in* $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ *if and only if* $\operatorname{rank}\alpha = \operatorname{rank}\beta$.

(*ix*) $\alpha\mathscr{H}\beta$ *in* $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$ *and* $\operatorname{rank}\alpha = \operatorname{rank}\beta$.

(*x*) $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ *is a bisimple semigroup.*

Proof. Statements (*i*) – (*iv*) are trivial and their proofs follow from the definition of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$.

(*v*) For the idempotents $\varepsilon = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \end{pmatrix}$ and $\iota = \begin{pmatrix} l_1 & l_2 & l_3 & l_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$ we put $\alpha = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$. Then $\alpha\alpha^{-1} = \varepsilon$ and $\alpha^{-1}\alpha = \iota$.

(*vi*) Let $\alpha = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \end{pmatrix}$ and $\beta = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$ be any elements of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$, where $n_i, m_i, k_i, l_i \in \mathbb{N}$ for $i = 1, 2, 3, \ldots$. We put $\gamma = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & \cdots \\ n_1 & n_2 & n_3 & n_4 & \cdots \end{pmatrix}$ and $\delta = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$. Then we have that $\gamma\alpha\delta = \beta$. Therefore $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) \cdot \alpha \cdot \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) = \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ for any $\alpha \in \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ and hence $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is a simple semigroup.

(*vii*) Let $\alpha, \beta \in \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ be such that $\alpha\mathscr{R}\beta$. Since $\alpha\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) = \beta\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ and $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is an inverse semigroup, Theorem 1.17 [7] implies that $\alpha\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) = \alpha\alpha^{-1}\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$, $\beta\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) = \beta\beta^{-1}\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ and $\alpha\alpha^{-1} = \beta\beta^{-1}$. Hence $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

Conversely, let $\alpha, \beta \in \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ be such that $\operatorname{dom}\alpha = \operatorname{dom}\beta$. Then $\alpha\alpha^{-1} = \beta\beta^{-1}$. Since $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is an inverse semigroup, Theorem 1.17 [7] implies that $\alpha\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) = \alpha\alpha^{-1}\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) = \beta\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ and hence $\alpha\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) = \beta\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$.

The proof of statement (*viii*) is similar to (*vii*).

Statement (*ix*) follows from (*vii*) and (*viii*).

(*x*) By statements (*vii*) and (*viii*) it is sufficient to show that every distinct idempotents of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ are $\mathscr{D}$-equivalent. For idempotents $\varepsilon = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ m_1 & m_2 & m_3 & m_4 & \cdots \end{pmatrix}$ and $\iota = \begin{pmatrix} l_1 & l_2 & l_3 & l_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$ we put $\alpha = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ l_1 & l_2 & l_3 & l_4 & \cdots \end{pmatrix}$. Then by statements (*vii*) and (*viii*) we have that $\varepsilon\mathscr{R}\alpha$ and $\alpha\mathscr{L}\iota$, and hence $\varepsilon\mathscr{D}\iota$. $\square$

Proposition 1.2. *For every $\alpha,\beta\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$, both sets $\{\chi\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})\mid\alpha\cdot\chi=\beta\}$ and $\{\chi\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})\mid\chi\cdot\alpha=\beta\}$ are finite. Consequently, every right translation and every left translation by an element of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is a finite-to-one map.*

Proof. We denote $A=\{\chi\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})\mid\alpha\cdot\chi=\beta\}$ and $B=\{\chi\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})\mid\alpha^{-1}\cdot\alpha\cdot\chi=\alpha^{-1}\cdot\beta\}$. Then $A\subseteq B$ and the restriction of any partial map $\chi\in B$ to $\operatorname{dom}(\alpha^{-1}\cdot\alpha)$ coincides with the partial map $\alpha^{-1}\cdot\beta$. Since every partial map from the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is almost monotone (i. e., almost non-decreasing) and co-finite, the set B is finite and hence so is A. □

For an arbitrary non-empty set X we denote by $\mathsf{S}_\infty(X)$ the group of all bijective transformations of X with finite supports (i. e., $\alpha\in\mathsf{S}_\infty(X)$ if and only if the set $\{x\in X\mid(x)\alpha\neq x\}$ is finite).

The definition of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ implies the following proposition:

Proposition 1.3. *Every maximal subgroup of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is isomorphic to $\mathsf{S}_\infty(\mathbb{N})$.*

The semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ contains $\mathscr{I}_{\infty}^{\!\nearrow}(\mathbb{N})$ as a subsemigroup and Theorem 2.9 of [19] states that if S is a semigroup and $h\colon\mathscr{I}_{\infty}^{\!\nearrow}(\mathbb{N})\to S$ is a non-annihilating homomorphism, then either h is a monomorphism or $(\mathscr{I}_{\infty}^{\!\nearrow}(\mathbb{N}))h$ is a cyclic subgroup of S. This arises the following problem: *To describe all homomorphisms of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$.*

The definition of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ implies the following proposition:

Proposition 1.4. *For every $\gamma\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ there exists $n_\gamma\in\mathbb{N}$ such that $i-n_\gamma=(i)\alpha-(n_\gamma)\alpha$ for all $i\geqslant n_\gamma$, $i\in\mathbb{N}$.*

Lemma 1.1. *For every $\gamma\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ there exists an idempotent $\varepsilon\in\mathscr{C}_{\mathbb{N}}(\pi,\sigma)$ such that $\gamma\cdot\varepsilon,\varepsilon\cdot\gamma\in\mathscr{C}_{\mathbb{N}}(\pi,\sigma)$. Consequently, for every idempotent $\iota\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ there exists $\varepsilon_0\in E(\mathscr{C}_{\mathbb{N}}(\pi,\sigma))$ such that $\iota\cdot\varepsilon_0=\varepsilon_0\cdot\iota=\varepsilon_0$.*

Proof. Let $n_\gamma\in\mathbb{N}$ be such as in the statement of Proposition 1.4. We put $m_\gamma=\max\{n_\gamma,(n_\gamma)\gamma\}$ and define

$$\varepsilon=\left(\begin{array}{cccc} m_\gamma & m_\gamma+1 & m_\gamma+2 & \cdots\\ m_\gamma & m_\gamma+1 & m_\gamma+2 & \cdots\end{array}\right).$$

Then we have that $\gamma\cdot\varepsilon,\varepsilon\cdot\gamma\in\mathscr{C}_{\mathbb{N}}(\pi,\sigma)$.

Let ι be an arbitrary idempotent of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$. By the first assertion of the lemma there exists $\varepsilon\in E(\mathscr{C}_{\mathbb{N}}(\pi,\sigma))$ such that $\iota\cdot\varepsilon\in\mathscr{C}_{\mathbb{N}}(\pi,\sigma)$. Since the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is inverse Theorem 1.17 [7] implies that $\varepsilon_0=\iota\cdot\varepsilon=\varepsilon\cdot\iota$ is an idempotent of $\mathscr{C}_{\mathbb{N}}(\pi,\sigma)$. Hence we have that $\iota\cdot\varepsilon_0=\varepsilon_0\cdot\iota=\varepsilon_0$. □

Lemma 1.2. *Let S be a semigroup and $h\colon\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})\to S$ be a non-annihilating homomorphism such that the set $\big(E(\mathscr{C}_{\mathbb{N}}(\pi,\sigma))\big)h$ is singleton. Then $\big(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})\big)h=\big(\mathscr{C}_{\mathbb{N}}(\pi,\sigma)\big)h$.*

Proof. Suppose that $\big(E(\mathscr{C}_{\mathbb{N}}(\pi,\sigma))\big)h=\{e\}$. Since $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is an inverse semigroup and $E(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}))=E(\mathscr{C}_{\mathbb{N}}(\pi,\sigma))$ we conclude that e is a unique idempotent in $\big(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})\big)h$. Fix an arbitrary element γ of $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$. Let ε be such as in Lemma 1.1. Then we have

$$(\gamma)h = (\gamma\cdot\gamma^{-1}\cdot\gamma)h = (\gamma)h\cdot(\gamma^{-1}\cdot\gamma)h = (\gamma)h\cdot(\varepsilon)h = (\gamma\cdot\varepsilon)h \in \big(\mathscr{C}_{\mathbb{N}}(\pi,\sigma)\big)h,$$

the assertion of the lemma holds. □

We need the following theorem from [19]:

Theorem 1 ([19, Theorem 2.9])**.** *Let S be a semigroup and $h\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to S$ a non-annihilating homomorphism. Then either h is a monomorphism or $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}))h$ is a cyclic subgroup of S.*

Lemma 1.3. *Let S be a semigroup and $h\colon \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) \to S$ be a homomorphism such that the restriction $h|_{\mathscr{C}_{\mathbb{N}}(\pi,\sigma)}\colon \mathscr{C}_{\mathbb{N}}(\pi,\sigma) \to \big(\mathscr{C}_{\mathbb{N}}(\pi,\sigma)\big)h \subseteq S$ is an isomorphism. Then h is an isomorphism.*

Proof. Suppose to the contrary that the map $h\colon \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) \to S$ is not an isomorphism. Then by Theorem 1 we have that the restriction $h|_{\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})}\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to (\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}))h \subseteq S$ is an isomorphism. Since $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is an inverse semigroup we conclude that if $(\alpha)h = (\beta)h$ for some $\alpha,\beta \in \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ then $\alpha\mathscr{H}\beta$. Otherwise if α and β are not $\mathscr{H}$-equivalent and $(\alpha)h \neq (\beta)h$ then $(\alpha^{-1})h \neq (\beta^{-1})h$ and therefore either $(\alpha\alpha^{-1})h \neq (\beta\beta^{-1})h$ or $(\alpha^{-1}\alpha)h \neq (\beta^{-1}\beta)h$, a contradiction to the assumption that the restriction $h|_{\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})}\colon \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \to (\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}))h \subseteq S$ is an isomorphism. Thus by the Green Theorem (see [7, Theorem 2.20]) without loss of generality we can assume that $(\mathbb{I})h = (\alpha)h$ for some $\alpha \in H(\mathbb{I})$. Since the group $\mathsf{S}_{\infty}(\mathbb{N})$ has only one proper normal subgroup and such subgroup is the group $\mathsf{A}_{\infty}(\mathbb{N})$ of even permutations of $\mathbb{N}$ (see [22] and [12, pp. 313–314, Example]) we conclude that $(\mathsf{A}_{\infty}(\mathbb{N}))h = (\mathbb{I})h$. We denote

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n & \cdots \\ 2 & 3 & 1 & 4 & 5 & \cdots & n & \cdots \end{pmatrix} \qquad \text{and} \qquad \varepsilon_{1,2} = \begin{pmatrix} 3 & 4 & 5 & \cdots & n & \cdots \\ 1 & 4 & 5 & \cdots & n & \cdots \end{pmatrix}.$$

Then $\beta \in \mathsf{A}_{\infty}(\mathbb{N})$. Therefore we have that

$$(\varepsilon_{1,2})h = (\varepsilon_{1,2}\cdot\mathbb{I})h = (\varepsilon_{1,2})h\cdot(\mathbb{I})h = (\varepsilon_{1,2})h\cdot(\beta)h = (\varepsilon_{1,2}\cdot\beta)h$$

and similarly $(\varepsilon_{1,2})h = (\beta\cdot\varepsilon_{1,2})h$. Since

$$\beta\cdot\varepsilon_{1,2} = \begin{pmatrix} 2 & 4 & 5 & \cdots & n & \cdots \\ 3 & 4 & 5 & \cdots & n & \cdots \end{pmatrix} \qquad \text{and} \qquad \varepsilon_{1,2}\cdot\beta = \begin{pmatrix} 3 & 4 & 5 & \cdots & n & \cdots \\ 1 & 4 & 5 & \cdots & n & \cdots \end{pmatrix}$$

we conclude that $\beta\cdot\varepsilon_{1,2}, \varepsilon_{1,2}\cdot\beta \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. Hence by Theorem 1 the set $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}))h$ contains only one idempotent and therefore the assertions of Lemma 1.2 hold. This completes the proof of the lemma. □

Theorem 1 and Lemmas 1.2 and 1.3 imply the following theorem:

Theorem 2. *Let S be a semigroup and $h\colon \mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}) \to S$ a non-annihilating homomorphism. Then either h is a monomorphism or $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}))h$ is a cyclic subgroup of S.*

2 Topologizations of some classes of countable semigroups

Definition 2.1. *We shall say that a semigroup S has:*

- *an* S-property *if for every $a, b \in S$ there exist $c, d \in S^1$ such that $c \cdot a \cdot d = b$;*
- *an* F-property *if for every $a, b, c, d \in S^1$ the sets $\{x \in S \mid a \cdot x = b\}$ and $\{x \in S \mid x \cdot c = d\}$ are finite or empty;*
- *an* FS-property *if S has F- and S-properties.*

Remark 2.1. *We observe that*

1) *every simple (resp., left simple, right simple) semigroup has* S-property*;*
2) *every free (Abelian) semigroup has* F-property*;*
3) *$\mathscr{I}_\infty^{\Rsh\!\!\nearrow}(\mathbb{N})$, $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and the bicyclic semigroup have FS-property.*

Lemma 2.1. *Let S be a Hausdorff semitopological semigroup with FS-property. If S has an isolated point then S is the discrete topological space.*

Proof. Let t be an isolated point in S. Since the semigroup S has the FS-*property* we conclude that for every $s \in S$ there exist $a, b \in S^1$ such that $a \cdot s \cdot b = t$ and the equation $a \cdot x \cdot b = t$ has a finite set of solutions. Therefore the continuity of translations in (S, τ) implies that the element s has a finite open neighbourhood, and hence Hausdorffness of (S, τ) implies that s is an isolated point of (S, τ). This completes the proof of the lemma. □

A topological space X is called *Baire* if for each sequence $A_1, A_2, \ldots, A_i, \ldots$ of nowhere dense subsets of X the union $\bigcup\limits_{i=1}^{\infty} A_i$ is a co-dense subset of X [10].

Theorem 3. *Let S be a countable semigroup with FS-property. Then every Baire topology τ on S such that (S, τ) is a Hausdorff semitopological semigroup is discrete.*

Proof. We consider countable cover $\Gamma = \{s \mid s \in S\}$ of the Baire space (S, τ). Then there exists an isolated point t in S. By Lemma 2.1 the topological space is discrete. □

A Tychonoff space X is called *Čech complete* if fot every compactification cX of X the remainder $cX \setminus c(X)$ is an F_σ-set in cX [10].

Since every Čech complete space (and hence every locally compact space) is Baire, Theorem 3 implies the following:

Corollary 2.1. *Every Hausdorff Čech complete (locally compact) countable semitopological semigroup with FS-property is discrete.*

A topological space X is called *hereditary Baire* if every closed subset of X is a Baire space [10]. Every Čech complete (and hence locally compact) space is hereditary Baire (see [10, Theorem 3.9.6]). We shall say that a Hausdorff semitopological semigroup S is an *I-Baire space* if either sS or Ss is a Baire space for every $s \in S$.

Remark 2.2. *We observe that every left ideal Ss and every right ideal sS of a regular semigroup S are generated by some idempotents of S. Therefore every principal left or right ideal of a regular Hausdorff semitopological semigroup S is a closed subset of S. Hence every regular Hausdorff hereditary Baire semitopological semigroup is the I-Baire space.*

Theorem 4. *Let S be a countable semilattice with F-property. Then every I-Baire topology τ on S such that (S,τ) is a Hausdorff semitopological semilattice is discrete.*

Proof. Let s be an arbitrary element of the semilattice S. We consider a countable cover $\Gamma=\{e\mid e\in sS\}$ of sS. Since (S,τ) is an I-Baire space we conclude that there exists an isolated point t in sS. Since S is a semilattice we have that $s\cdot t=t$. Then ${\uparrow_{sS}}t=\{x\in sS\mid x\cdot t=t\}$ is a finite subset of S which contains s and by Proposition VI-1.13 [11] we get that ${\uparrow_{sS}}t$ is an open subset of sS. Hence there exists an open neighbourhood $U(s)$ of s in S such that $U(s)\cap sS=\{s\}$. The continuity of translations in S implies that there exists an open neighbourhood $V(s)\subseteq U(s)$ such that $V(s)\subseteq\{x\in S\mid x\cdot s=s\}$. Since the semilattice S is Hausdorff and has F-property we have that s is an isolated point of S. □

Theorem 4 implies the following:

Corollary 2.2. *Every I-Baire topology τ on the countable free semilattice FSL_ω such that (FSL_ω,τ) is a Hausdorff semitopological semilattice is discrete.*

3 On topologizations and closures of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$

Theorem 3 implies the following two corollaries:

Corollary 3.1. *Every Baire topology τ on $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that $(\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N}),\tau)$ is a Hausdorff semitopological semigroup is discrete.*

Corollary 3.2. *Every Baire topology τ on $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ such that $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau)$ is a Hausdorff semitopological semigroup is discrete.*

We observe that Corollary 3.2 generalizes Theorem 3.3 from [19].

The following example shows that there exists a non-discrete topology τ_F on the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ such that $(\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N}),\tau_F)$ is a Tychonoff topological inverse semigroup.

Example 3.1. *We define a topology τ_F on the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ as follows. For every $\alpha\in\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$ we define a family*

$$\mathscr{B}_F(\alpha)=\{U_\alpha(F)\mid F \text{ is a finite subset of } \operatorname{dom}\alpha\},$$

where

$$U_\alpha(F)=\{\beta\in\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})\mid \operatorname{dom}\alpha=\operatorname{dom}\beta,\operatorname{ran}\alpha=\operatorname{ran}\beta \text{ and } (x)\beta=(x)\alpha \text{ for all } x\in F\}.$$

Since conditions (BP1)–(BP3) [10] hold for the family $\{\mathscr{B}_F(\alpha)\}_{\alpha\in\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})}$ we conclude that the family $\{\mathscr{B}_F(\alpha)\}_{\alpha\in\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})}$ is the base of the topology τ_F on the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\nearrow}(\mathbb{N})$.

Proposition 3.1. *$(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_F)$ is a Tychonoff topological inverse semigroup.*

Proof. Let α and β be arbitrary elements of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$. We put $\gamma=\alpha\beta$ and let $F=\{n_1,\ldots,n_i\}$ be a finite subset of $\operatorname{dom}\gamma$. We denote $m_1=(n_1)\alpha,\ldots,m_i=(n_i)\alpha$ and $k_1=(n_1)\gamma,\ldots,k_i=(n_i)\gamma$. Then we get that $(m_1)\beta=k_1,\ldots,(m_i)\beta=k_i$. Hence we have that

$$U_\alpha(\{n_1,\ldots,n_i\})\cdot U_\beta(\{m_1,\ldots,m_i\})\subseteq U_\gamma(\{n_1,\ldots,n_i\})$$

and

$$\big(U_\gamma(\{n_1,\ldots,n_i\})\big)^{-1}\subseteq U_{\gamma^{-1}}(\{k_1,\ldots,k_i\}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_F)$.

We observe that the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ with the induced topology $\tau_F(H(\mathbb{I}))$ from $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_F)$ is a topological group (see [12, pp. 313–314, Example] and [22]) and the definition of the topology τ_F implies that every $\mathscr{H}$-class of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ is an open-and-closed subset of the topological space $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_F)$. Therefore Theorem 2.20 [7] implies that the topological space $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_F)$ is homeomorphic to a countable topological sum of topological copies of $\big(H(\mathbb{I}),\tau_F(H(\mathbb{I}))\big)$. Since every T_0-topological group is a Tychonoff topological space (see [26, Theorem 3.10]) we conclude that the topological space $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_F)$ is Tychonoff too. This completes the proof of the proposition. □

Remark 3.1. *We observe that the topology τ_F on $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ induces discrete topologies on the subsemigroups $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ and $E(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}))$.*

Example 3.2. *We define a topology τ_{WF} on the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ as follows. For every $\alpha\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$ we define a family*

$$\mathscr{B}_{WF}(\alpha)=\{U_\alpha(F)\mid F \text{ is a finite subset of } \operatorname{dom}\alpha\},$$

where

$$U_\alpha(F)=\{\beta\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})\mid \operatorname{dom}\beta\subseteq\operatorname{dom}\alpha \text{ and } (x)\beta=(x)\alpha \text{ for all } x\in F\}.$$

Since conditions (BP1)–(BP3) [10] hold for the family $\{\mathscr{B}_{WF}(\alpha)\}_{\alpha\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})}$ we conclude that the family $\{\mathscr{B}_{WF}(\alpha)\}_{\alpha\in\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})}$ is the base of the topology τ_{WF} on the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$.

Proposition 3.2. *$(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_{WF})$ is a Hausdorff topological inverse semigroup.*

Proof. Let α and β be arbitrary elements of the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N})$. We put $\gamma=\alpha\beta$ and let $F=\{n_1,\ldots,n_i\}$ be a finite subset of $\operatorname{dom}\gamma$. We denote $m_1=(n_1)\alpha,\ldots,m_i=(n_i)\alpha$ and $k_1=(n_1)\gamma,\ldots,k_i=(n_i)\gamma$. Then we get that $(m_1)\beta=k_1,\ldots,(m_i)\beta=k_i$. Hence we have that

$$U_\alpha(\{n_1,\ldots,n_i\})\cdot U_\beta(\{m_1,\ldots,m_i\})\subseteq U_\gamma(\{n_1,\ldots,n_i\})$$

and

$$\big(U_\gamma(\{n_1,\ldots,n_i\})\big)^{-1}\subseteq U_{\gamma^{-1}}(\{k_1,\ldots,k_i\}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_{WF})$.

Later we shall show that the topology τ_{WF} is Hausdorff. Let α and β be arbitrary distinct points of the space $(\mathscr{I}_{\infty}^{\Rsh\!\!\nearrow}(\mathbb{N}),\tau_{WF})$. Then only one of the following conditions holds:

(i) $\operatorname{dom}\alpha = \operatorname{dom}\beta$;

(ii) $\operatorname{dom}\alpha \neq \operatorname{dom}\beta$.

In case $\operatorname{dom}\alpha = \operatorname{dom}\beta$ we have that there exists $x \in \operatorname{dom}\alpha$ such that $(x)\alpha \neq (x)\beta$. The definition of the topology τ_{WF} implies that $U_\alpha(\{x\}) \cap U_\beta(\{x\}) = \varnothing$.

If $\operatorname{dom}\alpha \neq \operatorname{dom}\beta$, then only one of the following conditions holds:

(a) $\operatorname{dom}\alpha \subsetneqq \operatorname{dom}\beta$;

(b) $\operatorname{dom}\beta \subsetneqq \operatorname{dom}\alpha$;

(c) $\operatorname{dom}\alpha \setminus \operatorname{dom}\beta \neq \varnothing$ and $\operatorname{dom}\beta \setminus \operatorname{dom}\alpha \neq \varnothing$.

Suppose that case (a) holds. Let $x \in \operatorname{dom}\beta \setminus \operatorname{dom}\alpha$ and $y \in \operatorname{dom}\alpha$. The definition of the topology τ_{WF} implies that $U_\alpha(\{y\}) \cap U_\beta(\{x\}) = \varnothing$.

Case (b) is similar to (a).

Suppose that case (c) holds. Let $x \in \operatorname{dom}\beta \setminus \operatorname{dom}\alpha$ and $y \in \operatorname{dom}\alpha \setminus \operatorname{dom}\beta$. The definition of the topology τ_{WF} implies that $U_\alpha(\{y\}) \cap U_\beta(\{x\}) = \varnothing$.

This completes the proof of the proposition. □

Remark 3.2. *We observe that the topology τ_{WF} on $\mathscr{I}_\infty^{\rlap{\tiny$\nearrow$}\tiny$\nearrow$}(\mathbb{N})$ induces non-discrete topologies on the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and the semilattice $E(\mathscr{I}_\infty^{\rlap{\tiny$\nearrow$}\tiny$\nearrow$}(\mathbb{N}))$. Moreover, every $\mathscr{H}$-class of the semigroup $(\mathscr{I}_\infty^{\rlap{\tiny$\nearrow$}\tiny$\nearrow$}(\mathbb{N}), \tau_{WF})$ is homeomorphic to every $\mathscr{H}$-class of the semigroup $(\mathscr{I}_\infty^{\nearrow}(\mathbb{N}), \tau_W)$*

The proof of the following proposition is similar to Theorem 3:

Proposition 3.3. *Every Hausdorff Baire topology τ on a countable group G such that left (right) translations in (G, τ) are continuous is discrete.*

Theorem 5. *Let S be a topological semigroup which contains an infinite dense discrete subspace A such that every equations $a \cdot x = b$ and $y \cdot c = d$ have finitely many solutions in A. Then $I = S \setminus A$ is an ideal of S.*

Proof. Suppose that I is not an ideal of S. Then at least one of the following conditions holds:

$$1)\ IA \nsubseteq I, \qquad 2)\ AI \nsubseteq I, \qquad \text{or} \qquad 3)\ II \nsubseteq I.$$

Since A is a discrete dense subspace of S, Theorem 3.5.8 [10] implies that A is an open subspace of S. Suppose there exist $a \in A$ and $b \in I$ such that $b \cdot a = c \notin I$. Since A is a dense open discrete subspace of S the continuity of the semigroup operation in S implies that there exists an open neighbourhood $U(b)$ of b in S such that $U(b) \cdot \{a\} = \{c\}$. But by Proposition 1.2 the equation $x \cdot a = c$ has finitely many solutions in A. This contradicts the assumption that $b \in S \setminus A$. Therefore $b \cdot a = c \in I$ and hence $IA \subseteq I$. The proof of the inclusion $AI \subseteq I$ is similar.

Suppose there exist $a, b \in I$ such that $a \cdot b = c \notin I$. Since A is a dense open discrete subspace of S the continuity of the semigroup operation in S implies that there exist open neighbourhoods $U(a)$ and $U(b)$ of a and b in S, respectively, such that $U(a) \cdot U(b) = \{c\}$.

But by Proposition 1.2 the equations $x \cdot a_0 = c$ and $b_0 \cdot y = c$ have finitely many solutions in A. This contradicts the assumption that $a, b \in S \setminus A$. Therefore $a \cdot b = c \in I$ and hence $II \subseteq I$. □

Theorem 5 implies Corollaries 3.3 and 3.4:

Corollary 3.3. *Let S be a topological semigroup which contains a dense discrete subsemigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$. If $I = S \setminus \mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N}) \neq \varnothing$ then I is an ideal of S.*

Corollary 3.4 ([19])**.** *Let S be a topological semigroup which contains a dense discrete subsemigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$. If $I = S \setminus \mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}) \neq \varnothing$ then I is an ideal of S.*

Proposition 3.4. *Let S be a topological semigroup which contains a dense discrete subsemigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$. Then for every $c \in \mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$ the set*

$$D_c(A) = \{(x, y) \in \mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N}) \times \mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N}) \mid x \cdot y = c\}$$

is a closed-and-open subset of $S \times S$.

Proof. Since $\mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$ is a discrete subspace of S we have that $D_c(A)$ is an open subset of $S \times S$.

Suppose that there exists $c \in \mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$ such that $D_c(A)$ is a non-closed subset of $S \times S$. Then there exists an accumulation point $(a, b) \in S \times S$ of the set $D_c(A)$. The continuity of the semigroup operation in S implies that $a \cdot b = c$. But $\mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N}) \times \mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$ is a discrete subspace of $S \times S$ and hence by Corollary 3.3 the points a and b belong to the ideal $I = S \setminus \mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$ and hence $p \cdot q \in S \setminus \mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$ cannot be equal to c. □

A topological space X is defined to be *pseudocompact* if each locally finite open cover of X is finite. According to [10, Theorem 3.10.22] a Tychonoff topological space X is pseudocompact if and only if each continuous real-valued function on X is bounded.

Theorem 6. *If a topological semigroup S contains $\mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$ as a dense discrete subsemigroup then the square $S \times S$ is not pseudocompact.*

Proof. Since the square $S \times S$ contains an infinite closed-and-open discrete subspace $D_c(A)$, we conclude that $S \times S$ fails to be pseudocompact (see [10, Ex. 3.10.F(d)] or [8]). □

Remark 3.3. *Recall that, a topological semigroup S is called Γ-compact if for every $x \in S$ the closure of the set $\{x, x^2, x^3, \ldots\}$ is a compactum in S (see [21]). Since the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ contains the bicyclic semigroup as a subsemigroup the results obtained in [2], [3], [4], [18], [21] imply that* if a topological semigroup S satisfies one of the following conditions: (i) S is compact; (ii) S is Γ-compact; (iii) the square $S \times S$ is countably compact; (iv) S is a countably compact topological inverse semigroup; or (v) the square $S \times S$ is a Tychonoff pseudocompact space, then S does not contain the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ and hence the semigroup $\mathscr{I}_{\infty}^{\Rsh\!\!\!\nearrow}(\mathbb{N})$.

The proof of the following theorem is similar to Theorem 6:

Theorem 7. *If a topological semigroup S contains $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ as a dense discrete subsemigroup then the square $S \times S$ is not pseudocompact.*

References

1. Andersen O. *Ein Bericht über die Struktur abstrakter Halbgruppen*, PhD Thesis, Hamburg, 1952.

2. Anderson L.W., Hunter R.P., Koch R.J. *Some results on stability in semigroups*, Trans. Amer. Math. Soc., **117** (1965), 521—529.

3. Banakh T., Dimitrova S., Gutik O. *The Rees-Suschkiewitsch Theorem for simple topological semigroups*, Mat. Stud., **31**, 2 (2009), 211—218.

4. Banakh T., Dimitrova S., Gutik O. *Embedding the bicyclic semigroup into countably compact topological semigroups*, Topology Appl. (to appear) (arXiv:0811.4276).

5. Beĭda A.A. *Continuous inverse semigroups of open partial homeomorphisms,* Izv. Vyssh. Uchebn. Zaved. Mat., **1** (1980), 64—65 (in Russian).

6. Carruth J.H., Hildebrant J.A., Koch R.J. The Theory of Topological Semigroups, Vol. I, Marcel Dekker, Inc., New York and Basel, 1983; Vol. II, Marcel Dekker, Inc., New York and Basel, 1986.

7. Clifford A.H., PrestonG.B. The Algebraic Theory of Semigroups, Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I., 1961; Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.

8. Colmez J. *Sur les espaces précompacts*, C. R. Acad. Paris, **233** (1951), 1552—1553.

9. Eberhart C., Selden J. *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc., **144** (1969), 115—126.

10. Engelking R. General Topology, 2nd ed., Heldermann, Berlin, 1989.

11. Gierz G., Hofmann K.H., Keimel K., Lawson J.D., Mislove M.W., Scott D.S. *Continuous Lattices and Domains*, Cambridge Univ. Press, Cambridge, 2003.

12. Guran I.I. *Topology of an infinite symmetric group and condensation*, Comment. Math. Univ. Carol., **22**, 2 (1981), 311—316 (in Russian).

13. Gutik O., Lawson J., Repovš D. *Semigroup closures of finite rank symmetric inverse semigroups*, Semigroup Forum, **78**, 2 (2009), 326—336.

14. Gutik O.V., Pavlyk K.P. *On topological semigroups of matrix units*, Semigroup Forum, **71**, 3 (2005), 389—400.

15. Gutik O., Pavlyk K., Reiter A. *Topological semigroups of matrix units and countably compact Brandt* λ^0*-extensions*, Mat. Stud., **32**, 2 (2009), 115—131.

16. Gutik O.V., Reiter A.R. *Symmetric inverse topological semigroups of finite rank* $\leqslant n$, Mat. Metody Phis.-Mech. Polya., **53**, 3 (2009), 7—14.

17. Gutik O., Reiter A. *On semitopological symmetric inverse semigroups of a bounded finite rank*, Visnyk Lviv Univ. Ser. Mech.-Math. (to appear).

18. Gutik O., Repovš D. *On countably compact* 0*-simple topological inverse semigroups*, Semigroup Forum, **75**, 2 (2007), 464—469.

19. Gutik O., Repovš D. *Topological monoids of monotone, injective partial selfmaps of* $\mathbb{N}$ *having cofinite domain and image*, Stud. Sci. Math. Hungar. (to appear).

20. Hewitt E., Ross K.A. Abstract Harmonic Analysis, Vol. 1, Springer, Berlin, 1963.

21. Hildebrant J.A., Koch R.J. *Swelling actions of* Γ*-compact semigroups*, Semigroup Forum, **33** (1988), 65—85.

22. Karras A., Solitar D. *Some remarks on the infinite symmetric groups*, Math. Z., **66** (1956), 64—69.

23. Orlov S.D. *Topologization of the generalized group of open partial homeomorphisms of a locally compact Hausdorff space*, Izv. Vyssh. Uchebn. Zaved. Mat., **11** (1974), 61—68 (in Russian).

24. Orlov S.D. *On the theory of generalized topological groups*, Theory of Semigroups and its Applications, Sratov Univ. Press, **3** (1974), 80—85 (in Russian).

25. Petrich M. Inverse Semigroups, John Wiley & Sons, New York, 1984.

26. Pontryagin L.S. Topological Groups, Gordon & Breach, New York ets, 1966.

27. Ruppert W. Compact Semitopological Semigroups: An Intrinsic Theory, Lecture Notes in Mathematics, Vol. 1079, Springer, Berlin, 1984.

28. Subbiah S. *The compact-open topology for semigroups of continuous self-maps,* Semigroup Forum, **35**, 1 (1987), 29—33.

29. Wagner V.V. *Generalized groups*, Dokl. Akad. Nauk SSSR, **84** (1952), 1119—1122 (in Russian).

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У статті вивчається напівгрупа $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ майже монотонних ін'єктивних коскінченних часткових перетворень множини натуральних чисел. Доведено, що напіврупа $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ має алгебраїчні властивості близькі до властивостей бициклічної напівгрупи: вона є біпростою та всі її нетривиальні гомоморфізми є або ізоморфізмами, або ж груповими гомоморфізмами. Доведено, що кожна берівська топологія τ на $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ така, що $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau)$ – напівтопологічна напівгрупа є дискретною та описано замикання напівгрупи $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau)$ в топологічній напівгрупі.

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В работе изучается полугруппа $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ почти монотонных инъективных коконечных частичных преобразований множества натуральных чисел. Доказано, что полугруппа $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ имеет алгебраические свойства близкие к свойствам бициклической полугруппы: она бипроста и все её нетривиальные гомоморфизмы являются или изоморфизмами, или групповыми гомоморфизмами. Доказано, что каждая бэровская топология τ на $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ такая, что $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau)$ – полутопологическая полугруппа дискретна и описано замыкание полугруппы $(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N}),\tau)$ в топологической полугруппе.