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RIEMANN TYPE ALGEBRAIC STRUCTURES AND THEIR DIFFERENTIAL-ALGEBRAIC INTEGRABILITY ANALYSIS

The differential-algebraic approach to studying the Lax type integrability of generalized Riemann type equations is devised. The differentiations and the associated invariant differential ideals are analyzed in detail. The approach is also applied to studying the Lax type integrability of the well known Korteweg-de Vries dynamical system.

1 INTRODUCTION

Nonlinear hydrodynamic equations are of constant interest since the classical works by B. Riemann in the general three-dimensional case, having paid special attention to their one-dimensional spatial reduction, for which he devised the generalized method of characteristics and Riemann invariants. These methods appeared to be very effective [21, 15] in investigating many types of nonlinear spatially one-dimensional systems of hydrodynamical type and, in particular, the characteristics method in the form of a "reciprocal" transformation of variables has been used recently in studying the so called Gurevich-Zybin system [6, 5] in [13] and the Whitham type system in [17, 2, 19]. Moreover, this method was further effectively applied to studying solutions to a generalized [4] (owing to D. Holm and M. Pavlov) Riemann type hydrodynamical system

$$D_t^N u = 0, \quad D_t := \partial/\partial t + u\partial/\partial x, \quad N \in \mathbb{Z}_+, \tag{1}$$

where $u \in C^\infty(\mathbb{R}^2; \mathbb{R})$ is a smooth function. The case $N = 2$ was recently analyzed in detail in [2, 4] making use of the standard symplectic theory techniques. In particular, there was demonstrated that the Riemann type hydrodynamical system (1) at $N = 2$, looking upon putting $z := D_t u$ equivalently as

$$\left.\begin{array}{l} u_t = z - uu_x \\ z_t = -uz_x \end{array}\right\}, \tag{2}$$

allows the following Lax type representation

$$\partial f/\partial x = \ell[u, z; \lambda] f, \quad \partial f/\partial t = p(\ell) f, \quad p(\ell) := -u\ell[u, z; \lambda] + q(\lambda),$$

$$\ell[u, z; \lambda] := \begin{pmatrix} -\lambda u_x & -z_x \\ 2\lambda^2 & \lambda u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} \lambda u_x u & z_x u \\ -\lambda - 2\lambda^2 u & -\lambda u_x u \end{pmatrix}, \tag{3}$$

where $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{C}^2)$ and $\lambda \in \mathbb{C}$ is an arbitrary spectral parameter. Making use of a method devised in [16, 11, 7] and based on the spectral theory and related very complicated symplectic theory relationships in [4, 2, 14] the corresponding Lax type representations for the cases $N = 3, 4$ were constructed in explicit form.

In this work a new and very simple differential-algebraic approach to studying the Lax type integrability of the generalized Riemann type hydrodynamic equations at $N = 3, 4$ is devised. It can be easily generalized for treating the problem for arbitrary integers $N \in \mathbb{Z}_+$. The approach is also applied to studying the Lax type integrability of the well known Korteweg-de Vries dynamical system.

2 The differential-algebraic description of the Lax type integrability of generalized Riemann type hydrodynamical equation at $N = 3$ and 4

2.1 The differential-algebraic preliminaries

Take the ring $\mathcal{K} := \mathbb{R}\{\{x, t\}\}$, $(x, t) \in \mathbb{R}^2$, of convergent germs of real-valued smooth functions from $C^{(\infty)}(\mathbb{R}^2; \mathbb{R})$ and construct [18, 9, 20, 3] the associated differential polynomial ring $\mathcal{K}\{u\} := \mathcal{K}[\Theta u]$ with respect to a functional variable u, where Θ denotes the standard monoid of all operators generated by commuting differentiations $\partial/\partial x := D_x$ and $\partial/\partial t$. The ideal $I\{u\} \subset \mathcal{K}\{u\}$ is called [18, 9] differential if the condition $I\{u\} = \Theta I\{u\}$ holds.

Consider now the additional differentiation

$$D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}, \tag{4}$$

depending on the functional variable u, which satisfies the Lie-algebraic commutator condition

$$[D_x, D_t] = (D_x u) D_x, \tag{5}$$

for all $(x, t) \in \mathbb{R}^2$. As a simple consequence of (5) the following general (suitably normalized) *representation* of the differentiation (4)

$$D_t = \partial/\partial t + u\partial/\partial x \tag{6}$$

in the differential ring $\mathcal{K}\{u\}$ holds. Impose now on the differentiation (4) a new Riemann type algebraic constraint

$$D_t^N u = 0, \tag{7}$$

defining some smooth functional set (or "*manifold*") $\mathcal{M}^{(N)}$ of functions $u \in \mathbb{R}\{\{x, t\}\}$, and which allows to reduce naturally the initial ring $\mathcal{K}\{u\}$ to the basic ring $\mathcal{K}\{u\}|_{\mathcal{M}_{(N)}} \subseteq$

$\mathbb{R}\{\{x,t\}\}$. In this case the following natural problem of constructing the corresponding representation of differentiation (4) arises: *to find an equivalent linear representation of the reduced differentiation* $D_t|_{\mathcal{M}_{(N)}} : \mathbb{R}^{p(N)}\{\{x,t\}\} \rightarrow \mathbb{R}^{p(N)}\{\{x,t\}\}$ *in the functional vector space* $\mathbb{R}^{p(N)}\{\{x,t\}\}$ *for some specially chosen integer dimension* $p(N) \in \mathbb{Z}_+$.

As it will be shown below for the cases $N = 3$ and $N = 4$, this problem is completely analytically solvable, giving rise to the corresponding Lax type integrability of the generalized Riemann type hydrodynamical system (1). Moreover, the same problem is also solvable for the more complicated constraint

$$D_t u - D_x^3 u = 0, \tag{8}$$

equivalent to the well known Lax type integrable nonlinear Korteweg-de Vries dynamical system.

2.2   The generalized Riemann type hydrodynamical equation: the case N=3

To proceed with analyzing the above formulated representation problem for the generalized Riemann type equation (7) at $N = 3$, we first construct an adjoint to the differential ring $\mathcal{K}\{u\}$ and invariant with respect to differentiation (6) so called "Riemann differential ideal" $R\{u\} \subset \mathcal{K}\{u\}$ as

$$\begin{aligned}
R\{u\} \quad &:= \quad \left\{ \lambda \sum_{n\in\mathbb{Z}_+} f_n^{(1)} D_x^n u - \sum_{n\in\mathbb{Z}_+} f_n^{(2)} D_t D_x^n u + \sum_{n\in\mathbb{Z}_+} f_n^{(3)} D_t^2 D_x^n u : D_t^3 u = 0, \right. \\
f_n^{(k)} \quad &\in \quad \left. \mathcal{K}\{u\}, k = \overline{1,3}, n \in \mathbb{Z}_+ \right\} \subset \mathcal{K}\{u\},
\end{aligned} \tag{9}$$

where $\lambda \in \mathbb{R}$ is an arbitrary parameter, and formulate the following simple but important lemma.

Lemma 2.1. *The kernel* $Ker\ D_t \subset R\{u\}$ *of the differentiation* $D_t : \mathcal{K}\{u\} \rightarrow \mathcal{K}\{u\}$, *reduced modulo the Riemann differential ideal* $R\{u\} \subset \mathcal{K}\{u\}$, *is generated by elements satisfying the following linear functional-differential relationships:*

$$D_t f^{(1)} = 0, \quad D_t f^{(2)} = \lambda f^{(1)}, \quad D_t f^{(3)} = f^{(2)}, \tag{10}$$

where, by definition, $f^{(k)} := f^{(k)}(\lambda) = \sum_{n\in\mathbb{Z}_+} f_n^{(k)} \lambda^n \in \mathcal{K}\{u\}|_{\mathcal{M}_{(3)}} = \mathbb{R}\{\{x,t\}\}$, $k = \overline{1,3}$, *and* $\lambda \in \mathbb{R}$ *is arbitrary.*

It is easy to see that equations (10) can be equivalently rewritten both in the matrix form as

$$D_t f = q(\lambda)f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{11}$$

where $f := (f^{(1)}, f^{(2)}, f^{(3)})^\intercal \in \mathcal{K}^3\{u\}|_{\mathcal{M}_{(3)}}$, $\lambda \in \mathbb{R}$ is an arbitrary "spectral" parameter, and in the compact scalar form as

$$D_t^3 f_3 = 0 \tag{12}$$

for an element $f_3 \in \mathcal{K}\{u\}|_{\mathcal{M}_{(3)}}$. Here it is worth to note that the Riemann differential ideal (9), satisfying the D_t-invariance condition, is in this case maximal. Now we can construct by means of relationship (12) a new invariant, the so-called "Lax differential ideal" $L\{u\} \subset \mathcal{K}\{u\}$, isomorphic to the Riemann differential ideal $R\{u\} \subset \mathcal{K}\{u\}$ and realizing the Lax type integrability condition of the Riemann type hydrodynamical equation (1). Namely, based on the result of Lemma 2.1 the following proposition holds.

Proposition 2.1. *The expression (11) is an adjoint linear matrix representation in the space $\mathbb{R}^3\{\{x,t\}\}$ of the differentiation $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced to the ideal $R\{u\} \subset \mathcal{K}\{u\}$. The related D_x- and D_t-invariant Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$, which is isomorphic to the invariant Riemann differential ideal $R\{u\} \subset \mathcal{K}\{u\}$, is generated by the element $f_3(\lambda) \in \mathcal{K}\{u\}, \lambda \in \mathbb{R}$, satisfying condition (12), and equals*

$$\begin{aligned} L\{u\} &:= \{g_1 f_3(\lambda) + g_2 D_t f_3(\lambda) + g_3 D_t^2 f_3(\lambda) : D_t^3 f_3(\lambda) = 0, \\ \lambda &\in \mathbb{R}, g_j \in \mathcal{K}\{u\}, j = \overline{1,3}\} \subset \mathcal{K}\{u\}. \end{aligned} \tag{13}$$

We now construct a related adjoint linear matrix representation in the functional vector space $\mathbb{R}^3\{\{x,t\}\}$ for the differentiation $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced modulo the Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$. For this problem to be solved, we need to take into account the commutator relationship (5) and the important invariance condition of the Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$ with respect to the differentiation $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$. As a result of simple but slightly tedious calculations one obtains the following matrix representation:

$$D_x f = \ell[u,v,z;\lambda]f, \quad \ell[u,v,z;\lambda] := \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ 6\lambda^2 r[u,v,z] & -3\lambda & \lambda u_x \end{pmatrix}, \tag{14}$$

where, by definition, $v := D_t u, z := D_t v, (...)_x := D_x(...)$, a vector $f \in \mathbb{R}^3\{\{x,t\}\}$, $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and a smooth functional mapping $r : \tilde{\mathcal{M}}_{(3)} \to \mathbb{R}\{\{x,t\}\}, \tilde{\mathcal{M}}_{(3)} := \sqcap_{j=1}^3 D_t^j \mathcal{M}_{(3)}$, solves the following functional-differential equation

$$D_t r + r D_x u = 1. \tag{15}$$

Moreover, the matrix $\ell := \ell[u,v,z;\lambda] : \mathbb{R}^3\{\{x,t\}\} \to \mathbb{R}^3\{\{x,t\}\}$ satisfies the following determining functional-differential equation:

$$D_t \ell + \ell D_x u = [q(\lambda), \ell], \tag{16}$$

where $[\cdot, \cdot]$ denotes the usual matrix commutator in the functional space $\mathbb{R}^3\{\{x,t\}\}$. The following proposition solving the representation problem posed above, holds.

Proposition 2.2. *The expression (14) is an adjoint linear matrix representation in the space $\mathbb{R}^3\{\{x,t\}\}$ of the differentiation $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced modulo the invariant Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$, given by (13).*

Remark 2.1. *Here it is necessary to mention that the matrix representation (11) coincides completely with that obtained before in the work [4] by means of completely different methods, based mainly on the gradient-holonomic algorithm, devised in [16, 11, 7]. The presented derivation of these representations (11) and (14) is much easier and simpler that can be explained by a deeper insight into the integrability problem, devised above using the differential algebraic approach.*

To proceed further, it is now worth to observing that the invariance condition for the Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$ with respect to the differentiations $D_x, D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ is also equivalent to the related Lax type representation for the generalized Riemann type equation 1 in the following dynamical system form:

$$\left.\begin{array}{l} u_t = v - uu_x \\ v_t = z - uv_x \\ z_t = -uz_x \end{array}\right\} := K[u, v, z], \tag{17}$$

Namely, the following theorem, summing up the results obtained above, holds.

Theorem 1. *The linear differential-matrix expressions (11) and (14) in the space $\mathbb{R}^3\{\{x, t\}\}$ for differentiations $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ and $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, respectively, provide us with the standard Lax type representation for the generalized Riemann type equation (1) in the equivalent dynamical system form (17), thereby implying its Lax type integrability.*

The next problem of great interest is to construct, making use of the differential-algebraic tools, the functional-differential solutions to the determining equation (19), and to construct the corresponding differential-algebraic analogs of the symplectic structures characterizing the differentiations $D_x, D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, as well as the local densities of the related conservation laws, which were derived in [4, 14].

A was shown above and in [4, 14], the dynamical system (17) possesses the following Lax type representation:

$$f_x = \ell[u, v, z; \lambda]f, \quad f_t = p(\ell)f, \quad p(\ell) := -u\ell[u, v, z; \lambda] + q(\lambda),$$

$$\ell[u, v, z; \lambda] = \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ 6\lambda^2 r[u, v, z] & -3\lambda & \lambda u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{18}$$

$$p(\ell) = \begin{pmatrix} -\lambda uu_x & uv_x & -uz_x \\ -3u\lambda^2 + \lambda & 2\lambda uu_x & -\lambda uv_x \\ -6\lambda^2 r[u, v, z]u & 1 + 3u\lambda & -\lambda uu_x \end{pmatrix},$$

where $f \in L_\infty(\mathbb{R}; \mathbb{E}^3)$, $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and a function $r : \mathcal{M} \to \mathbb{R}$ satisfies the following functional-differential equation:

$$D_t r + r D_x u = 1 \tag{19}$$

under the commutator condition (5).

Notwithstanding a slightly complicated form of the functional-differential equation 19, making use of the differential-algebraic approach devised above, one can easily derive the next supplementing theorem, describing its exact solutions on the corresponding functional manifold $\tilde{\mathcal{M}}_{(3)}$.

Theorem 2. *The following set of functional expressions*

$$\tilde{\mathcal{R}} = \left\{ r_1^{(1)} = \frac{v_x v^3}{6z^3} - \frac{u_x v^2}{2z^2} + \frac{u(uz - v^2)z_x}{6z^3} + \frac{v}{z}, \ r_1^{(2)} = [(xv - u^2/2)/z]_x, \right.$$

$$\left. r_2 = \frac{v_x}{z_x} - \frac{u_x^2}{2z_x}, r_3 = \left(\frac{u_x^3}{6z_x^2} - \frac{u_x v_x}{2z_x^2} + \frac{3}{4z_x} \right) \Big/ \left(\frac{u_x}{z_x} - \frac{v_x^2}{2z_x^2} \right) \right\} \subset \mathcal{R} \tag{20}$$

solves the functional-differential equation (15) on the corresponding manifold $\tilde{\mathcal{M}}_{(3)}$.

The set (20) coincides exactly with that constructed before in [4, 2, 14] by means of completely different techniques.

2.3 The generalized Riemann type hydrodynamical equation: the case N=4

Now consider the generalized Riemann type differential equation (1) at $N = 4$

$$D_t^4 u = 0 \tag{21}$$

on an element $u \in \mathbb{R}\{\{x, t\}\}$ and construct the related invariant Riemann differential ideal $R\{u\} \subset \mathcal{K}\{u\}$ as follows:

$$R\{u\} := \left\{ \lambda^3 \sum_{n \in \mathbb{Z}_+} f_n^{(1)} D_x^n u - \lambda^2 \sum_{n \in \mathbb{Z}_+} f_n^{(2)} D_t D_x^n u + \lambda \sum_{n \in \mathbb{Z}_+} f_n^{(3)} D_t^2 D_x^n u \right. \tag{22}$$

$$\left. - \sum_{n \in \mathbb{Z}_+} f_n^{(4)} D_t^3 D_x^n u \ : \ D_t^4 u = 0, \lambda \in \mathbb{R}, f_n^{(k)} \in \mathcal{K}\{u\}, k = \overline{1,4}, n \in \mathbb{Z}_+ \right\}$$

at a fixed function $u \in \mathbb{R}\{\{x, t\}\}$. The Riemann differential ideal (22), satisfying the D_t-invariance condition, is in this case also maximal. The corresponding kernel $Ker \, D_t \subset R\{u\}$ of the differentiation $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced upon the Riemann differential ideal (22), is given by the following linear differential relationships:

$$D_t f^{(1)} = 0, \ D_t f^{(2)} = \lambda f^{(1)}, \ D_t f^{(3)} = \lambda f^{(2)}, \ D_t f^{(4)} = \lambda f^{(3)}, \tag{23}$$

where $f^{(k)} := f^{(k)}(\lambda) = \sum_{n \in \mathbb{Z}_+} f_n^{(k)} \lambda^n \in \mathcal{K}\{u\}|_{\mathcal{M}_{(4)}} = \mathbb{R}\{\{x, t\}\}$, $k = \overline{1,4}$ and $\lambda \in \mathbb{R}$ is arbitrary. The linear relationships (23) can be easily represented in the space $\mathbb{R}^4\{\{x, t\}\}$ in the following matrix form:

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}, \tag{24}$$

where $f := (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)})^{\mathsf{T}} \in \mathbb{R}^4\{\{x, t\}\}$, and $\lambda \in \mathbb{R}$. Moreover, it is easy to observe that relationships (23) can be equivalently rewritten in the compact scalar form as

$$D_t^4 f^{(4)} = 0, \tag{25}$$

where an element $f_4 \in \mathcal{K}\{u\}$. Thus, now one can construct the invariant Lax differential ideal, isomorphically equivalent to (22), as follows:

$$\begin{aligned} L\{u\} &:= \{g_1 f^{(4)} + g_2 D_t f^{(4)} + g_3 D_t^2 f^{(4)} + g_4 D_t^3 f^{(4)} : D_t^4 f^{(4)} = 0, \\ g_j &\in \mathcal{K}\{u\}, j = \overline{1, 4}\} \subset \mathcal{K}\{u\}, \end{aligned} \tag{26}$$

whose D_x-invariance should be checked separately. The latter gives rise to the representation

$$D_x f = \ell[u, v, w, z; \lambda] f, \quad \ell[u, v, w, z; \lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^2 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix}, \tag{27}$$

where we put, by definition,

$$D_t u := v, D_t v := w, D_t w := z, D_t z := 0, \tag{28}$$

$(u, v, w, z)^{\mathsf{T}} \in \tilde{\mathcal{M}}_{(4)} \subset \mathbb{R}^3\{\{x, t\}\}$, and the mappings $r_j : \tilde{\mathcal{M}}_{(4)} \to \mathbb{R}\{\{x, t\}\}, j = \overline{1, 2}$, satisfy the following functional-differential equations:

$$D_t r_1 + r_1 D_x u = 1, \qquad D_t r_2 + r_2 D_x u = r_1, \tag{29}$$

similar to (15), considered above. The equations (29) possess many different solutions, amongst which are the functional expressions:

$$\begin{aligned} r_1 &= D_x \left(\frac{uw^2}{2z^2} - \frac{vw^3}{3z^3} + \frac{vw^4}{24z^4} + \frac{7w^5}{120z^4} - \frac{w^6}{144z^5} \right), \\ r_2 &= D_x \left(\frac{uw^3}{3z^3} - \frac{vw^4}{6z^4} + \frac{3w^6}{80z^5} + \frac{vw^5}{120z^5} - \frac{w^7}{420z^6} \right). \end{aligned} \tag{30}$$

Whence, we obtain the following proposition.

Proposition 2.3. *The expressions (24) and (27) are the linear matrix representations in the space* $\mathbb{R}^4\{\{x, t\}\}$ *of the differentiations* $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ *and* $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, *respectively, reduced upon the invariant Lax differential ideal* $L\{u\} \subset \mathcal{K}\{u\}$ *given by (13).*

Based now on the representations (24) and (27) one easily constructs a standard Lax type representation, characterizing the integrability of the nonlinear dynamical system

$$\left. \begin{aligned} u_t &= v - uu_x \\ v_t &= w - uv_x \\ w_t &= z - uw_x \\ z_t &= -uz_x \end{aligned} \right\} := K[u, v, w, z], \tag{31}$$

equivalent to the generalized Riemann type hydrodynamical system (21). Namely, the following theorem holds.

Theorem 3. *The dynamical system (31), equivalent to the generalized Riemann type hydrodynamical system (21), possesses the Lax type representation*

$$f_x = \ell[u, v, z, w; \lambda]f, \quad f_t = p(\ell)f, \quad p(\ell) := -u\ell[u, v, w, z; \lambda] + q(\lambda), \tag{32}$$

where $f \in \mathbb{R}^4\{\{x, t\}\}$, $\lambda \in \mathbb{R}$ *is a spectral parameter and*

$$\ell[u, v, w, z; \lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^4 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} \lambda u u_x & -\lambda^2 u v_x & \lambda u w_x & -u z_x \\ \lambda + 4\lambda^4 u & -3\lambda^3 u u_x & 2\lambda^2 u v_x & -\lambda u w_x \\ 10\lambda^5 u r_1 & \lambda - 6\lambda^4 u & 3\lambda^3 u u_x & -\lambda^2 u v_x \\ 20\lambda^6 u r_2 & -10\lambda^5 u r_1 & \lambda + 4\lambda^4 u & -\lambda^3 u u_x \end{pmatrix}, \tag{33}$$

so it is a Lax type integrable dynamical system on the functional manifold $\tilde{\mathcal{M}}_{(4)}$.

The result obtained above can be easily generalized on the case of an arbitrary integer $N \in \mathbb{Z}_+$, thereby proving the Lax type integrability of the whole hierarchy of the Riemann type hydrodynamical equation (1). The related calculations will be presented and discussed in other work. Here we only do the next remark.

Remark 2.2. *The Riemann type hydrodynamical equation (1) as* $N \to \infty$ *can be equivalently rewritten as the following Benney type [1, 10, 15] chain*

$$D_t u^{(n)} = u^{(n+1)}, \quad D_t := \partial/\partial t + u^{(0)}\partial/\partial x, \tag{34}$$

for the suitably constructed moment functions $u^{(n)} := D_t^n u^{(0)}, u^{(0)} := u \in \mathbb{R}\{\{x, t\}\}, n \in \mathbb{Z}_+$.

This aspect of the problem is very interesting and we plan to treat it in detail by means of the differential-geometric tools elsewhere.

3 The differential-algebraic analysis of the Lax type integrability of the Korteweg-de Vries dynamical system

3.1 The differential-algebraic problem setting

We consider the well known Korteweg-de Vries equation in the following (8) differential-algebraic form:

$$D_t u - D_x^3 u = 0, \tag{35}$$

where $u \in \mathcal{K}\{u\}$ and the differentiations $D_t := \partial/\partial t + u\partial/\partial x, D_x := \partial/\partial x$ satisfy the commutation condition (5):

$$[D_x, D_t] = (D_x u)D_x. \tag{36}$$

We will also interpret relationship (35) as a nonlinear dynamical system

$$D_t u = D_{xxx} u \qquad (37)$$

on a suitably chosen functional manifold $\mathcal{M} \subset \mathbb{R}\{\{x, t\}\}$.

Based on the expression (35) we can easily construct a suitable invariant KdV-differential ideal $KdV\{u\} \subset \mathcal{K}\{u\}$ as follows:

$$
\begin{aligned}
KdV\{u\} &:= \{ \sum_{k=\overline{0,2}} \sum_{n \in \mathbb{Z}_+} f_n^{(k)} D_x^k D_t^n u \in \mathcal{K}\{u\} : \ D_t u - D_x^3 u = 0, \\
f_n^{(k)} &\in \mathcal{K}\{u\}, k = \overline{0,2}, n \in \mathbb{Z}_+ \} \subset \mathcal{K}\{u\}. \qquad (38)
\end{aligned}
$$

The ideal (38) proves to be not maximal, that seriously influences on the form of the reduced modulo it representations of derivatives D_x and $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$. As the next step we need to find the kernel $Ker\, D_t \subset KdV\{u\}$ of the differentiation $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced upon the KdV-differential ideal (38). We obtain by means of easy calculations that it is generated by the following differential relationships:

$$
\begin{aligned}
D_t f^{(0)} &= -\lambda f^{(0)}, \quad D_t f^{(2)} = -\lambda f^{(2)} + 2 f^{(2)} D_x u, \\
D_t f^{(1)} &= -\lambda f^{(1)} + f^{(1)} D_x u + f^{(2)} D_{xx} u, \qquad (39)
\end{aligned}
$$

where, by definition, $f^{(k)} := f^{(k)}(\lambda) = \sum_{n \in \mathbb{Z}_+} f_n^{(k)} \lambda^n \in \mathcal{K}\{u\}|_{\mathcal{M}} = \mathbb{R}\{\{x, t\}\}, k = \overline{0,2}$, and $\lambda \in \mathbb{R}$ is an arbitrary parameter. Based on the relationships (39) the following proposition holds.

Proposition 3.1. *The differential relationships (39) can be equivalently rewritten in the following linear matrix form:*

$$
D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} D_x u - \lambda & D_{xx} u \\ 0 & 2 D_x u - \lambda \end{pmatrix}, \qquad (40)
$$

where $f := (f_1, f_2)^\intercal \in \mathbb{R}^2\{\{x, t\}\}$, $\lambda \in \mathbb{R}$, giving rise to the corresponding linear matrix representation in the space $\mathbb{R}^2\{\{x, t\}\}$ of the differentiation $D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$, reduced upon the KdV-differential ideal (38).

3.2   The Lax type representation

Now, making use of the matrix differential relationship (40), we can construct the Lax differential ideal related to the ideal (38)

$$
\begin{aligned}
L\{u\} &:= \{ < g, f >_{\mathbb{E}^2} \in \mathcal{K}\{u\} : D_t f = q(\lambda) f, \\
f, g &\in \mathcal{K}^2\{u\} \} \subset \mathcal{K}\{u\}, \qquad (41)
\end{aligned}
$$

where $< \cdot, \cdot >_{\mathbb{E}^2}$ denotes the standard scalar product in the Euclidean real space $\mathbb{E}^2$. Since the Lax differential ideal (41) is, by construction, D_t-invariant and isomorphic to the D_t-

and D_x-invariant KdV-differential ideal (38), it is necessary to check its D_x-invariance. As a result of this condition the following differential relationship

$$D_x f = \ell[u;\lambda]f, \; \ell[u;\lambda] := \begin{pmatrix} D_x\tilde{a} & 2D_{xx}\tilde{a}. \\ -1 & -D_x\tilde{a} \end{pmatrix} \tag{42}$$

holds, where the mapping $\tilde{a} : \mathcal{M} \to \mathbb{R}\{\{x,t\}\}$ satisfies the functional-differential relationships

$$D_t\tilde{a} = 1, D_t u - D_x^3 u = 0, \tag{43}$$

and the matrix $\ell := \ell[u;\lambda] : \mathbb{R}^2\{\{x,t\}\} \to \mathbb{R}^2\{\{x,t\}\}$ satisfies for all $\lambda \in \mathbb{R}$ the determining functional-differential equation

$$D_t\ell + \ell D_x u = [q(\lambda), \ell] + D_x q(\lambda), \tag{44}$$

generalizing the similar equation (16). The result obtained above we formulate as the following proposition.

Theorem 4. *The derivatives $D_t : \mathbb{R}\{\{x,t\}\} \to \mathbb{R}\{\{x,t\}\}$ and $D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\}$ of the differential ring $\mathcal{K}\{u\}$, reduced upon the Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$, which isomorphic to the KdV-differential ideal $KdV\{u\} \subset \mathcal{K}\{u\}$, allow the compatible Lax type representation (generated by the invariant Lax differential ideal $L\{u\} \subset \mathcal{K}\{u\}$)*

$$\begin{aligned} D_t f &= q(\lambda)f, \; q(\lambda) := \begin{pmatrix} D_x u - \lambda & D_{xx} u \\ 0 & 2D_x u - \lambda \end{pmatrix}, \\ D_x f &= \ell[u;\lambda]f, \; \ell[u;\lambda] := \begin{pmatrix} D_x\tilde{a} & 2D_{xx}\tilde{a}. \\ -1 & -D_x\tilde{a} \end{pmatrix}, \end{aligned} \tag{45}$$

where the mapping $\tilde{a} : \mathcal{M} \to \mathbb{R}\{\{x,t\}\}$ satisfies the functional-differential relationships (43), $f \in \mathbb{R}^2\{\{x,t\}\}$ and $\lambda \in \mathbb{R}$.

It is interesting to mention that the Lax type representation (45) strongly differs from that given by the well known [12] classical expressions

$$\begin{aligned} D_t f &= q_{cl}(\lambda)f, \; q_{cl}(\lambda) := \begin{pmatrix} D_x u/6 & -(2u/3 - 4\lambda) \\ D_{xx} u/6 - (u/6 - \lambda)\times & \\ \times(2u/3 - 4\lambda) & -11D_x u/6 \end{pmatrix}, \\ D_x f &= \ell_{cl}[u;\lambda]f, \; \ell_{cl}[u;\lambda] := \begin{pmatrix} 0 & 1 \\ u/6 - \lambda & 0 \end{pmatrix}, \end{aligned} \tag{46}$$

where, as above, the following functional-differential equation (equivalent to the nonlinear dynamical system (37) on the functional manifold $\mathcal{M}$)

$$D_t\ell_{cl} + \ell_{cl}D_x u = [q_{cl}(\lambda), \ell_{cl}] + D_x q_{cl}(\lambda), \tag{47}$$

holds for any $\lambda \in \mathbb{R}$. This fact, as we suspect, is related with the existence of different D_t-invarinat KdV-differential ideals of form (38), which are not maximal. Thus, a problem of

constructing a suitable KdV-differential ideal $KdV\{u\} \subset \mathcal{K}\{u\}$ generating the corresponding invariant Lax type differential ideal $L\{u\} \subset \mathcal{K}\{u\}$, invariant with respect to the differential representations (46), naturally arises, and we expect to treat this in detail elsewhere. There also is a very interesting problem of the differential-algebraic analysis of the related symplectic structures on the functional manifold $\mathcal{M}$, with respect to which the dynamical system (37) is Hamiltonian and suitably integrable. Here we need also to mention a very interesting work [22], where the integrability structure of the Korteweg-de Vries equation was analyzed from the differential-algebraic point of view.

4   Conclusion

The results presented provide convincing evidence that the differential-algebraic tools, when applied to a given set of differential relationships based on the derivatives D_t and D_x : $\mathcal{K}\{u\} \to \mathcal{K}\{u\}$ in the differential ring $\mathcal{K}\{u\}$ and parameterized by a fixed element $u \in \mathcal{K}\{u\}$, make it possible to construct the corresponding Lax type representation as that realizing the linear matrix representations of the derivatives reduced modulo the corresponding invariant Riemann differentail ideal. This scheme was elaborated in detail for the generalized Riemann type differential equation (1) and for the classical Korteweg-de Vries equation (37). As these equations are equivalent to the corresponding Hamiltonian systems with respect to suitable symplectic structures, this aspect presents a very interesting problem from the differential-algebraic point of view, which we plan to study in the near future.

5   Acknowledgements

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Прикарпатський А.К., Артемович О.Д., Попович З., Павлов М.В. *Алгебраїчні структури типу Рімана та диференціально-алгебраїчний аналіз їх інтегровності* // Карпатські математичні публікації. — 2010. — Т.2, №1. — С. 96–108.

Розвивається диференціально-алгебраїчний підхід до інтегровності узагальнених рівнянь типу Рімана. Аналізується структура диференціювань та асоційованих з ними інваріантних диференціальних ідеалів. Підхід також застосовується до вивчення інтегровності за Лаксом відомої динамічної системи Кортевега-де Фріза.

Прикарпатский А.К., Артемович О.Д., Попович З., Павлов М.В. *Алгебраические структуры типа Римана и дифференциально-алгебраический анализ их интегрируемости* // Карпатские математические публикации. — 2010. — Т.2, №1. — С. 96–108.

Развивается дифференциально-алгебраический подход к изучению интегрируемости обобщенных уравнений типа Римана. Анализируется структура дифференцирований и ассоциированных с ними инвариантных дифференциальных идеалов. Подход также применяется к исследованию интегрируемости известной динамической системы Кортевега-де Фриза.