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ON TWO LONG STANDING OPEN PROBLEMS ON L_p-SPACES

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The present note was written during the preparation of the talk at the International Conference dedicated to 70-th anniversary of Professor O. Lopushansky, September 16-19, 2019, Ivano-Frankivsk, Ukraine. We focus on two long standing open problems. The first one, due to Lindenstrauss and Rosenthal (1969), asks of whether every complemented infinite dimensional subspace of L_1 is isomorphic to either L_1 or ℓ_1. The second problem was posed by Enflo and Rosenthal in 1973: does there exist a nonseparable space $L_p(\mu)$ with finite atomless μ and $1 < p < \infty$, $p \neq 2$, having an unconditional basis? We analyze partial results and discuss on some natural ideas to solve these problems.

Key words and phrases: L_p-spaces, complemented subspace, unconditional basis.

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1 Introduction

Investigation of the geometry of Lebesgue spaces $L_p := L_p[0,1]$ has long and rich history (see [3]) due to famous mathematicians: D.E. Alspach, S. Banach, J. Bourgain, D.L. Burkholder, L.E. Dor, P. Enflo, W.B. Johnson, M.I. Kadets, N. Kalton, J. Lindenstrauss, B. Maurey, E. Odell, R.E.A.C. Paley, A. Pełczyński, H.P. Rosenthal, G. Schechtman, T.W. Starbird, S. Szarek, M. Talagrand, L. Tzafriri and others. More is known on the isomorphic structure of these classical spaces. Isomorphic embeddability of $L_r(\nu)$ into L_p is completely known. We use the notation $X \hookrightarrow Y$ to express that X embeds isomorphically into Y, and $X \simeq Y$ means that the Banach spaces X and Y are isomorphic. The relation $\ell_p \hookrightarrow L_p$, which is easily seen, was first noted by S. Banach [4, p. 175]. The embedding $\ell_2 \hookrightarrow L_p$ follows from Khintchin's inequality [30, p. 66]. It is not hard to see that $\ell_p \not\hookrightarrow L_2$ for $p \neq 2$ (for the proof, see [4, p. 175]). The relation $\ell_r \not\hookrightarrow L_p$ for $2 < p < r$ and $1 \leq r < p < 2$ was proved by S. Banach [4, p. 175]. Paley's results [37] imply $\ell_r \not\hookrightarrow L_p$ for $1 \leq r < 2 < p$, $2 < r < p$ and $1 \leq p < 2 < r$.

A special case is $1 \leq p < r < 2$, where isometric embeddings of L_r into L_p are possible. First it was proved by P. Levy [25] that ℓ_r is *finitely representable*[1] in L_p if $1 \leq p < r < 2$. Later M.I. Kadets proved that $\ell_r \hookrightarrow L_p$ for $1 \leq p < r < 2$ [20]. Then the latter result was strengthen to the embedding $L_r \hookrightarrow L_p$ by J. Bretagnolle, D. Dacunha-Castelle and J. L. Krivine [9] and independently by J. Lindenstrauss J. and A. Pełczyński [27], who proved more: if a Banach space X is finitely representable in L_p then $X \hookrightarrow L_p$.

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[1] A Banach space X is said to be finitely representable in a Banach space Y if for every $\varepsilon > 0$ and every finite dimensional subspace F of X there exists a subspace G of Y of the same dimension such that $d(F,G) < 1 + \varepsilon$, where $d(F,G)$ denotes the Banach-Mazur distance between F and G.

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As we see, the properties of the spaces L_p are different for the cases $p < 2$ and $p > 2$. Moreover, if $2 < p < \infty$ then every subspace of L_p possesses the following properties:

– either is isomorphic to a Hilbert space or contains a complemented subspace isomorphic to ℓ_p [21];

– either contains a subspace isomorphic to ℓ_2 or embeds isomorphically into ℓ_p [19].

On the other hand, if $1 \le p < 2$ then every subspace of L_p either contains a complemented subspace isomorphic to ℓ_p or embeds isomorphically into L_r for some $p < r \le 2$ [44].

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2 COMPLEMENTED SUBSPACES OF L_p

2.1 As p goes to 1, the complementability properties of subspaces of L_p, $p \neq 2$, get worse

By Khintchin's inequality, the closed linear span R of the Rademacher system in L_p is isomorphic to ℓ_2 and is actually independent on p, as a set. Remark that R is complemented in L_p for $1 < p < \infty$ [30, p. 66] and is uncomplemented in L_1, as well as any other subspace of L_1 isomorphic to ℓ_2 [38]. So, it became interesting, whether there exists an uncomplemented subspace of L_p isomorphic to ℓ_2 for $p > 1$. If $2 < p < \infty$ then every subspace X of L_p isomorphic to ℓ_2 is complemented, and, moreover, the L_p- and L_2-norms[2] on X are equivalent [21]. To the contrast, if $1 < p < 2$ then there exists an uncomplemented subspace of L_p isomorphic to ℓ_2 (first it was proved for $1 < p < 4/3$ in [42] and then for the rest of values in [5]).

It is clear that L_p contains a complemented subspace isomorphic to ℓ_p. If $1 \le p < \infty$, $p \neq 2$, then there is an uncomplemented subspace of L_p isomorphic to ℓ_p, and hence, it is not difficult to show that there is an uncomplemented subspace of L_p isomorphic to L_p itself (first it was proved for $2 < p < \infty$ and $1 < p < 4/3$ in [43], then in a different way for all $1 < p < 2$ in [5], and finally for $p = 1$ in [6]).

2.2 Primarity of L_p and Enflo operators

By the famous Enflo theorem, if $L_p = X \oplus Y$, $1 \le p < \infty$, is a decomposition into mutually complemented subspaces, then at least one of the subspaces X, Y is isomorphic to L_p (first it was announced by P. Enflo; then B. Maurey [34] published a proof, see also [2] for all p, [14] for $p = 1$ and [31, p. 179] for a generalization to rearrangement invariant spaces). This nice property of the spaces L_p is called the *primarity.*

Let X, Y be Banach spaces. Denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from X to Y, and write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. Recall that an operator $T \in \mathcal{L}(X, Y)$ is said to *fix a copy of a Banach space Z*, if there exists a subspace X_1 of X isomorphic to Z such that the restriction $T|_{X_1}$ of T to X_1 is an into isomorphism. An operator $T \in \mathcal{L}(L_p, Y)$, $1 \le p < \infty$, is called an *Enflo operator* provided T fixes a copy of L_p. Note that every Enflo operator $T \in \mathcal{L}(L_p)$ fixes a complemented copy of L_p, that is, there is a complemented subspace X_1 of X isomorphic to L_p such that the restriction $T|_{X_1}$ is an into isomorphism, because every subspace X of L_p, which is isomorphic to L_p, contains a further subspace $Y \subseteq X$ isomorphic to

[2] which are well defined for these values of p

L_p and complemented in L_p (see [18, p. 239] for $p > 1$ and [14] for $p = 1$). The Enflo theorem implies that, if the identity operator Id on L_p is a sum of two projections $Id = P + Q$, then at least one of the projections P, Q is an Enflo operator. Moreover, the range of a projection P on L_p is isomorphic to L_p if and only if P is an Enflo operator (to prove, use the mentioned above result from [18] and Pełczyński's decomposition method [31, p. 54]).

2.3 Isomorphic types of complemented subspaces of L_p

How many do there exist pairwise non-isomorphic complemented subspaces of L_p for $1 \le p < \infty$, $p \neq 2$? If $p > 1$ then there are obviously the following pairwise non-isomorphic Banach spaces isomorphic to complemented subspaces of L_p:

$$L_p, \ \ell_p, \ \ell_2, \ \ell_p \oplus \ell_2, \ \Big(\bigoplus_{n=1}^{\infty} \ell_2\Big)_p.$$

Further finitely many examples, different from the above obvious ones, was obtained by H.P. Rosenthal in [43]. Later G. Schechtman provided infinitely many pairwise non-isomorphic examples in [48], and then J. Bourgain, H.P. Rosenthal and G. Schechtman constructed uncountably many pairwise non-isomorphic complemented subspaces of L_p for $1 < p < \infty$, $p \neq 2$ in [8] (it is unknown, whether there exists continuum such subspaces).

The exceptional case is $p = 1$: there are only two known obvious examples of pairwise non-isomorphic infinite dimensional subspaces of L_1, they are L_1 itself and ℓ_1.

Problem 1 (Lindenstrauss and Rosenthal, 1969, [29]). *Is every complemented infinite dimensional subspace of L_1 isomorphic to either L_1 or ℓ_1?*

2.4 Progress in the solution of Problem 1

The following assertions have been established for an arbitrary complemented subspace E of L_1.

Theorem 1 (Pełczyński, 1960, [38]). *E contains a subspace isomorphic to ℓ_1 and complemented in L_1.*

Theorem 2 (Lindenstrauss, Pełczyński, 1968, [27]). *If E has an unconditional basis then E is isomorphic to ℓ_1.*

Recall that *the Radon-Nikodým property* (*RNP*) for a Banach space X means that for every finite measure space (Ω, Σ, μ) and every μ-continuous X-valued measure $G : \Sigma \to X$ of bounded variation there exists $g \in L_1(\mu, X)$ such that $G(A) = \int_A g\, d\mu$ for all $A \in \Sigma$. One can show that the characteristic function $G(A) = \mathbf{1}_A$ is an example of L_1-valued such measure for which the function g does not exist [12, p. 61]; thus, L_1 does not have the RNP. However, ℓ_1 has the RNP (this can be proved directly, using the Radon-Nikodým theorem for separate coordinates [12, p. 64]).

Theorem 3 (Lewis, Stegall, 1973, [26]). *If E has the RNP then E is isomorphic to ℓ_1.*

A Banach space X is said to have the *Schur property* if the weak convergence of a sequence in X implies its norm convergence. It is well known that ℓ_1 has the Schur property.

Theorem 4 (Rosenthal, 1975, [45]). *If E does not have the Schur property then ℓ_2 embeds into E.*

Theorem 5 (Enflo, Starbird, 1979, [14]). *If E contains a subspace isomorphic to L_1 then E is itself isomorphic to L_1.*

Simultaneously, W.B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri [18] obtained the same result as Theorem 5 asserts for L_p with $1 < p < \infty$.

The next result strengthens Theorem 4.

Theorem 6 (Bourgain, 1980, [7]). *If E does not have the Schur property then $(\bigoplus_{n=1}^{\infty} \ell_2)_1$ embeds into E.*

There is a natural idea to solve Problem 1. Obviously, **the hypothesis that every complemented infinite dimensional subspace of L_1 is isomorphic to either L_1 or ℓ_1, is equivalent to the hypothesis that the following two climes hold true**.

Let E be an infinite dimensional complemented subspace of L_1.

Claim 1. *If E has the Schur property then E is isomorphic to ℓ_1.*

Claim 2. *If E does not have the Schur property then E is isomorphic to L_1.*

As to the best of our knowledge, there is no information about Claim 1 in the literature. Remark that there is no direct way to prove Claim 1 without taking into account peculiarity of L_1, because there exists a Banach space with the Schur property but without the RNP, and so, not isomorphic to ℓ_1 (see J. Hagler [17]).

However, Claim 2 has been considered by different mathematicians as a weak version of Problem 1 in the sense that a positive solution to Problem 1 implies a positive answer to Problem 2.

Problem 2 ([45], [14] and [7]). *Must a non-Dunford-Pettis projection $P \in \mathcal{L}(L_1)$ be an Enflo operator? Equivalently, whether each non-Schur complemented subspace of L_1 is isomorphic to L_1?*

The most unclear thing concerning Problem 2 is how to use the information that P is a projection, not just a continuous linear operator. H.P. Rosenthal constructed an example of a non-Dunford-Pettis operator $T \in \mathcal{L}(L_1)$ failing to be an Enflo operator [45]. This is the so-called biased coin convolution operator. To explain the details, recall that the *Rademacher system* is defined by $r_n(t) = \text{sign}\,\sin(2^{n+1}\pi t)$ for each $n \in \mathbb{N}$ and $t \in [0,1]$. Denote by $\mathbb{N}^{<\omega}$ the set of all finite subsets of $\mathbb{N}$. The *Walsh system* $(w_I)_{I \in \mathbb{N}^{<\omega}}$ is defined by setting $w_I = \prod_{i \in I} r_i$, where $(r_n)_{n=1}^{\infty}$ is the Rademacher system (in particular, $w_\varnothing = \mathbf{1}$, by convention). The Walsh system with respect to the lexicographical order $w_\varnothing, w_{\{1\}}, w_{\{2\}}, w_{\{1,2\}}, w_{\{3\}}, w_{\{1,3\}}, w_{\{2,3\}}, w_{\{1,2,3\}}, \ldots$ is a Schauder basis of L_p for $1 < p < \infty$, an orthonormal basis of L_2, a conditional basis of L_p for $p \neq 2$, and a Markushevich basis of L_1.

Theorem 7 (H.P. Rosenthal, [45]). *There is $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is an operator $R_\varepsilon \in \mathcal{L}(L_1)$ possessing the equality $R_\varepsilon w_I = \varepsilon^{|I|} w_I$ for all $I \in \mathbb{N}^{<\omega}$, where $|I|$ is the cardinality of I.*

The operator R_ε is called the ε-biased coin convolution operator. Since $R_\varepsilon r_n = \varepsilon r_n$ for all $n \in \mathbb{N}$, the operator R_ε is not Dunford-Pettis. H.P. Rosenthal proved in [45] that R_ε is not an Enflo operator.

2.5 All operators on L_1 are regular

Recall some information. Let E, F be vector lattices. An operator $T : E \to F$ is called *positive* if $T(E^+) \subseteq F^+$, and $T : E \to F$ is called *regular* if T equals a difference of two positive operators. Obviously, every positive (and hence, every regular) operator $T : E \to F$ is *order bounded*, that is, T sends order bounded subsets of E to order bounded subsets of F. Two elements $x, y \in E$ are said to be disjoint (write $x \perp y$) if $|x| \wedge |y| = 0$. The notation $x = \bigsqcup_{k=1}^n x_k$ means that $x = \sum_{k=1}^n x_k$ and $x_i \perp x_j$ for $i \neq j$.

It is an amazing and seldom used fact on operators on L_1 that all of them are regular [47, p. 232]. More precisely, every operator $T \in \mathcal{L}(L_1)$ admits the representation $T = T^+ - T^-$, where for every $x \in L_1^+$ one has

$$T^+ x = \sup\Big\{ \sum_{k=1}^m T x_k : \ x = \bigsqcup_{k=1}^n x_k, \ n \in \mathbb{N} \Big\}.$$

As a consequence, we obtain that for any operator $T \in \mathcal{L}(L_1)$ the modulus $|T| = T^+ + T^- \in \mathcal{L}(L_1)$ exists and could be defined by setting for every $x \in E^+$

$$|T|\, x = \sup\Big\{ \sum_{k=1}^n \big|T x_i\big| : \ x = \sum_{k=1}^n x_k, \ x_k \in E^+, \ n \in \mathbb{N} \Big\}.$$

Moreover, $\| |T| \| = \|T\|$ for every $T \in \mathcal{L}(L_1)$ [47, p. 232].

As was noted by H.P. Rosenthal [46], the regularity of operators $T \in \mathcal{L}\big(L_1(\mu), L_1(\nu)\big)$ is a consequence of the following Grothendieck's inequality [16, Corollaire, p. 67]: given any $f_1, \ldots, f_1 \in L_1(\mu)$, one has

$$\int_{\Omega_\nu} \max_i |T f_i| \, d\nu \le \|T\| \int_{\Omega_\mu} \max_i |T f_i| \, d\mu.$$

A very useful development of Grothendieck's inequality is M. Lévy's extension theorem (see [24]) asserting that, for every subspace X of $L_1(\mu)$ every order bounded operator $T \in \mathcal{L}(X, L_1(\nu))$ has an extension to some operator $\widehat{T} \in \mathcal{L}(L_1(\mu), L_1(\nu))$, which is therefore order bounded as well. The latter fact was then generalized to regular operators from $L_p(\mu)$ to $L_p(\nu)$ for $1 \le p \le \infty$ by G. Pisier in [40].

The regularity of all operators on L_1 in fact means that there are few operators on L_1, only regular ones. This explains why common subspaces of all L_p (like the closed linear span of the Rademacher system), which are complemented in L_p for $p > 1$ becomes uncomplemented in L_1: they are complemented in L_p by means of non-regular projections. The same reason makes the Haar system a conditional basis in L_1. This argument made the authors of [33] and [41, Problem 10.45] to generalize Problem1 as follows. We say that a subspace X of a Banach lattice is regularly complemented if there is a regular projection of E onto X.

Problem 3. *Let $1 \le p < \infty$, $p \neq 2$. Is every regularly complemented subspace of L_p isomorphic to either ℓ_p or L_p?*

2.6 Complemented subspaces of L_p for $0 < p < 1$

Consider now the quasi-Banach spaces L_p for $0 < p < 1$. The list of known isomorphic types of complemented subspaces of these spaces becomes smaller by one space, namely by ℓ_p, because L_p has trivial dual and hence cannot have a complemented subspace with nontrivial dual, like those that are isomorphic to ℓ_p. So, the problem is as follows.

Problem 4. *Let $0 \le p < 1$. Is every complemented subspace of L_p isomorphic to L_p?*

This problem has been systematically studied by N.J. Kalton in a number of papers. The best progress is Kalton's theorem, which asserts that, if there exists a complemented subspace of L_p not isomorphic to L_p, then at most one, up to an isomorphism [22].

3 UNCONDITIONAL BASES IN $L_p(\mu)$

3.1 Preliminary information

For convenience of the reader, we recall some necessary information on bases [1, 30]. A sequence $(x_n)_{n=1}^\infty$ of elements of a Banach space X is called a *Schauder basis* (or just a *basis*) of X if for every $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that

$$x = \sum_{k=1}^\infty a_k x_k. \tag{1}$$

A sequence in X, which is a basis in its closed linear span, is called a *basic sequence*. The partial sums $P_n x = \sum_{k=1}^n a_k x_k$ of the expansion (1) are linear bounded projections on X with $K := \sup_n \|P_n\| < \infty$, and the number K is called the *basis constant* of $(x_n)_{n=1}^\infty$. In particular, the coefficients $x_k^*(x) := a_k$ of the expansion (1) are elements of X^* with $\sup_n \|x_n\| \|x_n^*\| \le 2K$ and are called the *biorthogonal functionals* to $(x_n)_{n=1}^\infty$. The best possible basis constant is 1; a basis with basis constant 1 is said to be *monotone*. The biorthogonal functionals $(x_n^*)_{n=1}^\infty$ form a basic sequence in X^* with the same basis constant K. A basis $(x_n)_{n=1}^\infty$ of X is called *unconditional* if for every $x \in X$ the series $x = \sum_{k=1}^\infty x_k^*(x) x_k$ converges unconditionally; otherwise the basis is said to be conditional. If $(x_n)_{n=1}^\infty$ is unconditional then for every sequence of signs $\Theta = (\theta_n)_{n=1}^\infty$, $\theta_n \in \{-1, 1\}$, and every $x \in X$ the series $T_\Theta x := \sum_{n=1}^\infty \theta_n x_n^*(x) x_n$ converges and T_Θ is a linear bounded operator. Moreover, $M := \sup_\Theta \|T_\Theta\| < \infty$. The number M is called the *unconditional constant* of the unconditional basis $(x_n)_{n=1}^\infty$.

Let $(x_n)_{n=1}^\infty$ be a basic sequence in X, $(a_n)_{n=1}^\infty$ be a sequence of scalars and $0 \le k_1 < k_2 < \dots$ be integers. A sequence $(u_n)_{n=1}^\infty$ of nonzero elements of X of the form

$$u_n = \sum_{i=k_n+1}^{k_{n+1}} a_i x_i$$

is called a *block basis* of $(x_n)_{n=1}^\infty$. It is not hard to see that $(u_n)_{n=1}^\infty$ is a basic sequence itself, the basis constant of which does not exceed that of $(x_n)_{n=1}^\infty$. Two basic sequences $(x_n)_{n=1}^\infty$ in X and $(y_n)_{n=1}^\infty$ in Y are called *λ-equivalent* if there exists an isomorphism $T : [x_n] \to [y_n]$ between the closed linear spans of these systems with $Tx_n = y_n$ for all n such that $\|T\| \|T^{-1}\| \le \lambda$. Basic sequences are said to be *equivalent* if they are λ-equivalent for some $\lambda \in [1, +\infty)$. Using the Closed Graph theorem, one can easily show that basic sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are equivalent if and only if for every sequence of scalars $(a_n)_{n=1}^\infty$ the convergence of the series $\sum_{n=1}^\infty a_n x_n$ and $\sum_{n=1}^\infty a_n x_n$ are equivalent. It is clear that if one of two λ-equivalent basic sequences is unconditional then the other one is unconditional as well, and the basic (unconditional) constants K_1, K_2 are estimated as follows: $\lambda^{-1} K_1 \le K_2 \le \lambda K_1$.

3.2 The Haar system in L_p

Define the *dyadic intervals* by setting $I_n^k = [\frac{k-1}{2^n}, \frac{k}{2^n})$ for $n = 0, 1, \dots$ and $k = 1, \dots, 2^n$. The L_∞-normalized *Haar system* is the following sequence in L_∞: $\overline{h}_1 = \mathbf{1}$ and

$$\overline{h}_{2^n+k} = \mathbf{1}_{I_{n+1}^{2k-1}} - \mathbf{1}_{I_{n+1}^{2k}} \tag{2}$$

for $n = 0, 1, 2, \dots$ and $k = 1, 2, \dots, 2^n$ (by $\mathbf{1}_A$ we denote the characteristic function of a set A). The Haar system is a monotone basis of every space L_p with $1 \le p < \infty$ [30, p. 3], and an unconditional basis of L_p for any $1 < p < \infty$ [31, p. 155] (the first fact one can obtain using a criterium of bases, and the second fact is a deep result of Paley [36] (1932), the proof of which was then simplified by Burkholder [11] (1985)). The unconditional constant of the Haar system in L_p equals $K_p = \max\{p, q\} - 1$, where $1/p + 1/q = 1$ [10].

The Haar system possesses the following useful property, called the *precise reproducibility* [28], [31, p. 158]: for every isomorphic embedding $T : L_p \to X$, $1 \le p < \infty$, where X is a Banach space with a basis $(x_n)_{n=1}^\infty$, and every $\varepsilon > 0$ there is a block basis $(u_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$, which is $(\|T\|\|T^{-1}\| + \varepsilon)$-equivalent to the Haar system in L_p. This gives that the Haar system is the "best" basis: once we have an unconditional basis in L_p, the Haar system is unconditional as well, and its unconditional constant is the minimal possible one. Since the Haar system is a conditional basis in L_1 [31, p. 156], we obtain that L_1 cannot be isomorphically embedded in a Banach space with an unconditional basis (initially this was proved by A. Pełczyński [39]).

3.3 Nonseparable $L_p(\mu)$-spaces

There is a nice complete isomorphic classification of the spaces $L_p(\mu)$ over finite atomless measure spaces (Ω, Σ, μ). A canonical representative of measure spaces (Ω, Σ, μ) with[3] $\dim L_p(\mu) = \aleph_\alpha$ for $0 < p < \infty$ is $\big(\{-1,1\}^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha}\big)$, where ω_α is the cardinal of cardinality $\aleph_\alpha$, Σ_{ω_α} is the Borel σ-algebra of subsets of $\{-1,1\}^{\omega_\alpha}$ endowed with the Tykhonov topology on the power of the discrete two-point space $\{-1,1\}$, and μ_{ω_α} is the corresponding power of the measure μ_0 on the subsets of $\{-1,1\}$ defined by $\mu_0\{-1\} = \mu_0\{1\} = 1/2$. In other words, μ_{ω_α} is the Haar measure on the compact Abelian group $\{-1,1\}^{\omega_\alpha}$ with the point-wise product. By the famous Maharam theorem (see [32] for the original paper, and [15,23] for different proofs), every finite atomless measure space (Ω, Σ, μ) is isomorphic (in the sense of measure spaces) to a unique (up to a permutation of summands) direct sum of the measure spaces $\bigoplus_{\alpha \in \mathcal{A}} \big(\{-1,1\}^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \varepsilon_\alpha \mu_{\omega_\alpha}\big)$, where $\mathcal{A}$ is an at most countable set of ordinals, called the *Maharam invariants* of (Ω, Σ, μ), and $\varepsilon_\alpha > 0$ are weights with $\sum_{\alpha \in \mathcal{A}} \varepsilon_\alpha = \mu(\Omega)$. The Lebesgue measure space $\big([0,1], \Sigma, \lambda\big)$, where λ is the Lebesgue measure on the Borel σ-algebra Σ of subsets of $[0,1]$, is isomorphic to $\big(\{-1,1\}^{\omega_0}, \Sigma_{\omega_0}, \mu_{\omega_0}\big)$. As a consequence, we obtain that every $L_p(\mu)$-space over a finite atomless measure μ with $0 < p \le \infty$ is isometrically isomorphic to the ℓ_p-sum $\big(\sum_{\alpha \in \mathcal{A}} L_p\{-1,1\}^{\omega_\alpha}\big)_p$.

A (not ordered) family $(x_i)_{i \in I}$ of elements of a (non-separable) Banach space X is called an *unconditional basis* of X if every $x \in X$ admits a unique representation $x = \sum_{i \in I} a_i x_i$, where the set of all indices $i \in I$ with $a_i \neq 0$ is at most countable, and the series converges unconditionally. One can show directly, that a family $(x_i)_{i \in I}$ with dense linear span is an unconditional basis of X if and only if every its countable subfamily is an unconditional basic sequence. If this is the case then the unconditional constants of countable subfamilies are bounded from

[3] By $\dim X$ we mean the smallest cardinality of subsets of X with dense linear span.

above, and their supremum equals the unconditional constant of the entire family, which is defined similarly.

P. Enflo and H.P. Rosenthal (1973) [13] proved that, if $\dim L_p(\mu) \geq \aleph_{\omega_0}$, where μ is finite atomless and $1 < p < \infty$, $p \neq 2$, then $L_p(\mu)$ does not embed isomorphically into a Banach space with an unconditional basis. They proved preliminarily that, for any $n \in \mathbb{N}$, assuming the isomorphic embedding $T : L_p\{-1,1\}^{\omega_n} \to X$ into a Banach space X with an unconditional basis $(x_i)_{i\in I}$, the finite Walsh system $(w_J)_{|J|\leq n}$ is $\|T\|\|T^{-1}\|$-reproducible in $(x_i)_{i\in I}$, even more, $\|T\|\|T^{-1}\|$-equivalent to a suitable block basis of $(x_i)_{i\in I}$. As a consequence, the unconditional constant M_n of $(w_J)_{|J|\leq n}$ does not exceed $M\|T\|\|T^{-1}\|$, where M is the unconditional constant of $(x_i)_{i\in I}$. Since for every $n \in \mathbb{N}$ the space $L_p\{-1,1\}^{\omega_n}$ isometrically embeds into $L_p(\mu)$, it then remained to show that $M_n \to \infty$ as $n \to \infty$, which is true. Unfortunately, their method could not give more, remaining the following problem to be open.

Problem 5 (P. Enflo and H.P. Rosenthal, 1973, [13])**.** *Let $1 \leq p < \infty$, $p \neq 2$, and let (Ω, Σ, μ) be a finite atomless measure space with $\aleph_0 < \dim L_p(\mu) < \aleph_{\omega_0}$. Is there an unconditional basis of $L_p(\mu)$?*

Below we describe two different possible ideas to solve this problem.

3.4 The Olevskii system

In 1966 A.M. Olevskii constructed a system of functions on $[0,1]$, which is a basis of L_1 containing the Rademacher system as a part [35]. This system, called in the literature the Olevskii system, is a conditional basis in L_p for $p \neq 2$, a result of E.M. Semenov [49]. If one tries to prove that $L_p\{-1,1\}^{\omega_1}$ (and therefore, $L_p\{-1,1\}^{\omega_n}$ for each $n \geq 1$) has no unconditional basis, then it would be enough to prove that the Olevskii system is reproducible in any unconditional basis of $L_p\{-1,1\}^{\omega_1}$. Let us present an author's description of the Olevskii system, which may be convenient for this purpose.

First, we represent the Haar system (2), collected by bunches, via the Rademacher system $(r_n)_{n=1}^{\infty}$ as follows:

$$\begin{aligned}
&\text{bunch } 1: \ \mathbf{1},\\
&\text{bunch } 2: \ r_1,\\
&\text{bunch } 3: \ \frac{r_1+1}{2}\cdot r_2, \ \frac{r_1-1}{2}\cdot r_2,\\
&\text{bunch } 4: \ \frac{r_1+1}{2}\cdot\frac{r_2+1}{2}\cdot r_3, \ \frac{r_1+1}{2}\cdot\frac{r_2-1}{2}\cdot r_3, \ \frac{r_1-1}{2}\cdot\frac{r_2+1}{2}\cdot r_3, \ \frac{r_1-1}{2}\cdot\frac{r_2-1}{2}\cdot r_3,\\
&\dots
\end{aligned}$$

The Olevskii system can be constructed using the following scheme. First, we take the function $\mathbf{1}$. Then, to obtain the $(n+1)$-th Olevskii bunch, we multiply the beginning of the Haar system including its n-th bunch by r_n.

$$\begin{aligned}
&\text{bunch } 1: \ \mathbf{1},\\
&\text{bunch } 2: \ r_1,\\
&\text{bunch } 3: \ r_2, \quad r_1\cdot r_2,
\end{aligned}$$

$$\text{bunch 4}: \quad r_3, \quad r_1 \cdot r_3, \quad \frac{r_1+1}{2} \cdot r_2 \cdot r_3, \quad \frac{r_1-1}{2} \cdot r_2 \cdot r_3,$$

$$\text{bunch 5}: \quad r_4, \quad r_1 \cdot r_4, \quad \frac{r_1+1}{2} \cdot r_2 \cdot r_4, \quad \frac{r_1-1}{2} \cdot r_2 \cdot r_4, \quad \frac{r_1+1}{2} \cdot \frac{r_2+1}{2} \cdot r_3 \cdot r_4,$$
$$\frac{r_1+1}{2} \cdot \frac{r_2-1}{2} \cdot r_3 \cdot r_4, \quad \frac{r_1-1}{2} \cdot \frac{r_2+1}{2} \cdot r_3 \cdot r_4, \quad \frac{r_1-1}{2} \cdot \frac{r_2-1}{2} \cdot r_3 \cdot r_4,$$

...

A partial question to this concern: *is the beginning* $\mathbf{1}$, r_1, r_2, $r_1 \cdot r_2$ *of the Olevskii system isometrically reproducible in any unconditional basis of* $L_p\{-1,1\}^{\omega_1}$*?* Remark that it is isometrically reproducible in any unconditional basis of $L_p\{-1,1\}^{\omega_2}$, by the Enflo-Rosenthal results, because it coincides with the Walsh system of order two.

3.5 A close separable problem

Consider the following important partial case of Problem 5.

Problem 6. *Let* $1 \le p < \infty$, $p \ne 2$. *Does there exist an unconditional basis in* $L_p\{-1,1\}^{\omega_1}$*?*

We now pose a separable problem and then provide arguments to show that it is close to Problem 6. Let $E_p = L_p[0,1]^2$ be the L_p-space over the Lebesgue measure space of Borel subsets of the square $[0,1]^2$, and let F_p be the subspace of E_p consisting of all functions depending only on the first variable.

Problem 7. *Let* $1 \le p < \infty$, $p \ne 2$. *Does there exist an unconditional basis* $(f_n) \cup (g_n)$ *in* E_p *consisting of two parts such that* $[f_n] = F_p$ *and the unconditional constant of* (f_n) *equals the unconditional constant of the entire basis* $(f_n) \cup (g_n)$*?*

Theorem 8. *An affirmative answer to Problem 6 implies an affirmative answer to Problem 7.*

Before the proof, we provide with some necessary information. Given an infinite set I, $i \in I$, $x \in \{-1,1\}^{I \setminus \{i\}}$, and $\theta \in \{-1,1\}$, we denote by $\theta \times x$ the element $y \in \{-1,1\}^I$ such that $y(i) = \theta$ and $y(j) = x(j)$ for all $j \in I \setminus \{i\}$. Following [13], a μ_I-measurable function $f : \{-1,1\}^I \to \mathbb{R}$ is said to be *independent* of $i \in I$, if $f(1 \times x) = f(-1 \times x)$ for $\mu_{I\setminus\{i\}}$-almost all values of $x \in \{-1,1\}^{I\setminus\{i\}}$. In the opposite case we say that f *depends* on i. For any measurable function $f : \{-1,1\}^I \to \mathbb{R}$, the set $\{i \in I : f \text{ depends on } i\}$ is at most countable. By the obvious reason, the same terminology we apply to equivalence classes of measurable functions.

Proof. Let $(f_\alpha)_{\alpha<\omega_1}$ be an unconditional basis of $L_p\{-1,1\}^{\omega_1}$ with unconditional constant M. For any $\alpha < \omega_1$ we denote by X_α the subspace of all $f \in L_p\{-1,1\}^{\omega_1}$ depending on coordinates $< \alpha$ only. Obviously, X_α is isometrically isomorphic to $L_p\{-1,1\}^{\alpha}$, which is separable and atomless, and hence, isometrically isomorphic to L_p.

Lemma 1. *There exists a strictly increasing* ω_1*-sequence of limited ordinals* $(\xi_\gamma)_{\gamma<\omega_1}$, $\xi_\gamma < \omega_1$, *such that* $[f_\alpha]_{\alpha<\xi_\gamma} = X_{\xi_\gamma}$ *for all* $\gamma < \omega_1$.

Proof of Lemma 1. Since every function $f \in L_p\{-1,1\}^{\omega_1}$ depends on at most countable set of ordinals $\alpha < \omega_1$, for every separable subspace Z of $L_p\{-1,1\}^{\omega_1}$ the value $\varphi(Z) = \min\{\alpha < \omega_1 : Z \subseteq X_\alpha\}$ is well defined. Then

$$Z \subseteq X_{\varphi(Z)} \quad \text{for every separable subspace } Z. \tag{3}$$

Since every function $f \in L_p\{-1,1\}^{\omega_1}$ has an expansion $f = \sum_{\alpha<\omega_1} a_\alpha f_\alpha$, where the set $\{\alpha < \omega_1 : a_\alpha \neq 0\}$ is at most countable, for every separable subspace Z of $L_p\{-1,1\}^{\omega_1}$ the value $\psi(Z) = \min\{\beta < \omega_1 : Z \subseteq [f_\alpha]_{\alpha<\beta}\}$ is well defined as well. Then

$$Z \subseteq [f_\alpha]_{\alpha<\psi(Z)} \text{ for every separable subspace } Z. \tag{4}$$

Now define recursively ω_1-sequences $(\alpha_\eta)_{\eta<\omega_1}$ and $(\beta_\eta)_{\eta<\omega_1}$ possessing the following properties for every $\eta < \zeta < \omega_1$:

1. $\alpha_\eta \le \beta_\eta < \alpha_\zeta$;
2. $[f_\alpha]_{\alpha<\alpha_\eta} \subseteq X_{\beta_\eta} \subseteq [f_\alpha]_{\alpha<\alpha_\zeta}$.

Set $\alpha_0 = \omega_0$ and $\beta_0 = \max\{\varphi([f_\alpha]_{\alpha<\alpha_0}), \omega_0\}$. Then $\alpha_0 \le \beta_0$ and $[f_\alpha]_{\alpha<\alpha_0} \subseteq X_{\beta_0}$. Given any $\delta < \omega_1$, we assume that δ-sequences $(\alpha_\eta)_{\eta<\delta}$ and $(\beta_\eta)_{\eta<\delta}$ possessing 1 and 2 for every $\eta < \zeta < \delta$ have been already constructed. To define α_δ and β_δ, we consider cases.

(i) δ is an isolated ordinal, that is, $\delta = \delta' + 1$. In this case we set

$$\alpha_\delta = \max\{\psi(X_{\beta_{\delta'}}), \beta_{\delta'} + 1\} \text{ and } \beta_\delta = \max\{\varphi([f_\alpha]_{\alpha<\alpha_\delta}), \alpha_\delta\}.$$

(ii) δ is a limited ordinal. In this case we set

$$\alpha_\delta = \beta_\delta = \bigcup_{\eta<\delta} \alpha_\eta = \bigcup_{\eta<\delta} \beta_\eta$$

(the latter equality is guaranteed by property 1 for every $\eta < \zeta < \delta$).

Property 1 for $\eta < \zeta \le \delta$ follows directly from the construction. To prove 2, observe that in case (i) by (3), (4), $X_{\beta_{\delta'}} \subseteq [f_\alpha]_{\alpha<\psi(X_{\beta_{\delta'}})} \subseteq [f_\alpha]_{\alpha<\alpha_\delta}$ and $[f_\alpha]_{\alpha<\alpha_\delta} \subseteq X_{[f_\alpha]_{\alpha<\alpha_\delta}} \subseteq X_{\beta_\delta}$. In case (ii) inclusions 2 are obvious. Thus, the desired ω_1-sequences $(\alpha_\eta)_{\eta<\omega_1}$ and $(\beta_\eta)_{\eta<\omega_1}$ are constructed.

By (ii) and 2, for every limited ordinal $\delta < \omega_1$ one has $[f_\alpha]_{\alpha<\alpha_\delta} = X_{\alpha_\delta}$. By (ii), for every limited ordinal $\delta < \omega_1$, the ordinal α_δ is limited as well. Since there are uncountably many such ordinals, we can renumber them to obtain the desired ω_1-sequence. □

Lemma 2. *Let $I \subset J$ be countable subsets with $J \setminus I$ infinite. Then there is an isometric isomorphism $T : E_p \to L_p\{-1,1\}^J$ such that $T(F_p)$ equals the subspace of $L_p\{-1,1\}^J$ consisting of all functions which depend on coordinates $i \in I$ only.*

Proof of Lemma 2. It is straightforward that the linear span of the Walsh system $(w_A)_{A\in\mathbb{N}^{<\omega}}$ coincides with that of the Haar system, hence it is dense in L_p. So, to define an isometrical isomorphism on the entire $L_p(\mu)$, it is enough to define it on the Walsh system and prove that it is an isometry on the linear span. Observe that the Walsh system in $E_p = L_p[0,1]^2$ is given by $w_A(x)w_B(y)$, where A, B are finite subsets of $\mathbb{N}$.

Let $I = \{i_1, i_2, \ldots\}$ and $J \setminus I = \{j_1, j_2, \ldots\}$ be any numerations. Given any $A, B \in \mathbb{N}^{<\omega}$, we define functions $\widehat{w}'_A, \widehat{w}''_B : \{-1,1\}^J \to \mathbb{R}$ by setting $\widehat{w}'_A(x) = \prod_{n\in A} x(i_n)$ and $\widehat{w}''_B(x) = \prod_{n\in B} x(j_n)$. Likewise, the Walsh system in $L_p\{-1,1\}^J$ can be represented as follows: $\widehat{w}'_A \cdot \widehat{w}''_B$, $A, B \in \mathbb{N}^{<\omega}$. Now we define $T : E_p \to L_p\{-1,1\}^J$, first on the Walsh system by $Tw_A(x)w_B(y) = \widehat{w}'_A \cdot \widehat{w}''_B$ for all $A, B \in \mathbb{N}^{<\omega}$, and then extend to the linear span of the Walsh system W by linearity. We omit a routine proof that the obtained mapping is an isometry on W. It remains to observe that $T(F_p) = L_p\{-1,1\}^I$. □

We continue the proof of the theorem. Take a sequence $(\xi_\gamma)_{\gamma<\omega_1}$ satisfying the claims of Lemma 1. Denote by M_γ the unconditional constant of the system $(f_\alpha)_{\alpha<\xi_\gamma}$. Then $M_\gamma \uparrow M$. Since there is no strictly increasing ω_1-sequence of reals, we obtain that there is $\gamma_0 < \omega_1$ such that $M_\gamma = M$ for all $\gamma_0 \leq \gamma < \omega_1$. Choose by Lemma 2 an isometric isomorphism $T : E_p \to X_{\xi_{\gamma_0+1}}$ with $T(F_p) = X_{\xi_{\gamma_0}}$. Since $(f_\alpha)_{\alpha<\xi_{\gamma_0+1}} = (f_\alpha)_{\alpha<\xi_{\gamma_0}} \cup (f_\alpha)_{\xi_{\gamma_0}\leq\alpha<\xi_{\gamma_0+1}}$ is an unconditional basis of $X_{\xi_{\gamma_0+1}}$ with unconditional constant M and $(f_\alpha)_{\alpha<\xi_{\gamma_0}}$ is an unconditional basis of $X_{\xi_{\gamma_0}}$ with with the same unconditional constant M, we obtain that $(T^{-1}f_\alpha)_{\alpha<\xi_{\gamma_0+1}} = (T^{-1}f_\alpha)_{\alpha<\xi_{\gamma_0}} \cup (T^{-1}f_\alpha)_{\xi_{\gamma_0}\leq\alpha<\xi_{\gamma_0+1}}$ is an unconditional basis of $T^{-1}(X_{\xi_{\gamma_0+1}}) = E_p$ with unconditional constant M and $(T^{-1}f_\alpha)_{\alpha<\xi_{\gamma_0}}$ is an unconditional basis of $T^{-1}(X_{\xi_{\gamma_0}}) = F_p$ with with the same unconditional constant M. $\square$

Remarks.

1. In Problem 7, one can equivalently replace the unconditional constants of unconditional bases with the supremum of norms of projections with respect to the bases.

2. We do not know of whether an affirmative solution to Problem 7 formally implies an affirmative solution to Problem 6, however, an affirmative solution to Problem 7 would give a possible way to construct an unconditional basis of $L_p\{-1,1\}^{\omega_1}$ by a recursive procedure.

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Дану замітку написано при підготовці доповіді на міжнародній конференції, присвяченій 70-річчю професора О. Лопушанського, 16-19 вересня 2019 р. Ми зосереджуємося на двох давніх відкритих проблемах. Перша, що належить Лінденштраусу і Розенталю (1969 р.), формулюється так: чи кожний доповнювальний нескінченновимірний підпростір простору L_1 ізоморфний до L_1 чи до ℓ_1? Друга проблема була поставлена Енфло і Розенталем у 1973 р.: чи існує несепарабельний простір $L_p(\mu)$ зі скінченною безатомною мірою μ та $1 < p < \infty$, $p \neq 2$, з безумовним базисом? У замітці наведено аналіз часткових результатів та природних ідей розв'язання даних проблем.

Ключові слова і фрази: простори L_p, доповнювальний підпростір, безумовний базис.