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ZERO PRODUCT PRESERVING BILINEAR OPERATORS ACTING IN SEQUENCE SPACES

Consider a couple of sequence spaces and a product function — a canonical bilinear map associated to the pointwise product — acting in it. We analyze the class of "zero product preserving" bilinear operators associated with this product, that are defined as the ones that are zero valued in the couples in which the product equals zero. The bilinear operators belonging to this class have been studied already in the context of Banach algebras, and allow a characterization in terms of factorizations through $\ell^r(\mathbb{N})$ spaces. Using this, we show the main properties of these maps such as compactness and summability.

Key words and phrases: sequence spaces, bilinear operators, factorization, zero product preserving map, product.

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1 INTRODUCTION

Let us fix a couple of Banach spaces having a characteristic operation involving couples of vectors for giving an element in other Banach space. For example, the pointwise product of functions from L^p and $L^{p'}$ for obtaining an element of L^1, or the internal product in a Banach algebra. Let us call "product" this bilinear map. Bilinear maps factoring through such a product preserve some of its good properties, and so it is interesting to know which bilinear operators satisfy such a factorization. This general philosophy is in the root of some current developments in mathematical analysis, mainly in the Banach algebras and vector lattices setting (see for example [1,5,7,12] and references therein).

In this paper we analyze the class of bilinear maps factoring through a product in a different context. We study the main characterizations and properties when the operators act in couples of classical Banach sequence spaces ($\ell^p(\mathbb{N})$-spaces). The essential result (Theorem 1) shows that the factorization is equivalent to a certain "zero product preservation" property. Concretely, bilinear maps satisfying this property are the ones that are 0-valued for couples of elements whose products are equal to zero.

Let us explain the relation of our class of maps with some notions and results that can be found in the current literature. Alaminos J. et al have studied zero product preserving bilinear maps defined on a product of Banach algebras and C^*-algebras to get a characterization for (weighted) homomorphisms and derivations. They have obtained a class of Banach algebras A that satisfy the equality $\varphi(ab,c) = \varphi(a,bc)$, $a,b,c \in A$, for every continuous zero product preserving bilinear map $\varphi : A \times A \to B$. By adding some conditions to the algebra, they have

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proved that $\varphi(ab,c) = \varphi(a,bc)$ gives a factorization for the bilinear operator φ as $\varphi(a,b) := P(ab)$ for a certain linear map $P : A \to B$ [1]. Recently, Alaminos J. et al have shown in [2] that there are some Banach algebras that do not satisfy the equality $\varphi(ab,c) = \varphi(a,bc)$ such as the algebra $C^1[0,1]$ of continuously differentiable functions from $[0,1]$ to $\mathbb{C}$, although the operator φ is zero product preserving map. In particular, this shows that any bilinear operator cannot be factored through the product.

In the meantime, some authors have studied the zero product preserving property for the bilinear maps acting in vector lattices and function spaces with the name *orthosymmetry*. This term is firstly used by Buskes G. and van Rooij A. to give a factorization for bilinear maps defined on vector lattices and they obtained the powers of vector lattices by orthosymmetric maps, see [6,7]. Recently, Ben Amor F. has studied the commutators of orthosymmetric maps in [4] and investigated an expanded class of orthosymmetric bilinear maps that are related to symmetric operators given by Buskes G. and van Rooij A. The interested reader can see the reference [5] for a detailed information about the orthosymmetric maps acting in vector lattices.

In a different direction, factorization of zero product preserving bilinear maps for the convolution product acting in function spaces has been studied by Erdoğan E. et al (see [10]). Recently, Erdoğan E. and Gök Ö. have studied a class of bilinear operators acting in a product of Banach algebras of integrable functions and showed a zero product preserving bilinear operator defined on the product of Banach algebras that factors through a subalgebra of absolutely integrable functions by convolution product (see [11]). Moreover, Erdoğan E. et al have obtained a class of zero product preserving bilinear operators acting in pairs of Banach function spaces that factor through the pointwise product and they have given characterizations by means of norm inequalities for these bilinear maps [12].

The aim of this paper is to give a new version of the factorization results given in the mentioned studies for the zero product preserving bilinear operators defined on the product of sequence spaces. We center our attention on bilinear operators B defined on the product of Banach spaces E and F satisfying the zero product preserving property

$$x \circledast y = 0 \text{ implies } B(x,y) = 0, \quad (x,y) \in E \times F,$$

where $\circledast$ is defined using the pointwise product of sequences, showing that they are exactly the ones that factors through $\circledast$.

This paper is organised as follows: Section 2 is devoted to giving some preliminary results on products and factorization through them. In Section 3, the main result of the paper on factorization of zero product preserving on sequence spaces is proved (Theorem 1). Using it, compactness and summability properties of product factorable operators are investigated and some applications are given.

2 Preliminaries: products and bilinear maps

We use standard notations and notions from Banach space theory. The sets of natural numbers and integers are denoted by $\mathbb{N}$ and $\mathbb{Z}$, respectively. For a Banach space E, B_E will denote the unit ball of E. We write χ_A for the characteristic function of a set A. *Operator* (linear or multilinear) indicates *continuous operator*. The space of all linear operators between Banach spaces X, Y is denoted by $L(X, Y)$, and we write $\mathcal{B}(X \times Y, Z)$ for the vector space of all bilinear Z-valued operators, where Z is also a Banach space.

For a positive real number p, $\ell^p(\mathbb{N})$ is the space of all complex valued absolutely p-summable sequences. It is a Banach space with the norm $\|(x_i)\|_p = \Big(\sum\limits_{i=1}^{\infty} |x_i|^p\Big)^{1/p}$ for $p \geq 1$, and $\ell^\infty(\mathbb{N})$ shows the Banach space of all bounded sequences endowed with the norm $\|(x_i)\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

If μ is a measure and $1 \leq p < \infty$, we write $L^p(\mu)$ for the Lebesgue space of classes of μ-a.e. equal p-integrable functions.

We call a continuous operator *(weakly) compact* if it maps the closed unit ball to a relatively (weakly) compact set.

A Banach space E has Dunford-Pettis property if every weakly compact linear operator $T : E \to F$ is completely continuous (that is, it maps every weakly compact set $A \in E$ into a compact set with respect to the norm topology of the Banach space F).

A linear operator $T : X \to Y$ is said to be *(p,q)-summing* ($T \in \Pi_{p,q}(X,Y)$) if there is a constant $k > 0$ such that for every $x_1, \ldots, x_n \in X$ and for all positive integers n

$$\Big(\sum_{i=1}^{n} \|T(x_i)\|_Y^p\Big)^{1/p} \leq k \sup_{x' \in B_{X'}} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^q\Big)^{1/q}.$$

For the summing operators we refer the reader to [9].

Throughout the paper we will use the term *product* for a specific bilinear map, typically with some special properties and being canonical in some sense. However, the only assumption on such a product is that it is a continuous bilinear map. We will need stronger properties for the products that are presented in [12] by Erdoğan E. et al.

Definition 1. *Consider a bilinear operator $\circledast : X \times Y \to Z$, $(x,y) \rightsquigarrow \circledast(x,y) =: x \circledast y$, where X, Y, Z are Banach spaces. We say that the bilinear operator $\circledast$ is a norm preserving product (n.p. product for short) if it satisfies the inclusion $B_Z \subseteq \circledast(B_X \times B_Y)$ and*

$$\|x \circledast y\|_Z = \inf\big\{\|x'\|_X \|y'\|_Y : x' \in X,\ y' \in Y,\ x \circledast y = x' \circledast y'\big\},$$

for every $(x,y) \in X \times Y$.

Now let us give some examples of bilinear operators that are n.p. product or not.

Example 1. *Let (Ω, Σ, μ) be a complete σ-finite measure space and let $(E, \|.\|_E)$ be a Banach function space over μ. (For the definition of Banach function space we refer to [14, Def 1.b.17]). We will write $E^{(p)}$, $p \geq 1$, for the p-convexification of the Banach lattice E in the sense of [14, Ch. 1.d] (see also the equivalent notion of pth power in [17, Ch.2] for a more explicit description). In the case that E is a Banach function space, $E^{(p)}$ is also a Banach function space with the norm $\|f\|_{E^{(p)}} = \||f|^p\|_E^{1/p}$ for $f \in E$ (see [16, Prop.1]).*

Let us consider the bilinear operator defined by the (μ-a.e.) pointwise product $\odot : E^{(p)} \times E^{(q)} \to E^{(r)}$, $(f,g) \rightsquigarrow f \cdot g$, where $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$ for $1 \leq r < p, q < \infty$. We claim that this bilinear map is a norm preserving product. Indeed, consider $f \in B_{E^{(r)}}$, $h := |f|^{r/p} sgn f \in E^{(p)}$ and $g := |f|^{r/q} \in E^{(q)}$, where $sgn f$ denotes the sign function of f. By the definition of the norm of *the p-convexification, it follows that* $\|h\|_{E^{(p)}} = \left\| \left| |f|^{r/p} sgnf \right|^{p} \right\|_{E}^{1/p} = \| |f|^{r} \|_{E}^{1/p} = \|f\|_{E^{(r)}}^{r/p} \leq 1$. *Similarly,* $\|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/q} \leq 1$. *Therefore,* $B_{E^{(r)}} \subseteq \odot(B_{E^{(p)}} \times B_{E^{(q)}})$ *is obtained.*

Let us show now that $\|h \cdot g\|_{E^{(r)}} = \inf\{\|h'\|_{E^{(p)}}\|g'\|_{E^{(q)}} : h' \in E^{(p)}, g' \in E^{(q)}, h \cdot g = h' \cdot g'\}$ *for* $h \in E^{(p)}$ *and* $g \in E^{(q)}$. *Indeed, by the generalized Hölder's inequality we have that* $h \cdot g \in E^{(r)}$ *and* $\|h \cdot g\|_{E^{(r)}} \leq \|h\|_{E^{(p)}}\|g\|_{E^{(q)}}$ *(see [16, Lemma 1]). Since this inequality holds for all couples* (h', g') *such that* $f = h \cdot g = h' \cdot g'$, *we obtain* $\|h \cdot g\|_{E^{(r)}} \leq \inf\{\|h'\|\|g'\| : h \cdot g = h' \cdot g'\}$. *Conversely, consider an arbitrary element* $f \in E^{(r)}$. *Then* f *has the following factorization:* $h = |f|^{r/p} sgnf \in E^{(p)}$, $g = |f|^{r/q} \in E^{(q)}$ *and* $h \cdot g \in E^{(r)}$. *Moreover,* $\|h\|_{E^{(p)}} = \|f\|_{E^{(r)}}^{r/p}$ *and* $\|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/q}$. *Therefore* $\|h\|_{E^{(p)}}\|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/p}\|f\|_{E^{(r)}}^{r/q} = \|f\|_{E^{(r)}}$. *This proves*

$$\|f\|_{E^{(r)}} = \|h \cdot g\|_{E^{(r)}} \geq \inf\left\{\|h'\|_{E^{(p)}}\|g'\|_{E^{(q)}} : h \cdot g = h' \cdot g'\right\},$$

and so $\odot$ *is an n.p. product.*

Note that if we consider $E = L^1(\mu)$ *we obtain that the pointwise product is an n.p. product from* $L^p(\mu) \times L^q(\mu)$ *to* $L^r(\mu)$. *In particular, if* μ *is the counting measure on* $\mathbb{N}$, *the pointwise product* $\odot : \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \to \ell^r(\mathbb{N})$ *is an n.p. product (for a more detailed information see [16, Lemma 1] or [17, Lemma 2.21(i)]).*

Example 2. *Let* E, F *be normed spaces and* $E \otimes F$ *denotes their algebraic tensor product. Projective norm* π *and injective norm* ε *on* $E \otimes F$ *are calculated by* $\pi(z) = \inf\Big\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \Big\}$, *and* $\varepsilon(z) = \sup\Big\{ \langle x' \otimes y', z\rangle : x' \in B_{E'}, y' \in B_{F'} \Big\}$, *respectively (see [8, Section 2,3]). It is well-known that any* reasonable *tensor norm* α *on the tensor product* $E \otimes F$ *satisfies the inequality* $\varepsilon \leq \alpha \leq \pi$. *For every* $(x, y) \in E \times F$, *it is seen that by the definitions of these norms*

$$\varepsilon(x \otimes y) \leq \alpha(x \otimes y) \leq \pi(x \otimes y) \leq \inf\{\|x'\|\|y'\| : x' \otimes y' = x \otimes y\}.$$

Besides, for every simple tensor $x \otimes y$ *it is known that for any reasonable tensor norm* α *we have* $\alpha(x \otimes y) = \|x\|_E\|y\|_F$ *(see [8, §12.1]). Then, any reasonable tensor norm satisfies the equality involving the norm in Definition 1. But the tensor product does not satisfy the inclusion, since clearly it is not surjective. So, it is not a norm preserving product.*

Example 3. *Let us define the following seminorm on* $X \otimes L(X, Y)$. *If* $z = \sum_{j=1}^{n} x_j \otimes T_j$ *is such that* $\sum_{j=1}^{n} T_j(x_j) = y_z \in Y$, *we define*

$$\pi_{\bullet}(z) = \inf\Big\{\pi(z') : z' = \sum_{j=1}^{m} x'_j \otimes T'_j, \text{ such that } \sum_{j=1}^{m} T'_j(x'_j) = y_z\Big\}.$$

That is, $\pi_{\bullet}$ *is the quotient norm given by the tensor contraction* $c : X\hat{\otimes}_{\pi}L(X, Y) \to Y$ *defined as* $c(z) = c\big(\sum_{j=1}^{n}\sum_{j=1}^{n} x_j \otimes T_j\big) = \sum_{j=1}^{n} \bullet(x_j, T_j) = \sum_{j=1}^{n} T_j(x_j)$ *associated to the following factorization.*

$$\begin{tikzcd} X \times L(X,Y) \arrow[d, "\otimes"'] \arrow[dr, dotted, "\bullet"] & \\ X\hat{\otimes}_{\pi}L(X,Y) \arrow[r, "c"] & Y. \end{tikzcd}$$

The description of this seminorm can be found in [18]. It defines a norm if we construct a quotient space $X\hat{\otimes}_{\pi_\bullet}L(X,Y)$ by identifying the equivalence classes of the projective tensor product $X\hat{\otimes}_\pi L(X,Y)$ with the range of c in Y, i.e. $c(X\hat{\otimes}_\pi L(X,Y)) \subset Y$. Thus, for $z = \sum_{j=1}^n x_j \otimes T_j$ and $z' = \sum_{j=1}^m x'_j \otimes T'_j$, $z \sim z'$ if and only if $\sum_{j=1}^m T_j(x'_j) = \sum_{j=1}^n T_j(x_j)$. The norm of a class $[z] = \{z' : z \sim z'\}$, for $z = \sum_{j=1}^n x_j \otimes T_j$, is given by

$$\pi_\bullet(z) = \inf\{\pi(z') : z \sim z'\}.$$

Let us show that $\bullet$ is a norm preserving product.

Fix $T \in L(X,Y)$ and $x \in X$ and consider $y_z = T(x)$; clearly the inequality $\|y_z\| \le \|T\|\|x\|$ holds. Now, consider another tensor $z = \sum_{i=1}^n x_i \otimes T_i$ such that $y_z = \sum_{i=1}^n T_i(x_i)$. Since $\|y_z\| = \|\sum_{i=1}^n T_i(x_i)\| \le \sum_{i=1}^n \|T_i\|\|x_i\|$, we obtain that $\|x \bullet T\| = \|y_z\| \le \pi_\bullet(z)$.

In the opposite direction, for $y \in Y$ there are elements $T_0 \in L(X,Y)$ and $x_0 \in X$ such that $T_0(x_0) = y$ and $\|y\| = \|T_0\|\|x_0\|$. To see this, just take a couple (x_0, x'_0) of norm one elements $x_0 \in X$ and $x'_0 \in X'$ such that $\langle x_0, x'_0\rangle = 1$. Now define $T_0(x) := \langle x, x'_0\rangle\, y$, $x \in X$, and note that $\|T_0\| = \|y\|$. Therefore, if $z = x_0 \otimes T_0$, we have that $y = y_z$. So, this gives in particular that $B_Y \subseteq \bullet(B_X \times B_{L(X,Y)})$, since $\pi_\bullet(z) \le \|y_z\|$. Together with the inequality in the previous paragraph this also gives $\|x_0 \bullet T_0\| = \|y_z\| = \pi_\bullet(z)$. More precisely, we have proven that

$$\|x \bullet T\|_Y = \inf\left\{\|x_0\|_X\|T_0\|_{L(X,Y)} : x_0 \in X,\ T_0 \in L(X,Y),\ x \bullet T = x_0 \bullet T_0\right\}$$

for all $T \in L(X,Y)$ and $x \in X$. Thus, $\bullet$ is a norm preserving product.

Since to find the factors of a Banach space is a current problem in the mathematical literature, there are found more examples of the norm preserving products including the Banach function spaces (see [13, 15, 19]).

Let X, Y, Z be Banach spaces. A bilinear operator $B : X \times Y \to Z$ is called $\circledast$-*factorable* for the Banach valued n.p. product $\circledast : X \times Y \to G$ if there exists a linear continuous map $T : G \to Z$ such that B factors through T and $\circledast$ (see [12, Definition 1]).

In this case, the following triangular diagram

$$\begin{tikzcd} X \times Y \arrow[r, "B"] \arrow[dr, dotted, "\circledast"'] & Z \\ & G \arrow[u, "T"'] \end{tikzcd}$$

holds. In the paper [12], Erdogan E. et al have proved a necessary and sufficint condition for $\odot$-factorability by a summability requirement as follows.

Lemma 1 (Lemma 1, [12]). *The bilinear operator $B : X \times Y \to Z$ is $\circledast$-factorable for the n.p. product $\circledast$ if and only if there exists a constant K such that for all $x_1, \dots, x_n \in X$ and $y_1, \dots, y_n \in Y$ we have*

$$\Big\|\sum_{i=1}^n B(x_i, y_i)\Big\|_Z \le K\Big\|\sum_{i=1}^n x_i \circledast y_i\Big\|_G.$$

Example 4. *Consider a bilinear continuous operator $B : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \ell^1(\mathbb{N})$. Let us use the result above for characterizing when B is $\odot$-factorable with respect to the pointwise product. It was shown in the first example that the pointwise product $\odot$ from $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$*

to $\ell^1(\mathbb{N})$ is an n.p. product. Let $(a,b) = \left(\sum_{k=1}^{\infty} \alpha_k \chi_{\{k\}}, \sum_{m=1}^{\infty} \beta_m \chi_{\{m\}}\right) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$. Then the image of this element under pointwise product is

$$a \odot b = \sum_{k=1}^{\infty} \alpha_k \sum_{m=1}^{\infty} \beta_m (\chi_{\{k\}} \odot \chi_{\{m\}}) = \sum_{k=1}^{\infty} \alpha_k \beta_k \chi_{\{k\}}.$$

Thus, for the finite sets of sequences $a_1, \dots, a_n, b_1, \dots, b_n$ we have

$$\sum_{i=1}^{n} a_i \odot b_i = \sum_{i=1}^{n} \sum_{k=1}^{\infty} \alpha_{ik} \beta_{ik} \chi_{\{k\}} = \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{n} \alpha_{ik} \beta_{ik} \Big) \chi_{\{k\}}.$$

The $\ell^1(\mathbb{N})$ norm of this sequence is $\|(z_k)\|_{\ell^1(\mathbb{N})} = \sum_{k=1}^{\infty} |\sum_{i=1}^{n} \alpha_{ik}\beta_{ik}|$. By Lemma 1, we obtain that the bilinear operator B factors through the pointwise product if and only if there is a constant K for all finite sequences $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n \subset \ell^2(\mathbb{N})$ such that

$$\Big\| \sum_{i=1}^{n} B(a_i, b_i) \Big\|_1 \leq K \sum_{k=1}^{\infty} \Big| \sum_{i=1}^{n} \alpha_{ik} \beta_{ik} \Big|.$$

Let us consider now a more specific bilinear operator $B : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \ell^1(\mathbb{N})$: a diagonal multilinear operator. Recall that a bilinear operator $B \in \mathcal{B}(\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}), \ell^1(\mathbb{N}))$ is called bilinear diagonal if there is a bounded sequence $\xi = (\xi_k)_k$ such that $B(a,b) = \sum_{k=1}^{\infty} \xi_k \alpha_k \beta_k \chi_{\{k\}}$. By Hölder inequality, it is easily seen that $B \in \mathcal{B}(\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}), \ell^1(\mathbb{N}))$ if and only if $\xi \in \ell^\infty(\mathbb{N})$. For arbitrary finite sequences $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n \subset \ell^2(\mathbb{N})$, we obtain

$$\Big\| \sum_{i=1}^{n} B(a_i, b_i) \Big\|_1 = \Big\| \sum_{i=1}^{n} \sum_{k=1}^{\infty} \xi_k \alpha_{ik} \beta_{ik} \chi_{\{k\}} \Big\|_1 \leq \|\xi_k\|_\infty \sum_{k=1}^{\infty} \Big| \sum_{i=1}^{n} \alpha_{ik} \beta_{ik} \Big| = K \sum_{k=1}^{\infty} \Big| \sum_{i=1}^{n} \alpha_{ik} \beta_{ik} \Big|.$$

Therefore, it is seen that every bilinear diagonal operator is factorable through $\odot$. Remark that a bilinear diagonal operator satisfies that $B(a,b) = 0$ whenever $a \odot b = 0$. We will prove in what follows that this is also a sufficient condition for factorability of bilinear operators defined on the topological product of sequence spaces.

3 The pointwise product in sequence spaces

Let us center our attention in this section in a particular product that is important in mathematical analysis. It is given by the pointwise product of sequences, functions and generalized sequences belonging to Banach lattices. In order to give a full generality to our results, we will consider several extensions of the bilinear map given by the pointwise products.

In the case of sequences, we will consider the following notion. The reference product is the pointwise product of sequences, that is $\odot : \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \to \ell^r(\mathbb{N})$, $a \odot b = a \cdot b = (a_i b_i)_{i=1}^{\infty} \in \ell^r(\mathbb{N})$, that is well-defined and continuous by Hölder's inequality. This is clearly an n.p. product, as have been explained in the previous section. Also, it has commutativity and associativity properties.

The following notion is crucial in this paper.

Let X, Y, Z be Banach spaces. We say that a bilinear continuous operator $B : X \times Y \to Z$ is *zero "product"-preserving* if it is 0-valued for couples of elements whose product equals 0.

Theorem 1. *Let $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$ for $1 \leq r < p, q < \infty$. Consider a bilinear operator $B : \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \to Y$. The following assertions are equivalent.*

(1) The operator B is zero $\odot$-preserving, i.e. $B(x,y) = 0$ whenever $x \odot y = 0$.

(2) The operator B is $\odot$-factorable. That is, there is a linear and continuous operator $T : \ell^r(\mathbb{N}) \to Y$ such that $B = T \circ \odot$, and so we have the factorization

$$\begin{tikzcd} \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \arrow[r, "B"] \arrow[dr, dotted, "\odot"] & Y \\ & \ell^r(\mathbb{N}). \arrow[u, dotted, "T"'] \end{tikzcd}$$

Proof. Let us show that there is a linear continuous operator T such that $B := T \circ \odot$ whenever the operator B is a zero $\odot$-preserving. Define the map $T_n : \ell^p(\mathbb{N}) \odot \ell^q(\mathbb{N}) \to Y$, $T_n(z) := B(z \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{1,2,\ldots,n\}})$ for all $n \in \mathbb{N}$, where $z \in \ell^p(\mathbb{N}) \odot \ell^q(\mathbb{N})$; note that $z \odot \chi_{\{1,2,\ldots,n\}} \in \ell^p(\mathbb{N})$, and $\chi_{\{1,2,\ldots,n\}} \in \ell^q(\mathbb{N})$, and so T_n is well defined for each $n \in \mathbb{N}$. The linearity of T_n is a consequence of the linearity of the bilinear operator B in the first variable. To show the boundedness of the map T_n, we give an equivalent formula for this operator. Since $\chi_{\{1,2,\ldots,n\}} = \sum_{i=1}^{n} \chi_{\{i\}}$ by the properties of characteristic function, we have

$$T_n(a \odot b) = B(a \odot b \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{1,2,\ldots,n\}}) = \sum_{i=1}^{n} B(a \odot b \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{i\}}).$$

The pointwise product of $a = (\alpha_k)_{k=1}^{\infty} \in \ell^p(\mathbb{N})$ and $b = (\beta_k)_{k=1}^{\infty} \in \ell^q(\mathbb{N})$ is $a \odot b = (\alpha_k \beta_k)_{k=1}^{\infty} = \sum_{k=1}^{\infty} \alpha_k \beta_k \chi_{\{k\}}$. By the continuity of B, the image of the couple $(a,b) \in \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N})$ under the bilinear operator B is

$$B(a,b) = B\Big(\sum_{k=1}^{\infty} \alpha_k \chi_{\{k\}}, \sum_{m=1}^{\infty} \beta_m \chi_{\{m\}}\Big) = \sum_{k=1}^{\infty} \alpha_k \sum_{m=1}^{\infty} \beta_m B(\chi_{\{k\}}, \chi_{\{m\}}).$$

Since $\chi_{\{k\}} \odot \chi_{\{m\}} = 0 \, (k \neq m)$ and by the zero $\odot$-preservation of the operator B, we have $B(a,b) = \sum_{k=1}^{\infty} \alpha_k \beta_k B(\chi_{\{k\}}, \chi_{\{k\}})$. Thus,

$$\begin{aligned} T_n(a \odot b) = \sum_{i=1}^{n} B(a \odot b \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{i\}}) &= \sum_{i=1}^{n} B\Big(\sum_{k=1}^{\infty} \alpha_k \beta_k \chi_{\{k\}} \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{i\}}\Big) \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_k \beta_k B(\chi_{\{k\}}, \chi_{\{i\}}). \end{aligned}$$

Using the zero $\odot$-preservation property once again, we obtain

$$T_n(a \odot b) = \sum_{i=1}^{n} \alpha_i \beta_i B(\chi_{\{i\}}, \chi_{\{i\}}) = B\Big(\sum_{i=1}^{n} \alpha_i \beta_i \chi_{\{i\}}, \sum_{i=1}^{n} \chi_{\{i\}}\Big) = B\Big(\sum_{i=1}^{n} \alpha_i \chi_{\{i\}}, \sum_{i=1}^{n} \beta_i \chi_{\{i\}}\Big).$$

By the boundedness of the bilinear operator B, it follows that

$$\begin{aligned} \sup_{z \in B_{\ell^r(\mathbb{N})}} \|T_n z\|_Y &= \sup_{\substack{(a,b) \in B_{\ell^p(\mathbb{N})} \times B_{\ell^q(\mathbb{N})} \\ z = a \odot b}} \Big\| B\Big(\sum_{i=1}^{n} \alpha_i \chi_{\{i\}}, \sum_{i=1}^{n} \beta_i \chi_{\{i\}}\Big) \Big\|_Y \\ &\leq \sup_{\substack{(a,b) \in B_{\ell^p(\mathbb{N})} \times B_{\ell^q(\mathbb{N})} \\ z = a \odot b}} \sum_{i=1}^{n} |\alpha_i \beta_i| \|B(\chi_{\{i\}}, \chi_{\{i\}})\|_Y < \infty. \end{aligned}$$

This shows that T_n is (uniformly) bounded, $n \in \mathbb{N}$, and therefore $(T_n)_{n=1}^{\infty}$ is a bounded sequence of linear operators acting in $\ell^r(\mathbb{N})$, since $\ell^r(\mathbb{N}) = \ell^p(\mathbb{N}) \odot \ell^q(\mathbb{N})$. Indeed, note that since $\odot$ is an n.p. product, we have that it is surjective and preserves the norm, and so for every $z \in \ell^r(\mathbb{N})$ we find adequate $a \in \ell^p(\mathbb{N})$ and $b \in \ell^q(\mathbb{N})$ such that $z = a \odot b$.

The sequence $\{T_n(a \odot b)\}_{n=1}^{\infty}$ is a Cauchy sequence for every $a \in \ell^p(\mathbb{N})$ and $b \in \ell^q(\mathbb{N})$, and it is convergent by completeness of the Banach space Y. Indeed, since $a \odot b \in \ell^r(\mathbb{N})$, then for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\Big\| \sum_{i=n}^{\infty} \alpha_i \chi_{\{i\}} \Big\|_{\ell^p(\mathbb{N})} \Big\| \sum_{i=n}^{\infty} \beta_i \chi_{\{i\}} \Big\|_{\ell^q(\mathbb{N})} < \frac{\varepsilon}{\|B\|} \quad \forall n > N.$$

Using again that $B(\chi_{\{i\}}, \chi_{\{j\}}) = 0$ if $i \neq j$, we obtain

$$\begin{aligned} \|T_m(a \odot b) - T_n(a \odot b)\|_Y &= \Big\| B\Big(\sum_{i=n+1}^{m} \alpha_i \beta_i \chi_{\{i\}}, \sum_{i=n+1}^{m} \chi_{\{i\}} \Big) \Big\|_Y \\ &\leq \|B\| \Big\| \sum_{i=n+1}^{m} \alpha_i \chi_{\{i\}} \Big\|_{\ell^p(\mathbb{N})} \Big\| \sum_{i=n+1}^{m} \beta_i \chi_{\{i\}} \Big\|_{\ell^q(\mathbb{N})} < \varepsilon \quad \forall m > n > N. \end{aligned}$$

Let us define now the limit operator $T : \ell^r(\mathbb{N}) \to Y$ of the operator sequence $\{T_n\}$, that is $T(a \odot b) = \lim_{n \to \infty} T_n(a \odot b)$. It is easily seen that T is well defined and linear. Since $T_n(a \odot b)$ converges for every $a \odot b \in \ell^r(\mathbb{N})$, then it is bounded for every $a \odot b$. By the Uniform Boundedness Theorem, it follows that T is continuous. Therefore, we obtain

$$B(a,b) = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i \beta_i B(\chi_{\{i\}}, \chi_{\{i\}}) = \lim_{n \to \infty} T_n(a \odot b) = T(a \odot b).$$

Besides, the image of an element is independent from its representation. Indeed, for the element $x = a_1 \odot b_1 = a_2 \odot b_2$, we obtain

$$\begin{aligned} T(a_1 \odot b_1) &= \lim_{n \to \infty} B(a_1 \odot b_1 \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{1,2,\ldots,n\}}) \\ &= \lim_{n \to \infty} B(a_2 \odot b_2 \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{1,2,\ldots,n\}}) = T(a_2 \odot b_2). \end{aligned}$$

Hence we obtain the factorization of the bilinear operator B through the pointwise product as $B = T \circ \odot$.

For the converse, assume that the map B is $\odot$-factorable. Then, by Lemma 1 given in [12] (see also page 59) it is obtained that there is a positive real number K such that, for all $x_1, \ldots, x_n \in \ell^p(\mathbb{N})$ and $y_1, \ldots, y_n \in \ell^q(\mathbb{N})$, the following inequality holds

$$\Big\| \sum_{i=1}^{n} B(x_i, y_i) \Big\|_Y \leq K \Big\| \sum_{i=1}^{n} x_i \odot y_i \Big\|_{\ell^r(\mathbb{N})}.$$

Clearly, this inequality implies zero $\odot$-preservation of the bilinear map B. This finishes the proof. □

Now we will give a generalization of our results. Consider two Banach spaces E and F that are isomorphic -as Banach spaces- to $\ell^p(\mathbb{N})$ and $\ell^q(\mathbb{N})$, respectively, and the isomorphisms are given by the operators $P : E \to \ell^p(\mathbb{N})$ and $Q : F \to \ell^q(\mathbb{N})$. We define the product $\odot_{P \times Q} : E \times F \to \ell^r(\mathbb{N})$ by

$$\odot_{P \times Q}(x, y) = P(x) \odot Q(y), \quad x \in E, \quad y \in F.$$

To make this definition more understandable, let us illustrate it by the following diagram

$$\begin{tikzcd} E \times F \arrow[r, "\odot_{P\times Q}"] \arrow[d, dotted, "P\times Q"'] & \ell^r(\mathbb{N}). \\ \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \arrow[ur, dotted, "\odot"'] & \end{tikzcd}$$

In this situation considered above of the product $\odot_{P\times Q} = P(.) \odot Q(.)$, a bilinear map $B : E \times F \to Y$ is zero $\odot_{P\times Q}$-preserving if

$$\odot_{P\times Q}(x, y) = 0 \quad \text{implies} \quad B(x, y) = 0$$

for all $x \in E$ and $y \in F$. Namely, the map B is said to be zero $\odot_{P\times Q}$-preserving if $B(x, y) = 0$ whenever $P(x) \odot Q(y) = 0$.

Theorem 2. *Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for $1 \leq r < p, q < \infty$. Let the Banach spaces E and F be isomorphic to $\ell^p(\mathbb{N})$ and $\ell^q(\mathbb{N})$ by means of the isomorphisms P and Q, respectively. Consider a Banach valued bilinear operator $B : E \times F \to Y$. The following assertions imply each other.*

(1) *The operator B is $\odot_{P\times Q}$-factorable. That is, there exists a linear continuous operator $T : \ell^r(\mathbb{N}) \to Y$ such that $B = T \circ \odot_{P\times Q}$, and the following diagram commutes.*

$$\begin{tikzcd} E \times F \arrow[r, "B"] \arrow[d, dotted, "P\times Q"'] & Y \\ \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \arrow[r, dotted, "\odot"] & \ell^r(\mathbb{N}). \arrow[u, dotted, "T"'] \end{tikzcd}$$

(2) *There is a positive real number K such that, for every finite set of elements $\{x_i\}_{i=1}^n \in E$ and $\{y_i\}_{i=1}^n \in F$, the following inequality holds*

$$\Big\| \sum_{i=1}^n B(x_i, y_i) \Big\|_Y \leq K \Big\| \sum_{i=1}^n P(x_i) \odot Q(y_i) \Big\|_{\ell^r(\mathbb{N})}.$$

(3) *The operator B is zero $\odot_{P\times Q}$-preserving, that is, $x \odot_{P\times Q} y = 0$ implies $B(x, y) = 0$.*

Proof. Let us prove that (3) implies (1). Under the conditions of the theorem, consider the bilinear map $\overline{B} = B \circ (P^{-1} \times Q^{-1}) : \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \to Y$. We have that for all $x \in E$ and $y \in F$, $x \odot_{P\times Q} y = P(x) \odot Q(y) = 0$ implies that $0 = B(x, y) = \overline{B}(P(x), Q(y)) = 0$. That is, since P and Q are isomorphisms, we have that for all $a \in \ell^p(\mathbb{N})$ and $b \in \ell^q(\mathbb{N})$, $a \odot b = 0$ implies that $\overline{B}(a, b) = 0$.

We are in situation of using Theorem 1 for $\overline{B}$. So we have that there is a linear operator $T : \ell^r(\mathbb{N}) \to Y$ such that $\overline{B} = T \circ \odot$. By the definition of $\overline{B}$, we obtain $B = \overline{B} \circ (P \times Q) = T \circ \odot \circ (P \times Q)$, the required factorization.

The equivalences among the three statements of the theorem follow directly using Lemma 1 in [12] and this factorization. $\square$

We will say a bilinear map $B : X \times X \to Y$ is *symmetric* if $B(f,g) = B(g,f)$ for every couple $(f,g) \in X \times X$.

It is easily seen that any $\odot$-factorable bilinear map $B : \ell^p(\mathbb{N}) \times \ell^p(\mathbb{N}) \to Y$ factorized through $\ell^r(\mathbb{N})$ for $2r = p$ is symmetric, since $B(a_n, b_n) = T(a_n \odot b_n) = T(b_n \odot a_n) = B(b_n, a_n)$ holds for all $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N})$ by the commutativity of the pointwise product.

Now, we will give symmetry condition for general product.

Corollary 1. *Let the Banach space X be isomorphic to $\ell^p(\mathbb{N})$ for $p \geq 2$. Then any zero $\odot_{P\times P}$-preserving bilinear map $B : X \times X \to Y$ satisfies the symetry condition, that is $B(x,y) = B(y,x)$ for all x, $y \in X$.*

Proof. Since the map B is zero $\odot_{P\times P}$-preserving, it is $\odot_{P\times P}$-factorable. Then, for $r = p/2$ there is a linear continuous map $T : \ell^r(\mathbb{N}) \to Y$ defined by $B(x,y) = T \circ \odot \circ (P \times P)(x,y) = T(P(x) \odot P(y))$. By the commutativity of the pointwise product we get the symmetry

$$B(x,y) = T(P(x) \odot P(y)) = T(P(y) \odot P(x)) = B(y,x).$$

□

Remark 1. *The extension of the result given in Theorem 1 from the case of $\odot$ to the case of $\odot_{P\times Q}$ products implicitly shows a fundamental fact about factorization through the pointwise product. The requirement "$a \odot b = 0$ implies $B(a,b) = 0$" can be understood as a lattice-type property: indeed, note that for sequences a and b in the corresponding spaces, $a \odot b = 0$ if and only if a and b are disjoint, and so we can rewrite the requirement of being zero $\odot$-preserving as "if $|a| \wedge |b| = 0$, then $B(a,b) = 0$". Since P and Q are just (Banach space) isomorphisms, we have shown that the property is primarely related to the pointwise product, and not to the lattice properties. The result is particularly meaningful if we consider P and Q to be the isomorphisms associated to changes of unconditional basis of $\ell^p(\mathbb{N})$ and $\ell^q(\mathbb{N})$ whose elements are not in general disjoint.*

Remark 2. *Consider the bilinear map $B : E \times E' \to Y$, where E' denotes the topological dual of E. This bilinear map can only be $\odot_{P\times Q}$-factorable through the sequence space $\ell^1(\mathbb{N})$. Indeed, let P denote the isomorphism between E and $\ell^p(\mathbb{N})$ $(p \geq 1)$. Since the duals of isomorphic spaces are isomorphic, it follows that E' is isomorphic to $(\ell^p(\mathbb{N}))' = \ell^{p'}(\mathbb{N})$ for $\frac{1}{p} + \frac{1}{p'} = 1$ by the isomorphism P' that is adjoint map of P. Therefore B can only be $\odot_{P\times P'}$-factorable and in this case it is factorized through $\ell^1(\mathbb{N})$.*

3.1 Compactness properties of zero $\odot_{P\times Q}$-preserving bilinear maps

Theorem 2 provides a useful tool to obtain the main properties of zero $\odot_{P\times Q}$-preserving bilinear maps. It is already clear that (weakly) compactness of the factorization map T is necessary and sufficient condition for the (weakly) compactness of the zero $\odot_{P\times Q}$-preserving map B by the definition of the norm preserving product. Indeed, for a zero $\odot_{P\times Q}$-preserving map B,

$$\begin{aligned} B \text{ is (weakly) compact} &\iff B(U_X \times U_Y) \text{ is relatively (weakly) compact} \\ &\iff \overline{B}(U_{\ell^p(\mathbb{N})} \times U_{\ell^q(\mathbb{N})}) \text{ is relatively (weakly) compact} \\ &\iff T(U_{\ell^r(\mathbb{N})}) \text{ is relatively (weakly) compact} \\ &\iff T \text{ is (weakly) compact.} \end{aligned}$$

Now, we will give more specific situations.

Proposition 1. *Let* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *for* $1 \leq r < p, q < \infty$. *Suppose that there are isomorphisms* P *and* Q *such that the bilinear operator* $B : E \times F \to Y$ *is zero* $\odot_{P \times Q}$*-preserving. Then*

(i) $B(E \times F)$ *is a linear space;*

(ii) if P *and* Q *are isometries, then* $B(B_E \times B_F)$ *is convex;*

(iii) if $r = 1$ *and* Y *is reflexive, then* $B(B_E \times y)$ *is a relatively compact set for every* $y \in F$ *as well as* $B(x \times B_F)$ *is relatively compact for every* $x \in E$;

(iv) if $r > 1$, *then* $B(B_E \times B_F)$ *is relatively weakly compact;*

(v) if $1 \leq s < r < \infty$ *and* $Y = \ell^s(\mathbb{N})$, *then* $B(B_E \times B_F)$ *is relatively compact.*

Proof. Consider the factorization for B given by $B = T \circ (P \odot Q)$.

(i) Since $\odot$ is an n.p. product and B factors through it by Theorem 2, we have that $B(E \times F) = T(\ell^p(\mathbb{N}) \odot \ell^q(\mathbb{N})) = T(\ell^r(\mathbb{N}))$, that is, the range of a linear map. So it is a linear space.

(ii) Clearly, $A = P \odot Q(B_E \times B_F) = B_{\ell^p(\mathbb{N})} \odot B_{\ell^q(\mathbb{N})} = B_{\ell^r(\mathbb{N})}$ is a convex set, and so $T(A)$ is also convex.

(iii) Note that there is a sequence $b = Q(y)$ such that $A = P \odot Q(B_E, y)$ is equivalent to $B_{\ell^p(\mathbb{N})} \odot b \subset \ell^1(\mathbb{N})$. Recall that $1 < p, q < \infty$. Note also that $T : \ell^1(\mathbb{N}) \to Y$ is weakly compact by the reflexivity of the range space Y. Since A is a weakly compact set in $\ell^1(\mathbb{N})$ we have that $T(A)$ is relatively compact by the Dunford-Pettis property of $\ell^1(\mathbb{N})$.

(iv) Since $B(B_E \times B_F) = T(P(B_E) \odot Q(B_F))$, and $P(B_E) \odot Q(B_F)$ is equivalent to the unit ball of the reflexive space $\ell^r(\mathbb{N})$, we get the result.

(v) Recall that by Pitt's Theorem (see [9, Ch. 12]), every bounded linear operator from $\ell^r(\mathbb{N})$ into $\ell^s(\mathbb{N})$ is compact whenever $1 \leq s < r < \infty$. The factorization gives directly the result. $\square$

3.2 Zero $\odot_{P \times Q}$-preserving bilinear operators among Hilbert spaces

In this section, assume that E, F and Y are separable Hilbert spaces. Our first result shows a summability property of zero product preserving bilinear maps, and is a direct consequence of Grothendieck's Theorem. It also provides an integral domination for B. The second corollary is obtained as a result of the Schur's property of $\ell^1(\mathbb{N})$ (recall that a Banach space has the Schur's property if weakly convergent sequences and norm convergent sequences are the same) and it is again an application of the compactness properties of the bounded subsets of $\ell^1(\mathbb{N})$.

Corollary 2. *Let* H_1, H_2 *and* H_3 *be separable Hilbert spaces. Let* $B : H_1 \times H_2 \to H_3$ *be a zero* $\odot_{P \times Q}$*-preserving bilinear operator. Then*

(i) for every $x_1, \ldots, x_n \in H_1$, $y_1, \ldots, y_n \in H_2$ *there is a constant* $K > 0$ *such that*

$$\sum_{i=1}^{n} \left\| B(x_i, y_i) \right\| \leq K \sup_{z' \in B_{\ell^\infty(\mathbb{N})}} \sum_{i=1}^{n} \left| \left\langle P(x_i) \odot Q(y_i), z' \right\rangle \right|,$$

(ii) and there is a regular Borel measure η *over* $B_{\ell^\infty(\mathbb{N})}$ *such that*

$$\|B(x, y)\| \leq K \int_{B_{\ell^\infty(\mathbb{N})}} |\langle P(x) \odot Q(y), z' \rangle| \, d\eta(z'), \quad x \in H_1, \ y \in H_2.$$

Proof. Let us consider the zero $\odot_{P\times Q}$-preserving bilinear map $B : H_1 \times H_2 \to H_3$. Since any separable Hilbert space is isomorphic to the sequence space $\ell^2(\mathbb{N})$, we can define a bilinear map $\overline{B} = B(P^{-1} \times Q^{-1}) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to H_3$. The zero $\odot_{P\times Q}$-preserving property of B implies the $\odot$-preserving property of the map $\overline{B}$. Therefore, by Theorem 2 we have the factorization $\overline{B} := T \circ \odot$, where $T : \ell^1(\mathbb{N}) \to H_3$. One of the results of Grothendieck's Theorem states that every linear operator from $\ell^1(\mathbb{N})$ to a Hilbert space is 1-summing. It follows that, for every $x_1, \dots, x_n \in H_1$, $y_1, \dots, y_n \in H_2$ there is a constant $K > 0$ such that

$$\sum_{i=1}^{n} \left\| B(x_i, y_i) \right\| = \sum_{i=1}^{n} \left\| \overline{B}(P(x_i), Q(y_i)) \right\| \leq K \sup_{z' \in B_{\ell^\infty(\mathbb{N})}} \sum_{i=1}^{n} \left| \left\langle P(x_i) \odot Q(y_i), z' \right\rangle \right|.$$

The second inequality of the corollary given above is clearly seen by Pietsch Domination Theorem (see [9, Theorem 2.12]). This theorem states that every 1-summable operator has such a regular Borel measure. Thus, we get a regular Borel measure η over $B_{\ell^\infty(\mathbb{N})}$ satisfying

$$\|B(x,y)\| = \|\overline{B}(P(x), Q(y))\| \leq K \int_{B_{\ell^\infty(\mathbb{N})}} |\langle P(x) \odot Q(y), z' \rangle| \, d\eta(z')$$

for $x \in H_1$, $y \in H_2$. □

Corollary 3. *Let H_1, H_2 and H_3 be separable Hilbert spaces. Let $B : H_1 \times H_2 \to H_3$ be a zero $\odot_{P\times Q}$-preserving bilinear operator. Then*

(i) for every couple of sequences $(x_i)_{i=1}^\infty$ in H_1 and $(y_i)_{i=1}^\infty$ in H_2 such that $(P(x_i) \odot Q(y_i))_{i=1}^\infty$ is weakly convergent, we have that $(B(x_i, y_i))_{i=1}^\infty$ converges in the norm;

(ii) for $S_1 \subseteq H_1$ and $S_2 \subseteq H_2$ such that $P(S_1) \odot Q(S_2) \subseteq \ell^1(\mathbb{N})$ is relatively weakly compact, we have that $B(S_1 \times S_2)$ is relatively compact.

We can obtain some (weaker) summability results if we consider the range space Y with some cotype-related properties. It is known that a Banach space has the Orlicz property, if it is of cotype 2 (see [8, 8.9]). Recall that a Banach space is said to have the Orlicz property if the identity map in it is $(2,1)$-summing. It follows that for any zero $\odot_{P\times Q}$-preserving bilinear map $B : E \times F \to Y$ whose range space Y has the Orlicz property, we get a domination as follows: for $f_1, \dots, f_n \in E$ and $g_1, \dots, g_n \in F$,

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|_Y^2 \Big)^{1/2} \leq k \sup_{\varepsilon_i \in \{-1,1\}} \Big\| \sum_{i=1}^{n} \varepsilon_i \left(P(f_i) \odot Q(g_i) \right) \Big\|_{\ell^r(\mathbb{N})}.$$

Let us finish the paper with an application by using convolution maps defined on sequence spaces and function spaces.

3.3 Application: convolution maps

Consider any bilinear map $B : L^2[0,2\pi] \times L^2[0,2\pi] \to Y$ such that $B(f,g) = 0$ whenever f, $g \in L^2[0,2\pi]$ are such that $f \odot_{\widehat{\ }\times\widehat{\ }} g = \widehat{f} \odot \widehat{g} = 0$, where $\widehat{\ }$ denotes the Fourier transform. Plancherel's well-known theorem states that the Banach space $L^2[0,2\pi]$ is isometrically isomorphic to $\ell^2(\mathbb{Z})$ by the Fourier transform. Therefore, the bilinear map B is zero

$\odot_{\widehat{\ }\times\widehat{\ }}$-preserving. The class of these bilinear maps was investigated by Erdoğan E. et al in [10] by the term $*$-continuous map and they gave a factorization for B such that

$$B = T \circ \check{} \circ \odot \circ (\widehat{\ }\times\widehat{\ }) = T \circ *,$$

where $\check{}$ is the inverse Fourier transform.

Now, we will give a more specific example. $\mathcal{H}$ and $\mathcal{H}^2$ stand for the holomorphic functions on the unit disc $\mathbb{D}$ and Hardy space of the functions, respectively. Recall that Hardy space $\mathcal{H}^2$ consists of the functions whose all Fourier coefficients are zero with negative index, besides, it is closed subspace of $L^2[0, 2\pi]$ which is isomorphically isomorphic to the sequence space $\ell^2(\mathbb{N})$ by Fourier transform. It is possible to represent any holomorphic function $f \in \mathcal{H}$ as a Taylor polynomial $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This representation is given by the Fourier coefficients for the elements of $\mathcal{H}^2$ whenever $f \in \mathcal{H}^2$.

Arregui and Blasco defined the u-convolution of the holomorphic functions f and g in $\mathcal{H}$ given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ as $f *_u g(z) = \sum_{n=0}^{\infty} u(a_n, b_n) z^n$, where $u : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a bilinear continuous map (see [3, Definition 1.1.]). If we consider the bilinear map u defined as $u(a_n, b_n) = a_n \odot b_n$, then we get $f *_u g(z) = \sum_{n=0}^{\infty} (a_n \odot b_n) z^n$. Therefore, it is seen that u-convolution defined on $\mathcal{H}^2 \times \mathcal{H}^2$ to $\mathcal{H}^2$ is a zero $\odot_{\widehat{\ }\times\widehat{\ }}$-preserving, since $f \odot_{\widehat{\ }\times\widehat{\ }} g = \widehat{f}(n) \odot \widehat{g}(n) = 0$ implies $f *_u g = 0$ for all $f, g \in \mathcal{H}^2$. By Theorem 2, it follows that there is a linear map $T : \ell^1(\mathbb{N}) \to \mathcal{H}^2$ such that $f *_u g = T(\widehat{f}(n) \odot \widehat{g}(n)) = \sum_{n=0}^{\infty} x_n z^n$, where $(x_n)_{n=1}^{\infty}$ is the sequence in $\ell^1(\mathbb{N})$ obtained by the pointwise product $\widehat{f}(n) \odot \widehat{g}(n)$. Also, by Corollary 1 it is obtained that u-convolution is a symmetric map.

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Ердоган Е. *Білінійні оператори, що зберігають нульовий добуток, на просторах послідовностей* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 55–68.

Розглянемо пару просторів послідовностей і функцію добутку (канонічне білінійне відображення, асоційоване з поточковим множенням), що діє на ньому. Ми аналізуємо клас білінійних операторів, що "зберігають нульовий добуток", асоційований з цим добутком, визначених таким чином, що вони дорівнюють нулю на парах, в яких добуток дорівнює нулю. Білінійні оператори, що належать цьому класу, вже досліджувалися в контексті банахових алгебр, вони можуть бути охарактеризовані в термінах факторизації $\ell^r(\mathbb{N})$ просторів. Використовуючи це, ми демонструємо основні властивості цих відображень, такі як компактність і сумовність.

Ключові слова і фрази: простори послідовностей, білінійні оператори, факторизація, зберігаюче нульовий добуток відображення, добуток.