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CONSTRUCTION OF THE FUNDAMENTAL SOLUTION OF A CLASS OF DEGENERATE PARABOLIC EQUATIONS OF HIGH ORDER

In the article, using the modified Levy method, a Green's function for a class of ultraparabolic equations of high order with an arbitrary number of parabolic degeneration groups is constructed. The modified Levy method is developed for high-order Kolmogorov equations with coefficients depending on all variables, while the frozen point, which is a parametrix, is chosen so that an exponential estimate of the fundamental solution and its derivatives is conveniently used.

Key words and phrases: degenerated parabolic equations, modified Levy's method, Kolmogorov's equation, fundamental solution, parametrix.

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INTRODUCTION

A fundamental solution of the inverse Cauchy problem for degenerate parabolic equations of second-order variables with smooth coefficients was constructed first by M. Weber [10]. Under the same conditions on the coefficients, a fundamental solution of the Cauchy problem was constructed in [5], in the case of Holder coefficients for second-order equations with two degenerate groups. The Levy method was modified in [7], and in Banach spaces in [8], for the second order Kolmogorov systems with one degeneracy group [4]. The Kolmogorov equation of high order has features that make it easy to use the Levy method for constructing a fundamental solution. The parametric method was applied to a degenerate parabolic equation of high order with one group of parabolic degeneracy variables in [2, 3, 9] and with two degenerate groups in [1] and with four degenerate groups for Kolmogorov type systems of the second order in [6]. We modified the Levy method with respect to the properties of a fundamental solution of high-order Kolmogorov-type equations with coefficients dependent only on t, in particular a selected point which is a parameter so that an exponential estimate of the fundamental solution and its derivatives is conveniently used.

1 DESIGNATION, TASK STATEMENT AND MAIN RESULTS

Let us denote by $n_j \in N$, $j = \overline{1,p}$, $n_1 \geq n_2 \geq \cdots \geq n_p$, $n_0 = \sum\limits_{j=1}^{p} n_j$, $x = \left(x_1, \ldots, x_p\right)$, $x_j = (x_{j1}, \ldots, x_{jn_j})$, $x_j \in R^{n_j}$, $x \in R^{n_0}$, $\xi = \left(\xi_1, \ldots, \xi_p\right)$, $\xi_j = (\xi_{j1}, \ldots, \xi_{jn_j})$, $\xi_j \in R^{n_j}$, $\xi \in R^{n_0}$,

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$x^{(j)} = \left(x_1, \ldots, x_j\right) \in R^{\sum\limits_{k=1}^{j} n_k}$, $\xi^{(j)} = \left(\xi_1, \ldots, \xi_j\right) \in R^{\sum\limits_{k=1}^{j} n_k}$, $j = \overline{2,p}$. $\Gamma(\alpha)$ is Euler's gamma function and $B(a,b)$ is Euler's beta function.

$$x_j - \xi_j + \sum_{k=1}^{j-1} x_k \frac{(t-\tau)^{j-k}}{(j-k)!} = \left(x_{j1} - \xi_{j1} + \sum_{k=1}^{j-1} x_{k1} \frac{(t-\tau)^{j-k}}{(j-k)!}, .., x_{jn_j} - \xi_{jn_j} + \sum_{k=1}^{j-1} x_{kn_j} \frac{(t-\tau)^{j-k}}{(j-k)!}\right),$$

$$\rho_1\left(t, x_1, \tau, \xi_1\right) = \left(\left|x_1 - \xi_1\right| (t-\tau)^{-\frac{1}{2b}}\right)^q, \; q = \frac{2b}{2b-1}, \; b \in N,$$

$$\rho_j\left(t, x^{(j)}, \tau, \xi^{(j)}\right) = \left(\left|x_j - \xi_j + \sum_{k=1}^{j-1} x_k \frac{(t-\tau)^{j-k}}{(j-k)!}\right| (t-\tau)^{-\left(j-1+\frac{1}{2b}\right)}\right)^q, \; j = \overline{2,p},$$

$$\xi(t,\tau) = \left(\xi_1, \xi_2 - \xi_1 (t-\tau), \ldots, \xi_p + \sum_{k=1}^{p-1} (-1)^{p-k} \xi_k \frac{(t-\tau)^{p-k}}{(p-k)!}\right).$$

We investigate the Cauchy problem for the equation

$$\partial_t u(t,x) - \sum_{j=1}^{p-1} \sum_{\mu=1}^{n_{j+1}} x_{j\mu} \partial_{x_{j+1}\mu} u(t,x) = \sum_{|k| \le 2b} a_k(t,x) D_{x_1}^k u(t,x), \tag{1}$$

with the initial condition

$$u(t,x)\Big|_{t=\tau} = u_0(x), \qquad 0 \le \tau \le t \le T, \tag{2}$$

where τ is a fixed number, and operator

$$\partial_t - \sum_{|k| \le 2b} a_k(t,x) D_{x_1}^k, \quad D_{x_1}^k = \frac{(-1)^k \partial^{k_1 + \cdots + k_{n_1}}}{\partial x_1^{k_1} \ldots \partial x_{n_1}^{k_{n_1}}}, \quad |k| = k_1 + \cdots + k_{n_1}, \tag{3}$$

is uniformly parabolic in the sense of Petrovsky in the strip $\Pi_{[0,T]} = (t,x), x \in R^{n_0}, 0 \le t \le T$.

Let us suppose that

1) $a_k(t,x), \partial_{x_j} a_k(t,x), j = \overline{2,p}$, are continuous and bounded in $\Pi_{[0,T]}$,

2) there are constants $\alpha \in (0,1]$, $r \in (0,1]$, such that for any $x \in R^{n_0}, \xi \in R^{n_0}$ and $t \in [0,T]$

$$\left|a_k(t,x) - a_k(t,\xi)\right| \le c_1 \left(\left|x_1 - \xi_1\right|^\alpha + \sum_{j=2}^{p} \left|x_j - \xi_j\right|\right),$$

$$\left|\partial_{x_j} a_k(t,x) - \partial_{x_j} a_k(t,\xi)\right| \le c_1 |x - \xi|^r, \qquad j = \overline{2,p}.$$

Theorem 1. *If conditions 1)–2) are satisfied, then equation* (1) *has a fundamental solution of the Cauchy problem* (1)–(2) $Z(t,x;\tau,\xi)$ *at* $t > \tau$ *and the following estimations hold:*

$$\left|\partial_{x_j} Z(t,x;\tau,\xi)\right| \le A(t-\tau)^{-\sum\limits_{s=1}^{p} \frac{2b(s-1)+1}{2b}(n_s + |m_s|)} \Phi(t,x;\tau,\xi),$$

$m_s = 0$, *at* $s \ne j$, $m_j = 1$, $j = \overline{2,p}$;

$$\left|\partial_{x_1}^{m_1} Z(t,x;\tau,\xi)\right| \le A_{m_1}(t-\tau)^{-\frac{n_1 + |m_1|}{2b} - \sum\limits_{s=2}^{p} \frac{2b(s-1)+1}{2b} n_s} \Phi(t,x;\tau,\xi),$$

$|m_1| \leq 2b,\ x \in R^{n_0},\ \ \xi \in R^{n_0},\ \ 0 \leq \tau < t \leq T$, *where*

$$\Phi(t,x;\tau,\xi) = \sum_{i=1}^{\infty} A^s \Gamma\left(1+\frac{s\alpha^*}{2b}\right)\Gamma\left(\frac{\alpha^*}{2b}\right)\Gamma^{-1}\left(1+\frac{\alpha^*(1+s)}{2b}\right) \times \exp\left\{-c_0\rho_1(t,x_1,\tau,\xi_1) - 2^{-2sp}c_0\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right\},$$

and positive constants A, A_{m_1}, c_0 depend on n_0, $2b$, c_1, α, r, and the constant of parabolicity of the operator (3) *is* $\sup\limits_{(t,x)\in\Pi_{[0,T]}} |a_k(t,x)|$ *and* $\alpha^* = \min(\alpha, r)$.

Proof. To prove the theorem, we write equation (1) in the form

$$\partial_t u(t,x) - \sum_{j=1}^{p-1}\sum_{\mu=1}^{n_{j+1}} x_{j\mu}\partial_{x_{j+1}\mu} u(t,x) = \sum_{|k|=2b} a_k(t,\xi(t,\tau)) D^k_{x_1} u(t,x) + \sum_{|k|=2b}[a_k(t,x) - a_k(t,\xi(t,\tau))]D^k_{x_1}u(t,x) + \sum_{|k|<2b} a_k(t,x)D^k_{x_1}u(t,x). \tag{4}$$

Let us denote by $Z_0(t,x;\tau,\xi;\xi(t,\tau))$ the fundamental solution of equation

$$\partial_t u(t,x) - \sum_{j=1}^{p-1}\sum_{\mu=1}^{n_{j+1}} x_{j\mu}\partial_{x_{j+1}\mu} u(t,x) = \sum_{|k|=2b} a_k(t,\xi(t,\tau)) D^k_{x_1} u(t,x). \tag{5}$$

Fundamental solution $Z_0(t,x,;\tau,\xi;\xi(t,\tau))$ of equation (5) is constructed in [5], where ξ is fixed. For derivatives of $Z_0(t,x;\tau,\xi;\xi(t,\tau))$ the following inequalities are performed

$$|\partial^m_x Z_0(t,x;\tau,\xi;\xi(t,\tau))| \leq C_m(t-\tau)^{-\sum\limits_{s=1}^{p}\frac{2b(s-1)+1}{2b}(n_s+|m_s|)} \times \exp\left\{-c_0\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right) + \rho_1(t,x_1,\tau,\xi_1)\right)\right\}, \tag{6}$$

where $|m| = \sum\limits_{j=1}^{p}|m_j|$, $|m_j| = \sum\limits_{k=1}^{n_j} m_{jk}$, $t > \tau, C_m > 0$.

Fundamental solution $Z(t,x;\tau,\xi)$ of equation (1) will be sought in the form

$$Z(t,x;\tau,\xi) = Z_0(t,x;\tau,\xi;\xi(t,\tau)) + \int_\tau^t d\beta \int_{R^{n_0}} Z_0(t,x;\beta,\gamma;\gamma(t,\beta))\,\varphi(\beta,\gamma;\tau,\xi)\,d\gamma, \tag{7}$$

where $\varphi(t,x;\tau,\xi)$ is an unknown absolutely integrable on R^{n_0} function at $t > \tau$.

We substitute (6) into equation (1) with respect to the function $\varphi(t,x;\tau,\xi)$, then

$$\varphi(t,x;\tau,\xi) = K(t,x;\tau,\xi) + \int_\tau^t K(t,x;\beta,\gamma)\varphi(\beta,\gamma;\tau,\xi)\,d\gamma, \tag{8}$$

where

$$K(t,x;\tau,\xi) = \sum_{|k|=2}(a_k(t,x) - a_k(t,\xi(t,\tau)))D^k_{x_1}Z_0(t,x;\tau,\xi;\xi(t,\tau)) + \sum_{|k|<2b} a_k(t,x)D^k_{x_1}Z_0(t,x;\tau,\xi;\xi(t,\tau)).$$

The solution of equation (8) can be represented by a Neumann series

$$\varphi\left(t,x;\tau,\xi\right)=\sum_{n=1}^{\infty}K_n\left(t,x;\tau,\xi\right), \tag{9}$$

where

$$K\left(t,x;\tau,\xi\right)=K_1\left(t,x;\tau,\xi\right);\quad K_n\left(t,x;\tau,\xi\right)=\int_{\tau}^{t}d\beta\int_{R^{n_0}}K(t,x;\beta,\gamma)K_{n-1}\left(\beta,\gamma;\tau,\xi\right)d\gamma. \tag{10}$$

Let us show the convergence of series (9) and the required estimation of the function for the Levy method $\varphi\left(t,x;\tau,\xi\right)$ and its increments.

Using the following lemma, which generalizes Lemma 2 and Lemma 1 in [3] for equation (1), we can obtain estimates for $K_n\left(t,x;\tau,\xi\right)$ and $K\left(t,x;\tau,\xi\right)$.

Lemma 1. *For any points* (t,x), (β,ξ), (τ,y), $0\le\tau<\beta<t$, $x\in R^{n_0}$, $\xi\in R^{n_0}$, $y\in R^{n_0}$, $b\in N$, $2b>2$ *the following inequality holds*

$$\begin{aligned}\rho_1\left(t,x_1,\beta,\xi_1\right)+\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\beta,\xi^{(j)}\right)+\rho_1\left(\beta,\xi_1,\tau,y_1\right)+\sum_{j=2}^{p}\rho_j\left(\beta,\xi^{(j)},\tau,y^{(j)}\right)\\ \ge 2^{-2p}\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,y^{(j)}\right)+\rho_1\left(t,x_1,\tau,y_1\right)\right).\end{aligned} \tag{11}$$

The proof of Lemma 1 is based on the inequalities

$$\begin{aligned}&\rho_p\left(t,x^{(p)},\beta,\xi^{(p)}\right)+\rho_p\left(\beta,\xi^{(p)},\tau,y^{(p)}\right)\\ &\ge 2^{-2}\left(\left|x_p-y_p+\sum_{j=1}^{p-1}\left[x_k(t-\beta)^{p-k}+\xi_k(\beta-\tau)^{p-k}\right]\frac{1}{(p-k)!}\right|(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q.\end{aligned} \tag{12}$$

From (12) we can get

$$\begin{aligned}&\left(\left|x_p-y_p+\sum_{k=1}^{p-1}\left[x_k(t-\beta)^{p-k}+\xi_k(\beta-\tau)^{p-k}\right]((p-k)!)^{-1}\right|(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q\\ &\ge 2^{-2}\left(\left|x_p-y_p+\sum_{k=1}^{p-1}\left[x_k\left(t-\beta\right)^{p-k}+\xi_k\left(\beta-\tau\right)^{p-k}\right]((p-k)!)^{-1}\right.\right.\\ &\qquad\left.\left.\times\ \frac{x_1((\beta-\tau)^{p-1}+(t-\beta)^{p-1})}{(p-1)!}\right|(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q\\ &-\sum_{\mu=1}^{n_p}\left(\left[|x_{1\mu}-\xi_{1\mu}|(\beta-\tau)^{p-1}\right]((p-k)!)^{-1}(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q.\end{aligned} \tag{13}$$

Applying (12) to the first part of (13) $(p-2)$ times, we have

$$\left(\left|x_p-y_p+\sum_{k=1}^{p-1}(x_k(t-\beta)^{p-k}+\xi_k\left(\beta-\tau\right)^{p-k})((p-k)!)^{-1}\right|(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q$$

$$\geq 2^{-2(p-1)}\rho_p\left(t,x^{(p)},\tau,y^{(p)}\right)-\sum_{\mu=1}^{n_p}\left(\left|x_{j\mu}-\xi_{j\mu}\right|(\beta-\tau)^{p-1})\left((p-1)!\right)^{-1}(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q$$
$$-\sum_{j=2}^{p-1}\sum_{\mu=2}^{n_j}2^{-2(j-1)}\left(\left|x_{1\mu}-\xi_{1\mu}+\sum_{k=2}^{j-1}x_{k\mu}(t-\beta)^{j-k}((j-k)!)^{-1}\right|\frac{(\beta-\tau)^{p-j}}{(p-j)!}(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q. \tag{14}$$

Taking into account the inequalities (11)–(14), we get

$$\rho_p\left(t,x^{(p)},\beta,\xi^{(p)}\right)+\rho_p\left(\beta,\xi^{(p)},\tau,y^{(p)}\right)\geq 2^{-2p}\rho_p\left(t,x^{(p)},\tau,y^{(p)}\right)$$
$$-\sum_{j=2}^{p-1}\sum_{\mu=1}^{n_p}2^{-2(j-1)}\left(\left|x_{j\mu}-\xi_{j\mu}+\sum_{k=2}^{j-1}x_{k\mu}(t-\beta)^{j-k}\ ((j-k)!)^{-1}\right|(\beta-\tau)^{p-j}((p-j)!)^{-1}\right. \tag{15}$$
$$\left.\times(t-\tau)^{-p+j-\frac{1}{2b}}\right)^q-2^{-2}\sum_{\mu=1}^{n_p}\left(\left|x_{1\mu}-\xi_{1\mu}\right|(\beta-\tau)^{p-1}((p-1)!)^{-1}(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q.$$

We will collect all of the terms that contain $x_{p-1}-\xi_{p-1}$

$$\rho_{p-1}\left(\beta,x^{(p-1)},\beta,\xi^{(p-1)}\right)-2^{-2(p-1)}\sum_{\mu=1}^{n_p}\left(\left|x_{p-1\mu}-\xi_{p-1\mu}+\sum_{k=1}^{p-2}x_k(t-\beta)^{p-1-k}\right.\right.$$
$$\left.\left.\times\frac{1}{((p-1-k)!)}\right|(\beta-\tau)\ (t-\beta)^{-p+1-\frac{1}{2b}}\right)^q\geq\sum_{\mu=n_{p+1}}^{n_{p-1}}\left(\left|x_{p-1\mu}-\xi_{p-1\mu}\right.\right.$$
$$\left.\left.+\sum_{k=1}^{p-2}x_{k\mu}\frac{(t-\beta)}{(p-1-k)!}^{p-1-k}\right|(t-\beta)^{-p+3-\frac{1}{2b}}\right)^q+\sum_{\mu=1}^{n_p}\left(1-2^{-2(p-1)}\right) \tag{16}$$
$$\times\left(\left|x_{p-1\mu}-\xi_{p-1\mu}+\sum_{k=1}^{p-2}x_{k\mu}(t-\beta)^{p-1-k}((p-1-k)!)^{-1}\right|(t-\tau)^{-p+2-\frac{1}{2b}}\right)^q.$$

Repeating all inequalities (12), (16) for the terms $\rho_j\left(t,x^{(j)},\beta,\xi^{(j)}\right)+\rho_j\left(\beta,\xi^{(j)},\tau,y^{(j)}\right)$, $j=\overline{1,p-1}$, and adding their together we have

$$\rho_1\left(t,x_1,\beta,\xi_1\right)+\rho_1\left(\beta,\xi_1,\tau,y_1\right)+\sum_{j=2}^{p}\left(\rho_j(t,x^(j),\beta,\xi^(j))+\rho_j\left(\beta,\xi^{(j)},\tau,y^{(j)}\right)\right)$$
$$\geq 2^{-2p}\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,y^{(j)}\right)+\rho_1\left(t,x_1,\tau,y_1\right)\right).$$

Lemma 2. *The following estimations are performed for reproducing kernels:*

$$\left|K_m\left(t,x;\tau,\xi\right)\right|\leq A_m^m(t-\tau)^{-\sum_{j=1}^{p}\frac{(1+2b(j-1))n_j}{2b}-1+\frac{m\alpha}{2b}}$$
$$\times\exp\left\{\rho_1\left(t,x_1,\beta,\xi_1\right)-2^{-2pm}c\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right\}, \tag{17}$$

at $m\leq m^*=\left[\sum_{j=1}^{p}\frac{\left((1+2b(j-1))n_j+2b\right)}{\alpha}\right]+1;$

$$|K_{m+l}(t,x;\tau,\xi)| \le A_m^{m+l}\prod_{k=0}^{l-1} B\left(\frac{\alpha}{2b},1+\frac{\alpha k}{2b}\right)(t-\tau)^{\frac{\alpha l}{2b}}$$
$$\times\exp\left\{-c\rho_1(t,x_1,\beta,\xi_1) - 2^{-2p(m+l)}\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right\}, \tag{18}$$

at $m+l>m^*$.

From (17), (18) it follows the convergence of a series (9) following for $\varphi(t,x;\tau,\xi)$

$$|\varphi(t,x;\tau,\xi)| \le A(t-\tau)^{-\sum\limits_{j=1}^{p}\frac{(1+2b(j-1))n_j+2b-\alpha}{2b}}\Phi(t,x;\tau,\xi). \tag{19}$$

Let us prove the existence of derivatives $\partial_{x_j}\varphi(t,x;\tau,\xi)$, $j=\overline{2,p}$, at $t>\tau$.

Under the assumption 1), there are continuous derivatives $\partial_{x_j}K(t,x;\tau,\xi)$, $j=\overline{2,p}$ satisfying the estimations

$$\left|\partial_{x_j}K(t,x;\tau,\xi)\right| \le A\exp\left\{-c\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)+\rho_1(t,x_1,\tau,\xi_1)\right)\right\}$$
$$\times(t-\tau)^{-\sum\limits_{s=1}^{p}\frac{(2b(s-1)+1)n_s}{2b}-\frac{j-(1-\alpha^*)}{2b}},\quad t>\tau. \tag{20}$$

To prove the existence of derivatives $\partial_{x_j}K(t,x;\tau,\xi)$, $j=\overline{2,p}$, we use the following property of the fundamental solution of equation (5) with $\xi(t,\tau)=y$, where y is a parameter

$$\partial_t u(t,x) - \sum_{j=1}^{p-1}\sum_{\mu=1}^{n_{j+1}} x_{j\mu}\partial_{x_{j+1}\mu}u(t,x) = \sum_{|k|\le 2b} a_k(t,\xi(t,\tau))D_{x_1}^k u(t,x).$$

Property 1. If $a_k(t,y)$ have continuous bounded derivatives by the parameter y up to the order r, then there are continuous derivatives by y, $\partial_y^s\partial_{x_1}^m Z_0(t,x;\tau,\xi;y)$, $s\in\overline{0,r}$, and

$$\left|\partial_{x_1}^m\partial_y^s Z_0(t,x;\tau,\xi;y)\right| \le C_m\exp\left\{-c\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)+\rho_1(t,x_1,\tau,\xi_1)\right)\right\}$$
$$\times(t-\tau)^{-\sum\limits_{j=1}^{p}\frac{(1+2b(j-1))n_j}{2b}-\frac{|m|}{2b}}. \tag{21}$$

Let us consider $\partial_{x_{p\mu}}K(t,x;\beta,\gamma)$, $\mu=\overline{1,n_p}$. Then

$$\partial_{x_{p\mu}}K(t,x;\beta,\gamma) = \sum_{|k|=2b}\left(\partial_{x_{p\mu}}a_k(t,x)-\partial_{\gamma_{p\mu}}a_k(t,\gamma(t,\beta))\right)$$
$$\times D_{x_1}^k Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) + \sum_{|k|=2b}(\partial_{\gamma_{p\mu}}a_k(t,\gamma(t,\beta)))D_{x_1}^k Z_0(t,x;\beta,\gamma;\gamma(t,\beta))$$
$$+\sum_{|k|=2b}(a_k(t,x)-a_k(t,\gamma(t,\beta)))\partial_{x_{p\mu}}D_{x_1}^k Z_0(t,x;\beta,\gamma;\gamma(t,\beta))$$
$$+\sum_{|k|<2b}\partial_{x_{p\mu}}a_k(t,x)\,D_{x_1}^k Z_0(t,x;\beta,\gamma;\gamma(t,\beta)). \tag{22}$$

Let us rewrite (22) by a convenient form for applications

$$\begin{aligned}
\partial_{x_{p\mu}} K(t,x;\beta,\gamma) = & \sum_{|k|=2b} \left(\partial_{x_{p\mu}} a_k(t,x) - \partial_{\gamma_{p\mu}} a_k(t,\gamma(t,\beta))\right) \\
& \times D^k_{x_1} Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) - \partial_{\gamma_{p\mu}} \left(\sum_{|k|=2b} \left(a_k(t,x) - a_k(t,\gamma(t,\beta))\right) \right. \\
& \times D^k_{x_1} Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) + \sum_{|k|=2b} \left(a_k(t,x) - a_k(t,\gamma(t,\beta))\right) \\
& \times D^k_{x_1} \partial_{\overline{\gamma}_{p\mu}} D^k_{x_1} Z_0(t,x;\beta,\gamma;\overline{\gamma}(t,\beta)) \left|_{\overline{\gamma}=\gamma} + \sum_{|k|<2b} \left(a_k(t,x)\right)'_{x_{p\mu}} D^k_{x_1} Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) \right. \\
& + \sum_{|k|<2b} a_k \partial_{\overline{\gamma}_{p\mu}} D^k_{x_1} Z_0(t,x;\beta,\gamma;\overline{\gamma}(t,\beta)) \left|_{\overline{\gamma}=\gamma} - \sum_{|k|<2b} a_k(t,x) D^k_{x_1} \partial_{\gamma_{p\mu}} Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) \right.,
\end{aligned} \tag{23}$$

where $\mu = \overline{1,n_p}$, $\overline{\gamma} = \left(\gamma_1,\ldots,\gamma_{p-1},\overline{\gamma_p}\right)$. Using the images (23), estimates (6) and (21) and integrating by parts of the terms with $\partial_{\gamma_{p\mu}}$, we can get $\partial_{x_{p\mu}} K_2(t,x;\tau,\xi) = \lim_{h\to 0} \int_0^{t-h} d\beta \int_{R^{n_0}} \partial_{x_{p\mu}} K(t,x;\ \beta,\gamma) K(\beta,\gamma;\tau,\xi)\, d\gamma$.

From the estimations of reproducing kernel (18), estimations of derivatives of the kernel (20) and Lemma 1, we obtain $\left|\partial_{x_{p\mu}} K_2(t,x;\tau,\xi)\right| \le A_2 \exp\Big\{ -c_2(1-\varepsilon)\Big(\rho_1(t,x_1,\tau,\xi_1) +2^{-2p} \sum_{j=2}^{p} \rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right) + \rho_1(t,x_1,\tau,\xi_1)\Big)\Big\} (t-\tau)^{-\sum_{j=1}^{p} \frac{(1+2b(j-1))n_j}{2b} - p - (1-\alpha^*)/2b}$ at $t > \tau$. By the method of mathematical induction we can prove the existence $\partial_{x_{p\mu}} K_m(t,x;\tau,\xi)$ for any m and evaluation

$$\begin{aligned}
\left|\partial_{x_{p\mu}} K_m(t,x;\tau,\xi)\right| \le & A_m(\varepsilon) \exp\Big\{-c_2(1-\varepsilon m)(\rho_1(t,x_1,\tau,\xi_1) + 2^{-mp} \sum_{j=2}^{p} \rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right) \\
& + \rho_1(t,x_1,\tau,\xi_1)\Big\} (t-\tau)^{-\sum_{j=1}^{p} \frac{(1+2b(j-1))n_j}{2b} - p - (1-\alpha m)/2b}, \qquad \mu = \overline{1,n_p}.
\end{aligned} \tag{24}$$

Taking into account the estimation (24), we can estimate the series $\sum_{m=1}^{\infty} \partial_{x_{p\mu}} K_m(t,x;\ \tau,\xi)$ by a converging series:

$$\begin{aligned}
\left|\sum_{m=1}^{\infty} \partial_{x_{p\mu}} K_m(t,x;\tau,\xi)\right| \le & \sum_{m=1}^{l} A_m (t-\tau)^{-\sum_{j=1}^{p} \frac{(1+2b(j-1))n_j}{2b} - p - (1-\alpha^* m)/2b} \\
& \times \exp\left\{-c_2(1-m\varepsilon)\left(c_m \rho_1(t,x_1,\tau,\xi_1) + 2^{-2mp}\left(\rho_1(t,x_1,\tau,\xi_1) + \sum_{j=2}^{p} \rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right)\right)\right\} \\
& + \sum_{k=1}^{\infty} A_0 \left(\Gamma\left(\frac{\alpha^*}{2b}\right) F A_0\right)^k (t-\tau)^{\frac{\alpha^* k}{2b}} \Gamma^{-1}\left(1 + \frac{k\alpha^*}{2b}\right) \\
& \times \exp\left\{-c_5\left(c_{4l+k+1}\rho_1(t,x_1,\tau,\xi_1) + 2^{-2p(l+k-1)}\left(\rho_1(t,x_1,\tau,\xi_1) + \sum_{j=2}^{p} \rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right)\right)\right\},
\end{aligned} \tag{25}$$

where $l = \left[\sum_{j=1}^{p} \frac{(1+2b(j-1))n_j + 2bp + 1}{\alpha^*}\right] + 1$, and A_0, F are positive constants,

$$F = \left(2\int_0^{\infty} \exp\left\{-\frac{\alpha^2}{2}\right\} d\alpha\right)^{n_0}.$$

The series $\sum\limits_{m=1}^{\infty} \partial_{x_p} K_m(t,x;\tau,\xi)$ at $0<\delta\le t-\tau\le T$ is convergent uniformly and absolutely. Then $\partial_{x_p}\varphi(t,x;\tau,\xi) = \sum\limits_{m=1}^{\infty}\partial_{x_p}K(t,x;\tau,\xi)$ and $\partial_{x_p}K_m(t,x;\tau,\xi)$ are continuous, then in the domain of convergence and $\partial_{x_p}\varphi(t,x;\tau,\xi)$ continuous function. Inequality (25) will be written in the form

$$\left|\partial_{x_p}\varphi(t,x;\tau,\xi)\right| \le A(t-\tau)^{-\sum\limits_{j=1}^{p}\frac{(1+2b(j-1))n_j}{2b}-p-(1-\alpha^*)/2b}\Phi(t,x;\tau,\xi).$$

Let us consider $\partial_{x_{j\mu}}K(t,x;\beta,\gamma)$, $j=\overline{2,p-1}$, $\mu=\overline{1,n_j}$. For $\mu=\overline{n_{j-1}+1,n_j}$ formula (23) is true with the corresponding replacing p by j. For $\mu=\overline{1,n_{j-1}}$, $\partial_{x_{j\mu}}K(t,x;\beta,\gamma)$ can be written in the form

$$\begin{aligned}
\partial_{x_{j\mu}}K(t,x;\beta,\gamma) &= \sum_{|k|=2b}\left[\partial_{x_{j\mu}}a_k(t,x)-\partial_{y_{j\mu}}a_k(t,y)\Big|_{y=\gamma(t,\beta)}\right]D^k_{x_1}Z_0(t,x;\beta,\gamma;\gamma(t,\beta))\\
&-\partial_{\gamma_{j\mu}}\left(\sum_{|k|=2b}\left[a_k(t,x)-a_k(t,\gamma(t,\beta))\right]D^k_{x_1}Z_0(t,x;\beta,\gamma;\gamma(t,\beta))\right)\\
&+\sum_{|k|=2b}\left[a_k(t,x)-a_k(t,\gamma(t,\beta))\right]\,\partial_{\overline{\gamma}_{j\mu}}D^k_{x_1}Z_0(t,x;\beta,\gamma;\overline{\gamma}(t,\beta))\,|_{\overline{\gamma}=\gamma}\\
&+\sum_{l=1}^{p-j}\sum_{|k|=2b}\left[a_k(t,x)-a_k(t,\gamma(t,\beta))\right]\partial_{\gamma_{j+l,\mu}}D^k_{x_1}Z_0(t,x;\beta,\gamma;\overline{\gamma}(t,\beta))\,|_{\overline{\gamma}=\gamma}\\
&\times(-1)^l\frac{(t-\tau)^{p-l-l}}{(p-j-l)!}\sum_{|k|=2b}\partial_{y_{j+l\ \mu}}a_k(t,y)\Big|_{y=\gamma(t,\beta)}(-1)^l\frac{(t-\beta)^{p-j-l}}{(p-j-l)!}\\
&\times D^k_{x_1}\partial_{\gamma_{j\mu}}Z_0(t,x;\beta,\gamma;\gamma(t,\beta))+\sum_{|k|<2b}\left(a_k(t,x)\right)'_{x_{j\mu}}D^k_{x_1}Z_0(t,x;\beta,\gamma;\gamma(t,\beta))\\
&+\sum_{|k|<2b}\partial_{\gamma_{j\mu}}a_k(t,x)D^k_{x_1}Z_0(t,x;\beta,\gamma;\gamma(t,\beta))\\
&+\sum_{|k|<2b}a_k(t,x)\,D^k_{x_1}\partial_{\overline{\gamma}_{j\mu}}Z_0(t,x;\beta,\gamma;\overline{\gamma}(t,\beta))\,|_{\overline{\gamma}=\gamma}\\
&+(-1)^l\frac{(t-\beta)^{p-j-l}}{(p-j-l)!}\sum_{l=1}^{p-j}\sum_{|k|=2b}a_k(t,x)D^k_{x_1}\partial_{\gamma_{j+l}}Z_0(t,x;\beta,\gamma;\gamma(t,\beta)).
\end{aligned}\tag{26}$$

Kernels have the highest singularity at the variable x_p. Also, using (26) we have the existence of $\partial_{x_j}\varphi(t,x;\tau,\xi)$, $j=\overline{2,p-1}$ and the following estimations

$$\left|\partial_{x_j}\varphi(t,x;\tau,\xi)\right| \le A(t-\tau)^{-\sum\limits_{s=1}^{p}\frac{(1+2b(j-1))n_s-\alpha^*+1}{2b}-j}\Phi(t,x;\tau,\xi),\qquad j=\overline{2,p-1}.$$

Using arguments like in [1] we can get

$$\Delta_{hx_1}\varphi(t,x;\tau,\xi)=\Delta_{hx_1}K(t,x;\tau,\xi)+\int_\tau^t d\beta\int_{R^{n_0}}\Delta_{hx_1}K(t,x;\beta,\gamma)K(\beta,\gamma;\tau,\xi)\,d\gamma.$$

Applying the technique developed for parabolic systems in [6], and the evaluation of reproducing kernels, we obtain

$$\left|\Delta_{hx_1}\varphi\left(t,x;\tau,\xi\right)\right| \leq |h_{x_1}|^{\alpha_1}(t-\tau)^{-\sum\limits_{s=1}^{p}\frac{(1+2b(s-1))n_s-(1-\alpha_2)}{2b}-j}\Phi\left(t,x;\tau,\xi\right),$$

$$\alpha_1 > 0,\ \alpha_2 > 0,\ \alpha_1+\alpha_1 = \alpha.$$

The existence and evaluation of $\partial^k_{x_1}Z\left(t,x;\tau,\xi\right)$, $|k| \leq 2b$, at $t > \tau$, are established for both of the cases of parabolic equations and systems in [6]. The theorem is proved. □

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У статті модифікованим методом Леві побудовано функцію Ґріна для одного класу ультрапараболічних рівнянь високого порядку з довільною кількістю груп виродження параболічності. Модифікований метод Леві розроблено для рівнянь Колмогорова високого порядку з коефіцієнтами залежними від усіх змінних, при цьому заморожена точка, яка є параметриксом, підібрана так, щоб зручно використовувалася експоненціальна оцінка фундаментального розв'язку та його похідних.

Ключові слова і фрази: вироджені параболічні рівняння, модифікований метод Леві, рівняння Колмогорова, фундаментальний розв'язок, параметрикс.