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HYPERCYCLIC OPERATORS ON SPACES OF BLOCK-SYMMETRIC ANALYTIC FUNCTIONS

The paper contains proof of the hypercyclicity of "symmetric translation" on the algebras of block-symmetric analytic functions of bounded type on an isomorphic copy of ℓ_1.

Key words and phrases: block-symmetric analytic functions, hypercyclic operator, symmetric translation operator.

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INTRODUCTION

Let X be a Fréchet linear space. A continuous linear operator $T : X \to X$ is called hypercyclic if there is a vector $x_0 \in X$ for which the orbit under T, $Orb(T, x_0) = \{x_0, Tx_0, T^2x_0, \ldots\}$, is dense in X. Every such vector x_0 is called a hypercyclic vector of T. The classical Birkhoff theorem [2] asserts that any operator of composition with translation $x \to x + a$, $T_a : f(x) \to f(x + a)$ is hypercyclic on the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$ if $a \neq 0$. The Birkhoff translation T_a has also been regarded as a differentiation operator

$$T_a(f) = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n f.$$

A generalization of the Birkhoff theorem was proved by Godefroy and Shapiro in [3]. They showed that if $\varphi(z) = \sum\limits_{|\alpha|\geq 1} c_\alpha z^\alpha$ is a non-constant entire function of exponential type on $\mathbb{C}^n$, then the operator

$$f \to \sum_{|\alpha|\geq 1} c_\alpha D^\alpha f, \qquad f \in H(\mathbb{C}^n),$$

is hypercyclic. Note that an analog of the Godefroy-Shapiro Theorem for weakly continuous analytic functions on Banach spaces which are bounded on bounded subsets was proved by Aron and Bés in [1]. A symmetric version of the Godefroy-Shapiro Theorem is proved in [9]. The purpose of this paper is to extend the results on hypercyclicity for the case of a special translation operator on the space of block-symmetric analytic functions of bounded type on ℓ_1.

The following proposition is well known (see [4, Proposition 4]).

Proposition 1. *Let T be a hypercyclic operator on X and A be an isomorphism on X. Then $A^{-1}TA$ is hypercyclic.*

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1 BLOCK-SYMMETRIC ANALYTIC FUNCTIONS

Let $\mathcal{X}^2 = \oplus_{\ell_1}\mathbb{C}^2$ be an infinite ℓ_1-sum of copies of Banach space $\mathbb{C}^2$. So any element $\overline{x} \in \mathcal{X}^2$ can be represented as a sequence $\overline{x} = (x_1, \ldots, x_n, \ldots)$, where $x_n \in \mathbb{C}^2$, $n \in \mathbb{N}$, and $\|\overline{x}\| = \sum\limits_{k=1}^{\infty} \|x_k\|_{\mathbb{C}^2} < \infty$.

A polynomial P on the space $\mathcal{X}^2$ is called block-symmetric (or vector-symmetric) if

$$P\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_1, \ldots, \begin{pmatrix} u_m \\ v_m \end{pmatrix}_m, \ldots\right) = P\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_{\sigma(1)}, \ldots, \begin{pmatrix} u_m \\ v_m \end{pmatrix}_{\sigma(m)}, \ldots\right)$$

for every permutation σ on the set $\mathbb{N}$, where $\begin{pmatrix} u_i \\ v_i \end{pmatrix} \in \mathbb{C}^2$, $i \in \mathbb{N}$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X}^2)$ the algebra of block-symmetric polynomials on $\mathcal{X}^2$.

In paper [8] it was shown that the following vectors form an algebraic basis of $\mathcal{P}_{vs}(\mathcal{X}^2)$

$$H^{k_1,k_2}(x,y) = \sum_{i=1}^{\infty} (x_i)^{k_1}(y_i)^{k_2},$$

where $(x_i, y_i) \in \mathbb{C}^2, i \geq 1$. Let us fix an enumerating of the basis functions $H_1 = H^{1,0}, \ldots, H_n = H^{m_1,m_2}, \ldots$.

Denote by $H^n_{bvs}(\mathcal{X}^2)$ the algebra of block-symmetric analytic functions on $\mathcal{X}^2$ which are topologically generated by polynomials $H_1, \ldots, H_n$. We will use the notation $\mathbf{H} := \{H_k\}_{k=1}^n$.

Lemma 1. *The map*

$$\mathcal{H}^{\mathbf{H}}_n : f(t_1, \ldots, t_n) \to f(H_1, \ldots, H_n)$$

is a topological isomorphism from the algebra $H(\mathbb{C}^n)$ to the algebra $H^n_{bvs}(\mathcal{X}^2)$.

Proof. Evidently, $\mathcal{H}^{\mathbf{H}}_n$ is a homomorphism. It is known [6, 7] that for every vector $(t_1, \ldots, t_n) \in \mathbb{C}^n$ there exists an element $(x, y) \in \mathcal{X}^2$ such that $H_1(x, y) = t_1, \ldots, H_n(x, y) = t_n$. Therefore the map $\mathcal{H}^{\mathbf{H}}_n$ is injective. Let us show that $\mathcal{H}^{\mathbf{H}}_n$ is surjective. Let $u \in H^n_{bvs}(\mathcal{X}^2)$ and $u = \sum u_k$ be the Taylor series expansion of u at zero. For every homogeneous polynomial u_k there exists a polynomial q_k on $\mathbb{C}^n$ such that $u_k = q_k(H_1, \ldots, H_n)$. Put $f(t_1, \ldots, t_n) = \sum\limits_{k=1}^{\infty} q_k(t_1, \ldots, t_n)$. Since f is a power series which converges for every vector $(t_1, \ldots, t_n)$, f is an entire analytic function on $\mathbb{C}^n$. Evidently, $\mathcal{H}^{\mathbf{H}}_n(f) = u$. From the known theorem about automatic continuity of an isomorphism between commutative finitely generated Fréchet algebras [5, p. 43] it follows that $\mathcal{H}^{\mathbf{H}}_n$ is continuous. □

Let $(x, y), (z, t) \in \mathcal{X}^2$,

$$(x,y) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots\right) \quad \text{and} \quad (z,t) = \left(\begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots\right)$$

where $(x_i, y_i), (z_i, t_i) \in \mathbb{C}^2$, $i \in \mathbb{N}$. We put

$$(x,y) \bullet (z,t) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots\right)$$

and define

$$\mathcal{T}_{(z,t)}(f)(x,y) := f((x,y) \bullet (z,t)). \tag{1}$$

We will say that $(x,y) \to (x,y) \bullet (z,t)$ is the symmetric translation and the operator $\mathcal{T}_{(z,t)}$ is the symmetric translation operator. Evidently, we have that

$$H^{k_1,k_2}((x,y) \bullet (z,t)) = H^{k_1,k_2}(x,y) + H^{k_1,k_2}(z,t)$$

for all k_1, k_2. It is easy to see that $\mathcal{T}_{(z,t)}$ is a continuous linear operator from $H^n_{bvs}(\mathcal{X}^2)$ to itself.

Theorem 1. *Let $(z,t) \in \mathcal{X}^2$ such that $(H_1(z,t), \dots, H_n(z,t))$ is nonzero vector in $\mathbb{C}^n$. Then the symmetric translation operator $\mathcal{T}_{(z,t)}$ is hypercyclic on $H^n_{bvs}(\mathcal{X}^2)$.*

Proof. Let $a = (H_1(z,t), \dots, H_n(z,t)) \in \mathbb{C}^n$. If $g \in H^n_{bvs}(\mathcal{X}^2)$, then

$$g(x,y) = \mathcal{H}^{\mathbf{H}}_n(f)(x,y) = f(H_1(x,y), \dots, H_n(x,y))$$

for some $f \in H^n_{bvs}(\mathcal{X}^2)$ and property (1) implies

$$\mathcal{T}_{(z,t)}(g)(x,y) = \mathcal{H}^{\mathbf{H}}_n T_a (\mathcal{H}^{\mathbf{H}}_n)^{-1}(g)(x,y).$$

Since T_a is hypercyclic on $H(\mathbb{C}^n)$, the operator $\mathcal{T}_{(z,t)}$ is hypercyclic on $H^n_{bvs}(\mathcal{X}^2)$ via Proposition 1, which completes the proof. □

2 THE INFINITY-DIMENSIONAL CASE

Let us recall a well known Kitai-Gethner-Shapiro theorem which is also known as the Hypercyclicity Criterion.

Theorem 2. *Let X be separable Fréchet space and $T : X \to X$ be a linear and continuous operator. Suppose there exist X_0, Y_0 dense subsets of X, a sequence (n_k) of positive integers and a sequence of mappings (possibly nonlinear, possibly not continuous) $S_n : Y_0 \to X$ so that*

1) $T^{n_k}(x) \to 0$ for every $x \in X_0$ as $k \to \infty$;

2) $S^{n_k}(y) \to 0$ for every $y \in Y_0$ as $k \to \infty$;

3) $T^{n_k} \circ S^{n_k}(y) = y$ for every $y \in Y_0$.

Then T is hypercyclic.

The operator T is said to satisfy the Hypercyclicity Criterion for full sequence if we can chose $n_k = k$. Note that T_a satisfies the Hypercyclicity Criterion for full sequence [3] and so the symmetric shift $\mathcal{T}_{(z,t)}$ on $H^n_{bvs}(\mathcal{X}^2)$ satisfies the Hypercyclicity Criterion for full sequence provided $(H_1(z,t), \dots, H_n(z,t)) \neq 0$.

In [9] it is proved the following lemma.

Lemma 2. *Let X be a Fréchet space and $X_1 \subset X_2 \subset \dots \subset X_n \subset \dots$ be a sequence of closed subspaces such that $\cup_{m=1}^{\infty} X_m$ is dense in X. Let T be an operator on X such that $T(X_m) \subset X_m$ for each m and each restriction $T|_{X_m}$ satisfies the Hypercyclicity Criterion for full sequence on X_m. Then T satisfies the Hypercyclicity Criterion for full sequence on X.*

We denote by $H_{bvs}(\mathcal{X}^2)$ the Fréchet algebra of all block-symmetric analytic functions on $\mathcal{X}^2$ which are bounded on bounded subset. This algebra is the completion of the space of block-symmetric polynomials on $\mathcal{X}^2$ endowed with the uniform topology on bounded subset.

Theorem 3. *The operator $\mathcal{T}_{(z,t)}$ is hypercyclic on $H_{bvs}(\mathcal{X}^2)$ for every $(z,t) \neq 0$.*

Proof. Since $(z,t) \neq 0$, we have $H_{m_0}(z,t) \neq 0$ for some m_0 [6, 7]. So, $\mathcal{T}_{(z,t)}$ is hypercyclic (and satisfies the Hypercyclicity Criterion for full sequence) on $H^n_{bvs}(\mathcal{X}^2)$ whenever $n \geq m_0$. The set $\cup_{n=m_0}^{\infty} H^n_{bvs}(\mathcal{X}^2)$ contains the space of all block-symmetric polynomials on $\mathcal{X}^2$ and so it is dense in $H_{bvs}(\mathcal{X}^2)$. Also $H^n_{bvs}(\mathcal{X}^2) \subset H^m_{bvs}(\mathcal{X}^2)$ if $m > n$. Hence $\mathcal{T}_{(z,t)}$ is hypercyclic via Lemma 2. □

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Кравців В.В., Можировська З.Г., Загороднюк А.В. *Гіперциклічні оператори на просторах блочно-симетричних аналітичних функцій* // Карпатські математичні публікації. — 2013. — Т.5, №1. — С. 59–62.

Стаття містить доведення гіперциклічності оператора симетричного зсуву на алгебрі блочно-симетричних аналітичних функцій обмеженого типу на ізоморфній копії простору ℓ_1.

Ключові слова і фрази: блочно-симетричні аналітичні функції, гіперциклічний оператор, оператор симетричного зсуву.

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Статья содержит доказательство гиперцикличности оператора симметрического сдвига на алгебре блочно-симметрических аналитических функций ограниченного типа на изоморфной копии пространства ℓ_1.

Ключевые слова и фразы: блочно-симметрические аналитические функции, гиперциклический оператор, оператор симметрического сдвига.