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ON BIDUAL BASES IN THE SPACE OF SYMMETRIC ANALYTIC FUNCTIONS ON ℓ_1

We consider a special Hilbert space of symmetric analytic functions on ℓ_1 and construct a pair of bidual bases of polynomials.

Key words and phrases: Hilbert space, symmetric analytic functions, bidual bases.

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1 INTRODUCTION AND PRELIMINARIES

A function f from complex ℓ_1 to $\mathbb{C}$ is said to be *symmetric* if

$$f(\sigma(x)) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, \dots) = f(x), \qquad x \in \ell_1,$$

for every permutation σ of positive integers $\mathbb{N}$.

The algebra of all continuous symmetric polynomials on ℓ_1 will be denoted by $P_s(\ell_1)$. Symmetric polynomials and analytic functions were investigated in [1, 2, 3, 4]. In particular, it is known that $P_s(\ell_1)$ admits algebraic bases. We need to use several standard bases which also are well known in the combinatorics.

The basis of power sums consists of polynomials

$$P_n(x) = \sum_{i=1}^{\infty} x_i^n, \quad x = (x_1, \dots, x_n, \dots) \in \ell_1$$

(see [3] for details). The elementary symmetric polynomials $G_n(x) = \sum x_{i_1} \dots x_{i_n}$ form an another basis in $P_s(\ell_1)$ and due to the Newton equality

$$nG_n = G_{n-1}P_1 - G_{n-2}P_2 + \dots + (-1)^n P_n.$$

Also, there is a basis of complex symmetric functions H_n which can be defined by

$$nH_n = H_{n-1}P_1 + H_{n-2}P_2 + \dots + H_1 P_{n-1} + P_n.$$

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition of a positive integer n, that is all $\lambda_k \in \mathbb{N}$ and

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = n.$$

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We denote by M_λ the symmetric polynomial in $P_s(\ell_1)$

$$M_\lambda(x) = \sum_{\sigma\in S} x_{\sigma(1)}^{\lambda_1} \dots x_{\sigma(n)}^{\lambda_n},$$

where S is the group of permutations of $\mathbb{N}$. It is known from the combinatorics (see [5]) that

$$H_n = \sum_{|\lambda|=n} M_\lambda.$$

For a given partition λ let $z_\lambda = \prod_{k\geq 1} k^{m_k} m_k!$, where m_k is the number of entries of k into λ. It is known that

$$H_n = \sum_{|\lambda|=n} z_\lambda^{-1} P_\lambda = \sum_{\nu_1+2\nu_2+\ldots+n\nu_n=n} \frac{1}{\nu_1! 1^{\nu_1} \dots \nu_n! n^{\nu_n}} P_1^{\nu_1} \dots P_n^{\nu_n}, \tag{1}$$

where $P_\lambda = P_{\lambda_1} \dots P_{\lambda_n}$. We will use also notations $G_\lambda = G_{\lambda_1} \dots G_{\lambda_n}$, $H_\lambda = G_{\lambda_1} \dots G_{\lambda_n}$. It is easy to see that each of system $\{P_\lambda\}$, $\{G_\lambda\}$, $\{H_\lambda\}$, and $\{M_\lambda\}$ form a linear basis in $P_s(\ell_1)$, where λ goes over all partitions of all positive integers.

Next we introduce an inner product on $P_s(\ell_1)$ so that $\{P_\lambda\}$ form an orthogonal basis. In this paper we consider the case when $\langle P_\lambda, P_\mu\rangle = \delta_{\lambda\mu} z_\lambda$, where $\delta_{\lambda\mu}$ is the Kronecker delta. Let $H_s = H_s^{z_\lambda}(\ell_1)$ be the completion of $P_s(\ell_1)$ with respect to the inner product. In the paper we will show that $\{H_\lambda\}$ and $\{M_\lambda\}$ are bidual bases in H_s.

2 MAIN RESULTS

Let $x\Diamond y$, $x, y \in \ell_1$ be an element in ℓ_1 with coordinates $(x_i y_j)_{i,j=1}^\infty$ ordered by a fixed way. It is easy to see, that $P_\lambda(x\Diamond y) = P_\lambda(x) P_\lambda(y)$ and

$$\sum_{n=0}^\infty H_n(x\Diamond y) = \prod_{i,j} (1 - x_i y_j)^{-1} = \prod_j \sum_{k=0}^\infty H_k(x) y_j^k = \sum_{n=0}^\infty \sum_{|\lambda|=n} H_\lambda(x) M_\lambda(y), \tag{2}$$

where $H_0 = 1$, $M_0 = 1$.

Theorem 1. $\langle H_\lambda, M_\mu\rangle = \delta_{\lambda\mu}$ *for all partitions* μ *and* λ.

Proof. By (1) and (2)

$$\sum_{|\lambda|=n} H_\lambda(x) M_\lambda(y) = H_n(x\Diamond y) = \sum_{|\lambda|=n} z_\lambda^{-1} P_\lambda(x\Diamond y) = \sum_{|\lambda|=n} z_\lambda^{-1} P_\lambda(x) P_\lambda(y).$$

So

$$\Big\langle \sum_{|\lambda|=n} H_\lambda(\cdot) M_\lambda(y), M_\mu(\cdot) \Big\rangle = \sum_{|\lambda|=n} z_\lambda^{-1} P_\lambda(y) \langle P_\lambda, M_\mu\rangle.$$

On the other hand, since $\frac{P_\lambda}{\sqrt{z_\lambda}}$ is an orthonormal basis,

$$M_\mu = \sum_\lambda \Big\langle \frac{P_\lambda}{\sqrt{z_\lambda}}, M_\mu \Big\rangle \frac{P_\lambda}{\sqrt{z_\lambda}} = \sum_{|\lambda|=|\mu|=n} \Big\langle \frac{P_\lambda}{\sqrt{z_\lambda}}, M_\mu \Big\rangle \frac{P_\lambda}{\sqrt{z_\lambda}}$$

and if y is such that R_y is continuous, then $R_y(M_\mu) = M_\mu(y) = \sum_{|\lambda|=n} z_\lambda^{-1} P_\lambda(y)\langle P_\lambda, M_\mu\rangle$. So

$$\Big\langle \sum_{|\lambda|=n} H_\lambda(\cdot)M_\lambda(y), M_\mu(\cdot)\Big\rangle = \sum_{|\lambda|=n} M_\lambda(y)\langle H_\lambda, M_\mu\rangle = M_\mu(y).$$

Since it is true for all y such that R_y is continuous, and since functionals R_y separate vectors in H_s we have that

$$\langle H_\lambda, M_\mu\rangle = \delta_{\lambda\mu}.$$

□

Let us make some computations. Taking into account that $\sum_{|\lambda|=n} z_\lambda^{-1} = 1$ (see [6, p. 49]) we have

$$\|H_n\|^2 = \langle H_n, H_n\rangle = \langle \sum_{|\lambda|=n} z_\lambda^{-1}P_\lambda, \sum_{|\lambda|=n} z_\lambda^{-1}P_\lambda\rangle = \sum_{|\lambda|=n} z_\lambda^{-2}\|P_\lambda\|^2 = \sum_{|\lambda|=n} z_\lambda^{-1} = 1.$$

Since $M_n = P_n$, $\|M_n\| = \sqrt{z_n} = \sqrt{n}$ and $\sup_\lambda \|M_\lambda\|\|H_\lambda\| = \infty$. So $\{H_\lambda\}$ does not form a Riesz basis.

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Голубчак О.М., Загороднюк А.В. *Двоїсті базиси в просторі симетричних аналітичних функцій на ℓ_1* // Карпатські математичні публікації. — 2013. — Т.5, №1. — С. 47–49.

Розглянуто спеціальний гільбертів простір симетричних аналітичних функцій на ℓ_1 і побудовано пару двоїстих базисів цього простору, які складаються з симетричних поліномів.

Ключові слова і фрази: гільбертів простір, симетричні аналітичні функції, дуальний базис.

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Рассмотрено специальное гильбертово пространство симметрических аналитических функций на ℓ_1 и построено пару дуальных базисов этого пространства, состоящих из симметрических полиномов.

Ключевые слова и фразы: гильбертово пространство, симметрические аналитические функции, дуальный базис.