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THE COMPLETENESS OF A NORMED SPACE IS EQUIVALENT TO THE HOMOGENEITY OF ITS SPACE OF CLOSED BOUNDED CONVEX SETS

We prove that an infinite-dimensional normed space X is complete if and only if the space $\mathrm{BConv}_H(X)$ of all non-empty bounded closed convex subsets of X is topologically homogeneous.

Key words and phrases: completeness, normed spaces, topological homogeneity, closed convex sets.

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INTRODUCTION

In this paper we shall prove that the completeness of an infinite-dimensional normed space X is equivalent to the topological homogeneity of its hyperspace $\mathrm{BConv}_H(X)$ of all non-empty bounded closed convex sets. The space $\mathrm{BConv}_H(X)$ is endowed with the Hausdorff metric

$$d_H(A,B) = \max\big\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\big\},\quad A,B\in \mathrm{BConv}_H(X).$$

Due to results of [5], [6], [2], the topological structure of the hyperspace $\mathrm{BConv}_H(X)$ is well-understood for each Banach space X. To formulate a classification result for the hyperspace $\mathrm{BConv}_H(X)$ we need to recall some notations.

All linear spaces considered in this paper are over the field of real numbers $\mathbb{R}$. For a linear topological space X its dimension $\dim(X)$ is defined as the smallest cardinality $|B|$ of a subset $B\subset X$ having dense linear hull in X. For a cardinal κ by $l_2(\kappa) = \{x\in\mathbb{R}^\kappa\colon \sum_{\alpha\in\kappa}|x(\alpha)|^2<\infty\}$ we denote the Hilbert space having an orthonormal base of cardinality κ. By ω we denote the smallest infinite cardinal. By $\bar{\mathbb{R}}_+$ and $\mathbb{I}$ we denote the closed half-line $[0,\infty)$ and the closed unit interval $[0,1]$, respectively.

The following classification theorem can be derived from [5], [6], [2].

Theorem 1. *For each Banach space X the hyperspace $\mathrm{BConv}_H(X)$ is homeomorphic to:*

1) $\{0\}$ *iff* $\dim(X)=0$;

2) $\bar{\mathbb{R}}_+\times\mathbb{R}$ *iff* $\dim(X)=1$;

3) $\mathbb{I}^\omega\times\bar{\mathbb{R}}_+$ *iff* $1<\dim(X)<\omega$;

4) $l_2(2^{\dim(X)})$ *iff* $\dim(X)\ge\omega$.

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In this paper we shall study the hyperspace $\mathrm{BConv}_H(X)$ for non-complete normed spaces X. In this case we shall show that $\mathrm{BConv}_H(X)$ has rather bad topological properties. In particular, it is neither topologically homogeneous nor even weakly homogeneous.

1 MAIN RESULT

A topological space X is defined to be

- *topologically homogeneous* if for any two points $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$;
- *weakly homogeneous* if for each non-empty open dense subset $U \subset X$ and each point $x \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) \in U$.

It is clear that each topologically homogeneous space is weakly homogeneous.

The main result of this note is the following theorem.

Theorem 2. *For an infinite-dimensional normed space X the following conditions are equivalent:*

(1) X is complete;

(2) $\mathrm{BConv}_H(X)$ is topologically homogeneous;

(3) $\mathrm{BConv}_H(X)$ is weakly homogeneous;

(4) $\mathrm{BConv}_H(X)$ is homeomorphic to $l_2(2^{\dim(X)})$.

Proof. We shall prove the following implications. (1)⇒(4)⇒(2)⇒(3)⇒(1). The implication (1)⇒(4) follows from Theorem 1 while (4) ⇒ (2) ⇒ (3) are trivial. So, it remains to prove (3) ⇒ (1).

In the space $\mathrm{BConv}_H(X)$ consider the open dense subspace

$$\mathrm{BCb}_H(X) = \{A \in \mathrm{BConv}_H(X) : Int(A) \neq \varnothing\}$$

consisting of bounded convex bodies (i.e., bounded convex sets with non-empty interior). Let $\bar{X}$ be the completion of the normed space X and $\mathrm{BCb}_H(\bar{X})$ be the space of bounded convex bodies in the Banach space $\bar{X}$. Observe that the map

$$\begin{array}{rccc} \mathrm{cl}: & \mathrm{BCb}_H(X) & \longrightarrow & \mathrm{BCb}_H(\bar{X}), \\ & B & \longmapsto & \bar{B}, \end{array}$$

is an isometric bijection. The space $\mathrm{BCb}_H(\bar{X})$, being open in the complete metric space $\mathrm{BConv}_H(\bar{X})$, is Čech-complete and so is its isometric copy $\mathrm{BCb}_H(X)$. Assuming that the space $\mathrm{BConv}_H(X)$ is weakly homogeneous, and taking into account that $\mathrm{BCb}_H(X)$ is an open dense Čech-complete subspace of $\mathrm{BConv}_H(X)$, we conclude that each point of the space $\mathrm{BConv}_H(X)$ has an open Čech-complete neighborhood. By a result of Arhangelski [1] and Frolik [4] (see also [3, 5.5.8(c)]), the space $\mathrm{BConv}_H(X)$, being locally Čech-complete and paracompact, is Čech-complete, and so is its closed subspace X. Being Čech-complete, the space X is a G_δ-set in its completion $\bar{X}$. Assuming that $X \neq \bar{X}$, we can find a point $x \in \bar{X} \setminus X$ and conclude that X and $X + x$ are two disjoint dense G_δ-subsets of Banach space $\bar{X}$, which is impossible according to the Baire Theorem. Consequently, $X = \bar{X}$ is a Banach space. □

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Гетьман I. *Еквівалентність повноти нормованого простору гомогенності гіперпростору його замкнених обмежених опуклих множин* // Карпатські математичні публікації. — 2013. — Т.5, №1. — С. 44–46.

Ми доводимо, що нескінченновимірний нормований простір X є повним тоді і лише тоді, коли гіперпростір $\mathrm{BConv}_H(X)$ усіх непорожніх замкнених опуклих підмножин простору X є топологічно гомогенним.

Ключові слова і фрази: повнота, нормовані простори, топологічна гомогенність, замкнені опуклі множини.

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Мы доказываем, что бесконечномерное нормированное пространство X полно тогда и только тогда, когда гиперпространство $\mathrm{BConv}_H(X)$, состоящее из всех непустых замкнутых выпуклых подмножеств пространства X, топологически гомогенно.

Ключевые слова и фразы: полнота, нормированные пространства, топологическая гомогенность, замкнутые выпуклые множества.