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ON ELEMENTARITY OF RADICAL CLASSES OF MODULES OVER NONCOMMUTATIVE DEDEKIND DUO-DOMAINS

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We find some sufficient conditions for a radical class of an idempotent radical in the category of modules over a Dedekind left bounded duo-domain to be axiomatisable. In the case of the integer numbers ring this result implies the Gorbachuk-Komarnitskii Theorem on axiomatizable radical classes of abelian groups.

Basic notions

We begin with recalling some basic facts and definitions. In this paper by A we denote an associative ring with the identity $1 \neq 0$, and assume that all modules are left unitary A-modules. The category of left A-modules we denote by $A - Mod$. Recall that a ring A is called a domain if it not contains left or right zero divisors ($a \neq 0$ is a left zero divisor if there exists $b \neq 0$ such that $ab = 0$). An ideal P of A is prime if, for all elements $a, b \in P$, $ab \in P$ implies that $a \in P$ or $b \in P$. A prime ring is a ring with the zero ideal to be a prime ideal. A ring A is called left hereditary if every left ideal is a projective module. A ring A is left Noetherian if every nonempty set of left ideals has a maximal element. Similarly we can define a right Noetherian and a right hereditary ring. A ring A is hereditary if it is right and left hereditary. Also a ring A is Noetherian if it is right and left Noetherian. Next recall that a ring Q is called a quotient ring if every regular element of Q is a unit. Given a quotient ring Q, a subring R, not necessarily containing 1, is called a left order in Q if each $q \in Q$ has the form $s^{-1}r$ for some $r, s \in R$.

Let Q be some fixed quotient ring and R_1, R_2 left orders of it. Then R_1 and R_2 are equivalent if there are units $a_1, a_2, b_1, b_2 \in Q$ such that $a_1R_1b_1 \subseteq R_2$ and $a_2R_1b_2 \subseteq R_1$. If Q is a quotient ring and R is a left order in Q, then R is called a maximal left order if it is maximal within its equivalence class. A ring A is a noncommutative Dedekind domain if it is a hereditary Noetherian prime ring and is a maximal order. A left duo-ring is a ring with every left ideal to be two-sided. For noncommutative Dedekind duo-domain (see [9]) is true the following

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Theorem 1. *If A is a noncommutative Dedekind duo-domain and P is a proper ideal of A, then there exist $a_1, a_2 \in A$ such that*

$$P = a_1A + a_2A.$$

Recall that an ideal I of a ring A is called essential if, for any ideal J of A, it holds that $I \cap J \neq (0)$. Moreover, a ring A is called bounded if its every essential ideal contains a two-sided ideal. An A-module M is said to be divisible if $Mc = M$ for any nonzero $c \in A$.

Let $r : A - Mod \to A - Mod$ be a functor. We say that r is a preradical of $A - Mod$ if r assigns to each object M a subobject $r(M)$ in such way that every morphism $M \to N$ induces $r(M) \to r(N)$. A preradical r is called a radical if $r(M/r(M)) = 0$ for every object M. A preradical r is idempotent if $r(r(M)) = r(M)$.

In this paper all radicals are idempotent. With every preradical r we can associate two classes of objects from $A - Mod$, namely

$$T_r = \{M \in A - Mod \mid r(M) = M\}$$

and

$$F_r = \{M \in A - Mod \mid r(M) = 0\}.$$

If r is a radical, then T_r is called a radical class and its objects are radical objects, while F_r is a torsion-free class consisting of torsion-free objects. This classes have such properties:

Theorem 2. *The class T_r is closed under quotient objects, coproducts and extensions, while F_r is closed under subobjects, products and extensions.*

We need (see [1]) also the following

Theorem 3. *If A is a Dedekind domain and P is its prime ideal, then, for every radical r of $A - Mod$, the module A/P is either radical or radical-free.*

Recall also some notions of the model theory. We use a language ${}_AL$ which is appropriate to the left A-modules first order language. A set of all sentences of the language which are true class of modules Ψ is called a theory of a class of modules Ψ and denoted by $Th(\Psi)$. A set of models of a theory T is any class of modules which satisfies all sentences from T. A class is axiomatisable (or elementary) if there is a set of sentences T such that it is exactly the class of models of T. Two modules are elementarily equivalent if every sentence which is true in one of them is true in other. Next we give notions about ultrafilters and ultraproducts.

Let I be a set. Then D is called a filter over I if D is some nonempty collection of subsets from I satisfying:

(1) $\varnothing \notin D$;

(2) if $S, T \in D$, then $S \cap T \in D$;

(3) if $S \in D$ and $S \subseteq T \subseteq I$, then $T \in D$.

A filter D is said to be an ultrafilter if, for every $S \subseteq I$, it holds $S \in D$ or $I \setminus S \in D$. If $\{A_i \mid i \in I\}$ is a family of all sets indexed by I, then an ultraproduct of A_i with respect to D is the quotient of $\prod_{i \in I} A_i$ by an equivalence relationship

$$f \equiv_D g \text{ if and only if } \{i \in I \mid f(i) = g(i)\} \ \in D$$

for any $f, g \in \prod_{i \in I} A_i$. An ultraproduct of the A_i with respect to D is denoted by $\prod_{i \in I} A_i / D$. Now we can formulate the following test of axiomatisability

Theorem 4. *A class of modules is axiomatisable if and only if it is closed under ultraproducts and an elementarily equivalency of modules.*

1 Eklof-Fisher Theorems

In this section we consider theorems from [6] that are proved for commutative Dedekind domains. In view of [1], this results can be used for noncommutative Dedekind duo-domains. First of all, we recall some designations from [6]. If M is left A-module, then M^α denote direct sum of α copies of a module M. If P is a prime ideal of a Dedekind domain A and M is a left A-module, then $M[P]$ will be the biggest submodule of M that has the annihilator P. Let us

$$U(P,n;M) = \begin{cases} dim(P^n M[P]/P^{n+1}M[P]) & \text{if this dimension is finite,} \\ \infty & \text{in else case.} \end{cases}$$

$$Tf(P;M) = \begin{cases} lim_{n\to\infty} dim(P^n M[P]/P^{n+1}M[P]) & \text{if it is finite,} \\ \infty & \text{in else case.} \end{cases}$$

$$D(P;M) = \begin{cases} lim_{n\to\infty} dim(P^n M[P]) & \text{if it is finite,} \\ \infty & \text{in else case.} \end{cases}$$

It is necessary to say that we consider dimension over A/P. Let

$$U^*(P,n;M) = \begin{cases} 0 & \text{if } U(P,n;M) = 0 \text{ and } A/P \text{ is infinite,} \\ \infty & \text{if } U(P,n;M) \neq 0 \text{ and } A/P \text{ is infinite,} \\ U(P,n;M) & \text{if } A/P \text{ is finite.} \end{cases}$$

$$Tf^*(P;M) = \begin{cases} 0 & \text{if } Tf(P;M) = 0 \text{ and } A/P \text{ is infinite,} \\ \infty & \text{if } Tf(P;M) \neq 0 \text{ and } A/P \text{ is infinite,} \\ Tf(P,n;M) & \text{if } A/P \text{ is finite.} \end{cases}$$

$$D^*(P;M) = \begin{cases} 0 & \text{if } D(P;M) = 0 \text{ and } A/P \text{ is infinite,} \\ \infty & \text{if } D(P;M) \neq 0 \text{ and } A/P \text{ is infinite,} \\ D(P;M) & \text{if } A/P \text{ is finite.} \end{cases}$$

We say that a module M has a bounded order if there exists $0 \neq \lambda \in A$ such that $\lambda M = 0$. Ω will denote the set of all nonzero prime ideals of a ring A. If $P \in \Omega$, then M_P will be a localization of a module M over P.

Theorem 5. *Let A be Dedekind domain and M be a left A-module. Then M is elementarily equivalent to a module $\oplus_{P\in\Omega}M_P \oplus M_d$, where*

$$M_P = \oplus_n (A/P^n)^{(\alpha_{P,n})} \oplus A_P^{(\beta_P)} \text{ and } M_d = \oplus_{P\in\Omega}(A/P)^{\gamma_P} \oplus K^{(\delta)}.$$

Here K is a field of fractions of a domain and

$$\alpha_{P,n} = \alpha_{P,n}(M) = \begin{cases} U^*(P, n-1; M) & \text{if it is finite,} \\ \geqslant k = Card A + \aleph_0 & \text{in other case.} \end{cases}$$

$$\beta_P = \beta_P(M) = \begin{cases} Tf^*(P; M) & \text{if it is finite,} \\ \geqslant k & \text{in other case.} \end{cases}$$

$$\gamma_P = \gamma_P(M) = \begin{cases} D^*(P; M) & \text{if it is finite,} \\ \geqslant k & \text{in other case.} \end{cases}$$

$$\delta = \delta(M) = \begin{cases} 0 & \text{if } M \text{ have bounded order,} \\ \geqslant k & \text{in other case.} \end{cases}$$

According to the fact that a direct sum and a direct product are elementarily equivalent this theorem can be formulated as follows

Theorem 6. *Let A be a Dedekind domain. Then every left A-module M is elementarily equivalent to a module*

$$(\oplus_n (A/P^n)^{(\alpha_{P,n})}) \oplus A_P^{(\beta_P)} \oplus (\oplus_{P\in\Omega}(A/P)^{\gamma_P}) \oplus K^{(\delta)},$$

where $\alpha_{P,n}, \beta_P, \gamma_P, \delta$ are the same as in the previous theorem.

Theorem 7. *Modules M and N over a Dedekind domain are elementarily equivalent if and only if*

$$U^*(P, n; M) = U^*(P, n; N),\ Tf^*(P; M) = Tf^*(P; N),\ D^*(P; M) = D^*(P; N),$$

where modules M and N have a bounded or unbounded order in the same time.

2 LEMMAS

Lemma 2.1. *Let A be a noncommutative Dedekind duo-domain and let r be a nontrivial radical for which the radical class T_r is axiomatisable. If the class T_r contains a module A/P, where P is some nonzero prime ideal of A, then it also contains such modules:*

1) *the localization A_P of A at a prime ideal P;*

2) *the field of fractions ${}_AK$ of a ring A that is considered as a left A-module;*

3) *$\widehat{A/P'}$, where P' is an arbitrary nonzero ideal of a ring A.*

Proof. The class T_r is closed under extensions, therefore $A/P^n \in T_r$ for arbitrary $n \in \mathbb{N}$.

Let D be a countably-incomplet ultrafilter over the set of natural numbers $\mathbb{N}$. Then, according to the fact that T_r is axiomatisable, we obtain that

$$M = (\prod A/P^n)/D$$

belong to T_r. A module M has an unbounded order, and so

$$\delta(M) \geq CardA + \aleph_0.$$

By the Eklof-Fisher Theorem (see Theorem 2) the class T_r contains a module for which module ${}_AK$ is a direct summand. Thus K is contained in T_r as an epimorphic image. Similarly, $K/A \in T_r$. But $K/A \cong \bigoplus_{P\in\Pi} A/P$, hence $A/P \in T_r$ for every $P \in \Omega$.

Consider the case when A/P is a finite module. Then

$$\beta_P = dim_{A/P} M/(t(M) + PM),$$

where $t(M)$ is the periodic part of a module M. Now we show that $\beta_P(M) \neq 0$. For this, we have to check that $t(M) + PM \neq M$. Let us denote by 1_n the coset in A/P^n with representative 1. We have to prove that the element $(1_1, 1_2, ...1_n, ...)$ of a module M do not belongs to the submodule $t(M) + PM$. By Theorem 2.1, $P = p_1A + p_2A$, where $p_1, p_2 \in A$. Thus $x = t + p_1a_1 + p_2a_2$, where $t \in t(M)$, $a_1, a_2 \in M$. Since the annihilator of an element t is power of an ideal P, for some $k \in \mathbb{N}$ we obtain that $P^k t = 0$. Consequently,

$$P^k x \subseteq P^{k+1}a_1 + P^{k+1}a_2.$$

Let

$$a_1 = \overline{(\overline{a'_1}, ...\overline{a'_n}, ...)},\ a_2 = \overline{(\overline{a''_1}, ...\overline{a''_n}, ...)},$$

where $a'_i, a''_i \in A$ for $i \in \mathbb{N}$. Therefore from previous inclusion for some set of indexes $U \in D$ is true that

$$P^k \subseteq P^{k+1}a'_i + P^{k+1}a''_i + P^i \subseteq P^i.$$

Hence from $P^i \subseteq P^k$ we obtain that $P^i = P^k$, $i \geq k+1$. But in a Dedekind ring a decomposition into a product of prime ideals is unique, so we obtain contradiction. Thus $\beta_P(M) \neq 0$. Then, from Theorem 2, T_r contains a module with A_P as a direct summand and, using previous thoughts, A_P lies in T_r.

Next if A/P is an infinite module, then, according to definition of $T^*f(P, M)$, the equality $\beta_P = 0$ is true only if

$$lim_{n\to\infty} dim P^n M/P^n + 1M = 0$$

for all k, and therefore $P^k M/P^{k+1}M = 0$. From the last equality we obtain

$$P^k = P^{k+1}P^k M = P^k M$$

for some k. This equation is false for a module M and arbitrary $k \in \mathbb{N}$. Verifying of this fact is similar to those we have done early in this proof. Therefore $A/P \in T_r$. □

Let Π be some set of prime ideals in a ring A. A module M is $\Pi-$divisible if $IM = M$ for every ideal I from Π. For every Π, a class of all Π-divisible modules is a radical class for some radical of the category $A - Mod$. This radical we will denote by r_Π.

Lemma 2.2. *A module M is Π-divisible if and only if it is elementarily equivalent to the module of the form*

$$\oplus_{(P\in\Omega\setminus\Pi, n\in N)}((A/P^n)^{\alpha_{P,n}}) \oplus (\oplus_{P\in\Omega} A_P^{\beta_P}) \oplus (\oplus_{P\in\Omega}(\widehat{A/P})^{(\gamma_P)}) \oplus K^{(\delta)}, \quad (1)$$

where $\alpha_{P,n}, \beta_P, \delta, \gamma_P$ are some cardinal numbers.

Proof. We consider a set of sentences of the language ${}_AL$

$$\mathcal{C} = \{(\forall x)(\exists y_1)(\exists y_2)(x = p_1y_1 + p_2y_2)p_1A + p_2A = P \in \Pi\}.$$

It is obvious that M is Π-divisible if and only if M is a model of a system of formulas C. Therefore the class of Π-divisible modules is axiomatisable and, consequently, this class is elementarily closed. Since A/P^n, A_P, A/P are Π-divisible for $P \in \Pi$ and K is Ω-divisible, using the fact that class of Π-divisible groups is closed under direct sums we obtain that modules of the form 1 are Π-divisible. If $\alpha_{P,n} \neq 0$ or $\beta_P \neq 0$ for some $P \in \Pi$, then a module is not Π-divisible. Hence all modules which are elementarily equivalent to it are not Π-divisible too. □

3 MAIN RESULT

Theorem 8. *The radical class of a nontrivial radical r in the category of left modules over a noncommutative Dedekind duo-ring A is axiomatisable if and only if $r = r_\Pi$ for some nonempty subset Π of the set of nonzero prime ideals in a ring A.*

Proof. It is well known that for every prime ideal $P \in \Omega$ the module A/P is r-radical or r-radical-free. So we have in Ω two subsets:

$$\Pi = \{P \in \Omega \mid A/P \notin F_r \Leftrightarrow A/P \in T_r\}$$

and

$$\Omega \setminus \Pi = \{P \in \Omega \mid A/P \in T_r\}.$$

We have to show that if the class T_r is axiomatisable, then it contains all Π-divisible modules. It is obvious that A/P' is Π-divisible for some $P' \in \Omega$ if and only if $P' \in \Omega \setminus \Pi$. Thus every Π-divisible module of the form A/P belongs to T_r. In view Lemma 1, the class T_r contains all modules of the form: A_P, A/Q, $P \in \Omega \setminus \Pi$, $Q \in \Omega$ and a module K. The class T_r is closed under extensions, and so therefore A/P^n, for $P \in \Omega \setminus \Pi$, $n \in \mathbb{N}$, belongs to the class T_r. Hence the class T_r contains every module of the form 1. The class T_r is axiomatisable, and so it contains all modules that are elementarily equivalent to the module of such form. Therefore T_r contains all Π-divisible modules. Let M be any module from the class T_r. We have to prove that M is Π-divisible. If we suppose that this is not true, then, by the Eklof-Fisher Theorem and Lemma 1, there exists $P \in \Pi$ such that one of the invariants $\alpha_{(P,1)}(M)$,

$\beta_P(M)$ of some module M from the class T_r is nonzero. As a consequence, A/P or A_P belongs to the class T_r, where $P \in \Pi$. Since $A_P/PA_P \cong A/P$, we deduce that $A/P \in T_r$, a contradiction with the definition of the set Π. Thus $r = r_\Pi$. $\square$

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Знайдено деякі достатні умови аксіоматизовності радикального класу ідемпотентного радикалу в категорії модулів над дедекіндовою лівою дуо-областю. У випадку кільця цілих чисел цей результат має наслідком теорему Горбачука-Комарницького про аксіоматизовність радикальних класів абелевих груп.

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Найдены некоторые достаточные условия аксиоматизируемости радикального класса идемпотентного радикала в категории модулей над дедекиндовой левой дуо-областю. В случае кольца целых чисел этот результат имеет следствием теорему Горбачука-Комарницкого о аксиоматизовности радикальных классов абелевых групп.