SANDHU G. S.[1], KUMAR D.[2]

ON ADDITIVITY OF DERIVATIONS

Let R be a ring and M be an R-bimodule. A mapping $d : R \to M$ (not necessarily additive) is called multiplicative derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In this paper, we intend to establish the additivity of d under some suitable restrictions. Moreover, we introduce multiplicative semi-derivations of rings and discuss their additivity.

Key words and phrases: derivation, multiplicative derivation, multiplicative semi-derivation, additivity, Peirce decomposition.

[1] Patel Memorial National College Rajpura, 140401, Punjab, India
[2] Punjabi University, 147002, Patiala, India
E-mail: gurninder_rs@pbi.ac.in (Sandhu G. S.), deep_math1@yahoo.com (Kumar D.)

INTRODUCTION

All through this paper, R denotes an associative ring (not necessarily with unity). A mapping $d : R \to R$ is called a derivation of R if for any $x, y \in R$

$$d(x + y) = d(x) + d(y) \tag{1}$$

$$\text{and } d(xy) = d(x)y + xd(y). \tag{2}$$

If d satisfies (2) but not necessarily (1), then d is called a multiplicative derivation of R (see [3]). In [2] Bergen extended the notion of a derivation by introducing *semi derivation* of a ring. Accordingly, a semi derivation (d, g) of a ring R is an additive mapping $d : R \to R$ associated with a ring endomorphism g of R such that $d(xy) = d(x)y + g(x)d(y) = d(x)g(y) + xd(y)$ and $d(g(x)) = g(d(x))$ for all $x, y \in R$. Clearly, every derivation is a semi derivation but the converse is not true always. We denote the Lie commutator $xy - yx$ by the symbol $[x, y]$. A non-zero element $e \in R$ is said to be *idempotent* if $e^2 = e$ and by a non-trivial idempotent we mean an idempotent element e different from the multiplicative identity of R. Let M be an $R-$bimodule and $e_1 \in R$ be a non-trivial idempotent element. For any $x \in M \cup R$ we shall write $x(1 - e_1)$ instead of $x - xe_1$, $(1 - e_1)x$ instead of $x - e_1x$ and e_2 instead of $(1 - e_1)$. Then we set $R_{ij} = e_iRe_j$ and $M_{ij} = e_iMe_j$, where $i, j \in \{1, 2\}$. Therefore, R and M can be factorized as follows: $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$ and $M = M_{11} \oplus M_{12} \oplus M_{21} \oplus M_{22}$. This representation of R and M is called *Peirce decomposition relative to e_1* (see [[5], pg. 48]). Further, the following are some well-known facts related to this decomposition of R:

(i) $R_{ij}R_{jk} \subseteq R_{ik}$, where $i, j, k \in \{1, 2\}$.

(ii) $R_{ij}R_{kl} = 0$, where $j \neq k$, and $i, j, k, l \in \{1, 2\}$.

(iii) $x_{ij}^2 = 0$ for all $x_{ij} \in R_{ij}$, where $i \neq j$ and $i, j \in \{1, 2\}$.

The structure of rings is tightly connected with the additive mapping like isomorphisms, derivations, centralizers etc. Therefore, the problem of exploring the conditions under which these mappings become additive on rings (or algebras) has naturally grown as a fascinating area of research and has been attracted many algebraists for the last six decades. In this direction, Martindale [8] considered the so called problem "When a multiplicative mapping is additive?" He gave a remarkable technique and established a set of conditions on a ring that forces a multiplicative isomorphism to be additive. In particular, every multiplicative isomorphism from a prime ring containing a non-trivial idempotent onto any ring is additive. Inspired by this, Daif [3] obtained the additivity of multiplicative derivations of rings and consequently introduced the notion of multiplicative derivations. After that a number of results has been obtained in associative as well as alternative rings and algebras (see [4, 6, 7, 9–11]) and references therein). Recently, Wang [11] explored the additivity of $n-$multiplicative isomorphisms and $n-$multiplicative derivations of rings. As a consequence, one may deduce the theorem of Martindale and theorem of Daif from corollary 3.1 and 3.3 of [11] respectively. In this paper, we will continue the study of analogue problems for some derivable mappings on associative rings.

1 MAIN RESULTS

1.1 Additivity of multiplicative derivations

In view of *Peirce decomposition*, we see that any mapping $\delta : R \rightarrow M$ can be expressed as

$$\delta(x) = \delta_{11}(x) + \delta_{12}(x) + \delta_{21}(x) + \delta_{22}(x)$$

for all $x \in R$, where $\delta_{ij} : R \rightarrow M_{ij}$ be a mapping defined as $x \mapsto e_i x e_j$ for all $i, j \in \{1, 2\}$. For any $x, y \in R$, we have $x = x_{11} + x_{12} + x_{21} + x_{22}$ and $y = y_{11} + y_{12} + y_{21} + y_{22}$. Further,

$$xy = (x_{11}y_{11} + x_{12}y_{21}) + (x_{11}y_{12} + x_{12}y_{22}) + (x_{21}y_{11} + x_{22}y_{21}) + (x_{21}y_{12} + x_{22}y_{22}).$$

Now, we extend the notion of multiplicative derivation of a ring R as follows:

Definition 1. *Let R be a ring (not necessarily with unity) and M be a bimodule over R. A mapping $d : R \rightarrow M$ (not necessarily additive) is said to be a multiplicative derivation of R into M if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.*

Since $d(e_1) \in M_{11} \oplus M_{21} \oplus M_{12} \oplus M_{22}$ i.e., $d(e_1) = m_{11} + m_{12} + m_{21} + m_{22}$, where $m_{ij} \in M_{ij}$ for all $i, j \in \{1, 2\}$. Also $d(e_1) = d(e_1^2) = d(e_1)e_1 + e_1 d(e_1)$. By using the value of $d(e_1)$ we obtain that $m_{11} = 0 = m_{22}$ and hence $d(e_1) \in M_{12} \oplus M_{21}$. For some fixed $x \in M$ and $z \in R$, we define a function $f : R \rightarrow M$ by $a \mapsto [z, x]a + a[x, z]$. Clearly, f is a derivation. Fix $x = m_{12} + m_{21}$ and $z = e_1$. Re-defining f as $a \mapsto [e_1, m_{12} + m_{21}]a + a[m_{12} + m_{21}, e_1]$. Thus, we have

$$
\begin{aligned}
f(e_1) &= [e_1, m_{12} + m_{21}]e_1 + e_1[m_{12} + m_{21}, e_1] \\
&= (m_{12} - m_{21})e_1 + e_1(m_{12} - m_{21}) = -m_{12} - m_{21} = -d(e_1).
\end{aligned}
$$

Hence, $(f + d)(e_1) = 0$. We set $f + d = D$. That means $D(e_1) = 0$. Now, we have the following relations:

$$D_{11}(xy) = D_{11}(x)y_{11} + x_{11}D_{11}(y) + D_{12}(x)y_{21} + x_{12}D_{21}(y), \tag{3}$$

$$D_{12}(xy) = D_{11}(x)y_{12} + D_{12}(x)y_{22} + x_{11}D_{12}(y) + x_{12}D_{22}(y), \tag{4}$$

$$D_{21}(xy) = x_{21}D_{11}(y) + D_{21}(x)y_{11} + x_{22}D_{21}(y) + D_{22}(x)y_{21},$$

$$D_{22}(xy) = x_{21}D_{12}(y) + D_{21}(x)y_{12} + D_{22}(x)y_{22} + x_{22}D_{22}(y).$$

Further, it is easy to check that $D_{ij}(e_1) = 0$ and $D_{ij}(xy) = D_{ij}(x)y + xD_{ij}(y)$ for all $i, j \in \{1, 2\}$.

Lemma 1. *Let R be a ring (not necessary with unity) and M be a bimodule over R. Suppose that R contains a non-trivial idempotent e_1 such that for any $m \in M$, the following are satisfied:*

(H1) $e_1 m e_1 R_{12} = (0)$ implies $e_1 m e_1 = 0$,

(H2) $e_1 m e_2 R_{22} = (0)$ implies $e_1 m e_2 = 0$,

(H3) $e_1 m e_2 R_{21} = (0)$ implies $e_1 m e_2 = 0$.

Then D_{11} and D_{12} are additive.

Proof. Firstly, we shall show that D_{11} is additive on $R_{11} \oplus R_{12} \oplus R_{22}$ and that D_{12} is additive on $R_{11} \oplus R_{12} \oplus R_{21}$. We begin with

$$D_{11}(x_{11} + x_{12} + x_{21} + x_{22}) = e_1 D(x_{11} + x_{12} + x_{21} + x_{22})e_1 = e_1 D((x_{11} + x_{12} + x_{21} + x_{22})e_1)e_1$$
$$= e_1 D(x_{11} + x_{21})e_1 = D_{11}(x_{11} + x_{21}).$$

That is

$$D_{11}(x_{11} + x_{12} + x_{21} + x_{22}) = D_{11}(x_{11} + x_{21}). \tag{5}$$

In particular, we have

$$D_{11}(x_{11} + x_{12} + x_{22}) = D_{11}(x_{11}). \tag{6}$$

For any $y_{12} \in R_{12}$, we have $D_{11}(x_{12})y_{12} = D_{11}(x_{12}y_{12}) - x_{12}D_{11}(y_{12}) = 0$. That means $D_{11}(x_{12})R_{12} = (0)$. By (H1), we obtain $D_{11}(x_{12}) = 0$ for all $x_{12} \in R_{12}$. Likewise $D_{11}(x_{22})R_{12} = (0)$ for all $x_{22} \in R_{22}$. Again by (H1), we find $D_{11}(x_{22}) = 0$ for all $x_{22} \in R_{22}$. Now, we can rewrite (6) as

$$D_{11}(x_{11} + x_{12} + x_{22}) = D_{11}(x_{11}) + D_{11}(x_{12}) + D_{11}(x_{22}).$$

It means that D_{11} is additive on $R_{11} \oplus R_{12} \oplus R_{22}$. On the other hand, for any $r \in R$, we find that

$$(D_{12}(x_{11} + x_{12} + x_{21} + x_{22}) - D_{12}(x_{12} + x_{22}))r$$
$$= D_{12}(x_{11} + x_{12} + x_{21} + x_{22})r - D_{12}(x_{12} + x_{22})r$$
$$= D_{12}(x_{11} + x_{12} + x_{21} + x_{22})(r_{21} + r_{22}) - D_{12}(x_{12} + x_{22})(r_{21} + r_{22})$$
$$= D_{12}((x_{11} + x_{12} + x_{21} + x_{22})(r_{21} + r_{22})) - (x_{11} + x_{12} + x_{21} + x_{22})$$
$$\quad D_{12}(r_{21} + r_{22}) - D_{12}((x_{12} + x_{22})(r_{21} + r_{22})) + (x_{12} + x_{22})D_{12}(r_{21} + r_{22})$$
$$= -(x_{11} + x_{21})D_{12}(r_{21} + r_{22}) = -(x_{11} + x_{21})e_1 D_{12}(r_{21} + r_{22})$$
$$= -(x_{11} + x_{21})D_{12}(e_1(r_{21} + r_{22})) + (x_{11} + x_{21})D_{12}(e_1)(r_{21} + r_{22}) = 0.$$

Hence $(D_{12}(x_{11} + x_{12} + x_{21} + x_{22}) - D_{12}(x_{12} + x_{22}))R = (0)$. In particular, $(D_{12}(x_{11} + x_{12} + x_{21} + x_{22}) - D_{12}(x_{12} + x_{22}))R_{22} = (0)$. By (H2), we find

$$D_{12}(x_{11} + x_{12} + x_{21} + x_{22}) = D_{12}(x_{12} + x_{22}).$$

Consequently

$$D_{12}(x_{11} + x_{12} + x_{21}) = D_{12}(x_{12}). \tag{7}$$

Now, for any $z_{22} \in R_{22}$, we get $D_{12}(x_{11})z_{22} = D_{12}(x_{11}z_{22}) - x_{11}D_{12}(z_{22}) = -x_{11}e_1D_{12}(z_{22}) = -x_{11}D_{12}(e_1z_{22}) + x_{11}D_{12}(e_1)z_{22} = 0$. That is $D_{12}(x_{11})R_{22} = (0)$ for all $x_{11} \in R_{11}$. Thus we may apply hypothesis (H2), which forces that $D_{12}(x_{11}) = 0$ for all $x_{11} \in R_{11}$. In the similar manner, we find that $D_{12}(x_{21})R_{22} = (0)$ for all $x_{21} \in R_{21}$. Again applying (H2), we get $D_{12}(x_{21}) = 0$ for all $x_{21} \in R_{21}$. Thus expression (7) assures that D_{12} is additive on $R_{11} \oplus R_{12} \oplus R_{21}$.

We now proceed to show that D_{11} is additive on R_{21} and D_{12} is additive on R_{22}. For any $x, y \in R$, we have

$$\begin{aligned} D_{11}(xy) &= D_{11}((x_{11} + x_{12} + x_{21} + x_{22})(y_{11} + y_{12} + y_{21} + y_{22})) \\ &= D_{11}((x_{11}y_{11} + x_{12}y_{21}) + (x_{21}y_{11} + x_{22}y_{21}) + (x_{11}y_{12} + x_{12}y_{22}) \\ &+ (x_{21}y_{12} + x_{22}y_{22})) = D_{11}((x_{11}y_{11} + x_{12}y_{21}) + (x_{21}y_{11} + x_{22}y_{21}))(\text{ using (5) }). \end{aligned}$$

and

$$\begin{aligned} D_{11}(x)y_{11} + x_{11}D_{11}(y) + D_{12}(x)y_{21} + x_{12}D_{21}(y) &= D_{11}(x_{11} + x_{21})y_{11} \\ &+ x_{11}D_{11}(y_{11} + y_{21}) + D_{12}(x_{12} + x_{22})y_{21} + x_{12}D_{21}(y_{11} + y_{21}). \end{aligned}$$

Now, relation (3) can be expressed as

$$\begin{aligned} D_{11}((x_{11}y_{11} + x_{12}y_{21}) + (x_{21}y_{11} + x_{22}y_{21})) &= D_{11}(x_{11} + x_{21})y_{11} \\ &+ x_{11}D_{11}(y_{11} + y_{21}) + D_{12}(x_{12} + x_{22})y_{21} + x_{12}D_{21}(y_{11} + y_{21}). \end{aligned} \tag{8}$$

In particular, putting $x_{11} = 0 = x_{12}$ in (8), we obtain

$$D_{11}(x_{21}y_{11} + x_{22}y_{21}) = D_{11}(x_{21})y_{11} + D_{12}(x_{22})y_{21}. \tag{9}$$

It follows that

$$D_{11}(x_{21}y_{11}) = D_{11}(x_{21})y_{11}, \quad D_{11}(x_{22}y_{21}) = D_{12}(x_{22})y_{21}. \tag{10}$$

Thus, (9) can be written as

$$D_{11}(x_{21}y_{11} + x_{22}y_{21}) = D_{11}(x_{21}y_{11}) + D_{11}(x_{22}y_{21}). \tag{11}$$

Replacing y_{11} by $x_{12}y_{21}$ and x_{22} by $z_{21}x_{12}$ in (11), we get

$$\begin{aligned} D_{11}(x_{21}x_{12}y_{21} + z_{21}x_{12}y_{21}) &= D_{11}(x_{21}x_{12}y_{21}) + D_{11}(z_{21}x_{12}y_{21}), \\ D_{11}((x_{21} + z_{21})x_{12}y_{21}) &= D_{11}((x_{21})(x_{12}y_{21})) + D_{11}((z_{21})(x_{12}y_{21})). \end{aligned}$$

Application of (10) yields that

$$D_{11}(x_{21} + z_{21})x_{12}y_{21} = D_{11}(x_{21})(x_{12}y_{21}) + D_{11}(z_{21})(x_{12}y_{21}).$$

That is,

$$(D_{11}(x_{21} + z_{21}) - D_{11}(x_{21}) - D_{11}(z_{21}))R_{12}R_{21} = (0).$$

Application of (H3) and (H1) respectively yields

$$D_{11}(x_{21} + z_{21}) = D_{11}(x_{21}) + D_{11}(z_{21}) \text{ for all } x_{21}, z_{21} \in R_{21}.$$

From (10), we have $D_{12}(x_{22})y_{21} = D_{11}(x_{22}y_{21})$. Therefore

$$\begin{aligned}
D_{12}(x_{22} + z_{22})y_{21} &= D_{11}((x_{22} + z_{22})y_{21}) = D_{11}(x_{22}y_{21} + z_{22}y_{21}) \\
&= D_{11}(x_{22}y_{21}) + D_{11}(z_{22}y_{21}) = D_{12}(x_{22})y_{21} + D_{12}(z_{22})y_{21}.
\end{aligned}$$

It implies that

$$(D_{12}(x_{22} + z_{22}) - D_{12}(x_{22}) - D_{12}(z_{22}))R_{21} = (0).$$

We may apply (H3) in order to obtain $D_{12}(x_{22} + z_{22}) = D_{12}(x_{22}) + D_{12}(z_{22})$. Hence, D_{12} is additive on R_{22}.

Next, we shall show that D_{11} is additive on R_{11} and D_{12} is additive on R_{11}. It is straight forward to check that, for any $x_{12}, y_{12} \in R_{12}$

$$(D_{11}(x_{12} + y_{12}) - D_{11}(x_{12}) - D_{11}(y_{12}))R_{12} = (0).$$

Thus, hypothesis (H1) forces $D_{11}(x_{12} + y_{12}) = D_{11}(x_{12}) + D_{11}(y_{12})$. Let $r_{12} \in R_{12}$. Then

$$\begin{aligned}
D_{11}(x_{11} + y_{11})r_{12} &= D_{11}((x_{11} + y_{11})r_{12}) - (x_{11} + y_{11})D_{11}(r_{12}) = D_{11}(x_{11}r_{12} + y_{11}r_{12}) \\
&\quad - x_{11}D_{11}(r_{12}) - y_{11}D_{11}(r_{12}) = D_{11}(x_{11}r_{12}) + D_{11}(y_{11}r_{12}) - x_{11}D_{11}(r_{12}) - y_{11}D_{11}(r_{12}) \\
&= D_{11}(x_{11})r_{12} + D_{11}(y_{11})r_{12}.
\end{aligned}$$

That is $(D_{11}(x_{11} + y_{11}) - D_{11}(x_{11}) - D_{11}(y_{11}))r_{12} = 0$ for all $r_{12} \in R_{12}$. Again we apply (H1) in order to obtain

$$D_{11}(x_{11} + y_{11}) = D_{11}(x_{11}) + D_{11}(y_{11}) \text{ for all } x_{11}, y_{11} \in R_{11}.$$

In like manner, for any $r_{21} \in R_{21}$, we see $(D_{12}(x_{11} + y_{11}) - D_{12}(x_{11}) - D_{12}(y_{11}))r_{21} = 0$. Thus $(D_{12}(x_{11} + y_{11}) - D_{12}(x_{11}) - D_{12}(y_{11}))R_{21} = (0)$. On utilizing (H3), D_{12} is additive on R_{11}. Further, we consider

$$\begin{aligned}
(D_{12}(x_{12} + y_{12}) - D_{12}(x_{12}) - D_{12}(y_{12}))r_{21} &= D_{12}(x_{12} + y_{12})r_{21} - D_{12}(x_{12})r_{21} - D_{12}(y_{12})r_{21} \\
&= D_{12}(x_{12}r_{21} + y_{12}r_{21}) - D_{12}(x_{12}r_{21}) - D_{12}(y_{12}r_{21}) = 0.
\end{aligned}$$

Therefore, we obtain $(D_{12}(x_{12} + y_{12}) - D_{12}(x_{12}) - D_{12}(y_{12}))R_{21} = (0)$. Hypothesis (H3) yields

$$D_{12}(x_{12} + y_{12}) = D_{12}(x_{12}) + D_{12}(y_{12}).$$

Now, we are well occupied to prove that D_{11} and D_{12} are additive on R. Observe that, as per the results derived above it is enough to show that $D_{11}(x_{11} + x_{21}) = D_{11}(x_{11}) + D_{11}(x_{21})$ and $D_{12}(x_{12} + x_{22}) = D_{12}(x_{12}) + D_{12}(x_{22})$.

Firstly, note that

$$\begin{aligned}
D_{21}(y) &= D_{21}(y_{11} + y_{12} + y_{21} + y_{22}) = e_2D(y_{11} + y_{12} + y_{21} + y_{22})e_1 \\
&= e_2D((y_{11} + y_{12} + y_{21} + y_{22})e_1)e_1 = e_2D(y_{11} + y_{21})e_1 = D_{21}(y_{11} + y_{21}).
\end{aligned}$$

and

$$(D_{22}(x_{11} + x_{12} + x_{21} + x_{22}) - D_{22}(x_{12} + x_{22}))r$$
$$= D_{22}(x_{11} + x_{12} + x_{21} + x_{22})(r_{21} + r_{22}) - D_{22}(x_{12} + x_{22})(r_{21} + r_{22})$$
$$= D_{22}((x_{11} + x_{12} + x_{21} + x_{22})(r_{21} + r_{22})) - (x_{11} + x_{12} + x_{21} + x_{22})D_{22}(r_{21}$$
$$+ r_{22}) - D_{22}((x_{12} + x_{22})(r_{21} + r_{22})) + (x_{12} + x_{22})D_{22}(r_{21} + r_{22}) = 0.$$

Let us rewrite expression (4) as

$$\begin{aligned} D_{12}((x_{11}y_{12} + x_{12}y_{22}) + (x_{21}y_{12} + x_{22}y_{22})) &= D_{11}(x_{11} + x_{21})y_{12} \\ &+ D_{12}(x_{12} + x_{22})y_{22} + x_{11}D_{12}(y_{12} + y_{22}) + x_{12}D_{22}(y_{12} + y_{22}). \end{aligned} \tag{12}$$

In particular, we put $x_{12} = 0 = x_{21}$ in (12), we find

$$D_{12}(x_{11}y_{12} + x_{22}y_{22}) = D_{11}(x_{11})y_{12} + D_{12}(x_{22})y_{22} + x_{11}D_{12}(y_{12} + y_{22}). \tag{13}$$

On substituting $x_{11} = e_1, y_{12} = z_{12}y_{22}$ in (13), we get

$$\begin{aligned} D_{12}((z_{12} + x_{22})y_{22}) &= D_{11}(e_1)z_{12}y_{22} + D_{12}(x_{22})y_{22} + e_1D_{12}(z_{12}y_{22} + y_{22}) \\ &= D_{12}(x_{22})y_{22} + D_{12}(e_1(z_{12}y_{22} + y_{22})) - D_{12}(e_1)(z_{12}y_{22} + y_{22}) \\ &= D_{12}(x_{22})y_{22} + D_{12}(z_{12}y_{22}) = D_{12}(x_{22})y_{22} + D_{12}(z_{12})y_{22}. \end{aligned}$$

That gives

$$D_{12}((z_{12} + x_{22})y_{22}) = D_{12}(z_{12})y_{22} + D_{12}(x_{22})y_{22}. \tag{14}$$

We next put $y_{12} = 0 = x_{11}$ in (12), we get

$$D_{12}((x_{12} + x_{22})y_{22}) = D_{12}(x_{12} + x_{22})y_{22} + x_{12}D_{22}(y_{22}). \tag{15}$$

On combining (14) and (15), it follows that

$$D_{12}(x_{12} + x_{22})y_{22} + x_{12}D_{22}(y_{22}) = (D_{12}(z_{12}) + D_{12}(x_{22}))y_{22}.$$

On substituting $y_{22} = y_{21}t_{12}$ in the above expression in order to obtain

$$\begin{aligned} (D_{12}(z_{12}) + D_{12}(x_{22}))y_{21}t_{12} &= D_{12}(x_{12} + x_{22})y_{21}t_{12} + x_{12}D_{22}(y_{21}t_{12}) \\ &= D_{12}(x_{12} + x_{22})y_{21}t_{12} + x_{12}D_{22}(y_{21})t_{12} + x_{12}y_{21}D_{22}(t_{12}) = D_{12}(x_{12} + x_{22})y_{21}t_{12}. \end{aligned}$$

That is $(D_{12}(x_{12} + x_{22}) - D_{12}(z_{12}) - D_{12}(x_{22}))y_{21}t_{12} = 0$ for all $y_{21} \in R_{21}$ and $t_{12} \in R_{12}$. Thus $(D_{12}(x_{12} + x_{22}) - D_{12}(z_{12}) - D_{12}(x_{22}))R_{21}R_{12} = (0)$. An application of (H1) and (H3) successively yields $D_{12}(z_{12} + x_{22}) = D_{12}(z_{12}) + D_{12}(x_{22})$. Moreover, we put $x_{12} = 0 = y_{22}$ in (14) in order to obtain

$$\begin{aligned} D_{11}(x_{11} + x_{21})y_{12} + x_{11}D_{12}(y_{12}) &= D_{12}(x_{11}y_{12} + x_{21}y_{12}) \\ &= D_{12}(x_{11}y_{12}) + D_{12}(x_{21}y_{12}). \end{aligned} \tag{16}$$

It follows that

$$D_{12}(x_{11}y_{12}) = D_{11}(x_{11})y_{12} + x_{11}D_{12}(y_{12}), \quad D_{12}(x_{21}y_{12}) = D_{11}(x_{21})y_{12}. \tag{17}$$

By utilizing (17) in (16), we find $(D_{11}(x_{11} + x_{21}) - D_{11}(x_{11}) - D_{11}(x_{21}))y_{12} = 0$ for all $y_{12} \in R_{12}$. That means $(D_{11}(x_{11} + x_{21}) - D_{11}(x_{11}) - D_{11}(x_{21}))R_{12} = (0)$. By (H1), we get $D_{11}(x_{11} + x_{21}) = D_{11}(x_{11}) + D_{11}(x_{21})$. $\square$

Analogously, we can prove the following lemma:

Lemma 2. *Let R be a ring (not necessary with unity) and M be a bimodule over R. Suppose that R contains a non-trivial idempotent e_1 such that for any $m \in M$, the following are satisfied:*

(H4) $e_2 m e_2 R_{21} = (0)$ *implies* $e_2 m e_2 = 0$,

(H5) $e_2 m e_1 R_{11} = (0)$ *implies* $e_2 m e_1 = 0$,

(H6) $e_2 m e_1 R_{12} = (0)$ *implies* $e_2 m e_2 = 0$.

Then D_{21} and D_{22} are additive.

Since $D = D_{11} + D_{12} + D_{21} + D_{22}$, Lemma 1 and Lemma 2 proves our main result:

Theorem 1. *Let R be a ring and M be a bimodule over R. If e_1 is a non-trivial idempotent in R such that for all $m \in M$ the conditions (H1)-(H6) hold. Then every multiplicative-derivation $d : R \to M$ is additive.*

Recall that R is said to be a prime ring if $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is called semiprime if $aRa = (0)$ for all $a \in R$. Let R be a semiprime ring and Q be the two sided Martindale quotient ring of R. The maximal left ring of quotients (also called left Utumi quotient ring) of R is denotes by Q_{ml}. The center C of Q is called the extended centroid of R. If R happens to be prime, then C is a field. Moreover, the extended centroid C of R coincides with the center of Q_{ml} and is reduced in the sense that C does not have nonzero nilpotent elements. For more information of these objects, we refer the reader to [1]. As an application of Theorem 1, we obtain the following consequent results:

Corollary 1. *Let R be a semiprime ring containing a non-trivial idempotent e. Suppose that for any $a \in Q_{ml}$ the following holds:*

(I) $e_1 a e_1 R e_2 = (0)$ *implies* $e_1 a e_1 = 0$,

(II) $e_2 a e_2 R e_1 = (0)$ *implies* $e_2 a e_2 = 0$.

Then any multiplicative-derivation $d : R \to Q_{ml}$ is additive.

Proof. Let $a \in Q_{ml}$ be an element such that $e_i a e_j R e_k = (0)$ for all $i, j, k \in \{1, 2\}$. We have the following possible cases:
Case 1. If $i = k$, then we have $(e_i a e_j R e_i) a e_j = 0$. It yields that $e_i a e_j = 0$ for all $i, j \in \{1, 2\}$.
Case 2. Suppose that $j = k$. In the view of proposition 2.1.7 (ii) of [1], there exist a dense left ideal D of R such that $D e_i a \subseteq R$. It implies that $(D e_i a e_j) R (D e_i a e_j) \subseteq (D e_i a e_j) R e_j = (0)$. It follows that $D e_i a e_j = (0)$ for all $i, j \in \{1, 2\}$. With the aid of proposition 2.1.7 (iii) of [1], we obtain $e_i a e_j = 0$ for all $i, j \in \{1, 2\}$.
Case 3. In latter case $i = j$. By our hypothesis $e_i a e_i R e_k = (0)$ implies $e_i a e_i = 0$ for all $i \in \{1, 2\}$. Now, we see that the condition (H1)-(H6) hold here. Therefore, d is additive by Theorem 1. $\square$

In case R is a prime ring, every derivation $d : R \to Q_{ml}$ is additive automatically, since if for any $q_1, q_2 \in Q_{ml}$, $q_1 R q_2 = (0)$ implies $q_1 = 0$ or $q_0 = 0$. Thus, we obtain

Corollary 2. *Let R be a prime ring containing a non-trivial idempotent e. Then every multiplicative-derivation $d : R \to Q_{ml}$ is additive.*

1.2 Additivity of multiplicative semi-derivations

In [8] Martindale give a set of conditions that are sufficient for the additivity of ring isomorphisms. Precisely, he proved that "Let R be a ring containing a family $\{e_\lambda : \lambda \in \Lambda\}$ of idempotents satisfying (Martindale's conditions)

(I) $xR = (0)$ implies $x = 0$,

(II) If for each $\lambda \in \Lambda$, $e_\lambda Rx = (0)$, then $x = 0$ (hence $Rx = (0)$ implies $x = 0$),

(III) If $e_\lambda x e_\lambda R(1 - e_\lambda) = (0)$ for each $\lambda \in \Lambda$, then $e_\lambda x e_\lambda = 0$.

Then any multiplicative isomorphism of R onto an arbitrary ring S is additive". It is natural to think of a unified notion of multiplicative derivation and a semi derivation. In view of this idea, we now give the notion of multiplicative semi-derivation, as follows:

Definition 2. *Let R be a ring. A mapping $g : R \to R$ (not necessarily additive) defined by $g(xy) = g(x)g(y)$ for all $x, y \in R$ is called a multiplicative homomorphism of R. Then the mapping $\delta : R \to R$ (not necessarily additive) together with g is called multiplicative semi-derivation of R if*

$$\delta(xy) = \delta(x)g(y) + x\delta(y) = \delta(x)y + g(x)\delta(y).$$

holds for all $x, y \in R$.

Example 1. *Let $R = \{\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} : u, v, w \in \mathbb{R}\}$, where $\mathbb{R}$ denotes the field of real numbers. Define a mapping $g : R \to R$ by $g\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & det\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \end{pmatrix}$, which is clearly a ring endomorphism of R. Now, it can be easily verified that $\delta = g - I$ is the multiplicative semi-derivation of R.*

In this section, our aim is to obtain the additivity of multiplicative semi-derivations of rings under certain conditions. Precisely, we obtain the following result:

Theorem 2. *Let R be a ring satisfying Martindale's conditions (I)-(III). If $d : R \to R$ is a multiplicative semi-derivation of R associated with a multiplicative isomorphism $g : R \to R$, then d is additive.*

Let us define a function $\varphi : R \times R \to R$ that $\varphi(x, y) = d(x + y) - d(x) - d(y)$, where d is a multiplicative semi-derivation of R. Clearly, φ is a well-define mapping and $\varphi(x, 0) = 0 = \varphi(0, x)$ for all $x \in R$. Now, it is clear that d is additive if and only if $\varphi = 0$. This observation motivated the technique opted in this paper. We prove Theorem 2 through a sequence of lemmas.

Lemma 3. *For any $x, y, k \in R$, $k\varphi(x, y) = \varphi(kx, ky)$ and $\varphi(x, y)k = \varphi(xk, yk)$.*

Proof. In the view of [[8], Theorem], g must be additive on R. For any $x, y, k \in R$, we have $\varphi(kx, ky) = d(k(x + y)) - d(kx) - d(ky) = d(k)g(x + y) + kd(x + y) - d(k)g(x) - kd(x) - d(k)g(y) - kd(y) = k(d(x + y) - d(x) - d(y)) = k\varphi(x, y)$. On the other hand, let us consider $\varphi(xk, yk) = d((x + y)k) - d(xk) - d(yk) = d(x + y)k + g(x + y)d(k) - d(x)k - g(x)d(k) - d(y)k - g(y)d(k) = (d(x + y) - d(x) - d(y))k = \varphi(x, y)k.$ □

Lemma 4. $\varphi(x_{ii}, x_{jk}) = 0 = \varphi(x_{jk}, x_{ii}); j \neq k,$ *where* $i, j, k \in \{1, 2\}.$

Proof. In case $i = j$. For any $r_{il} \in R_{il}$, we find $\varphi(x_{ii}, x_{jk})r_{il} = \varphi(x_{ii}r_{il}, x_{jk}r_{il}) = \varphi(z_{il}, 0) = 0$ for all $i, j, , k, l \in \{1, 2\}$, by Lemma 3. For any $r_{kl} \in R_{kl}$, we have $\varphi(x_{ii}, x_{jk})r_{kl} = \varphi(x_{ii}r_{kl}, x_{jk}r_{kl}) = \varphi(0, w_{jl}) = 0$ for all $i, j, k, l \in \{1, 2\}$. Since $i = j \neq k$, it implies $\varphi(x_{ii}, x_{jk}) \quad R = (0)$. By hypothesis (I), we obtain $\varphi(x_{ii}, x_{jk}) = 0$. In the latter case, we assume $i \neq j$. For any $r_{mi} \in R_{mi}$, we have $r_{mi}\varphi(x_{ii}, x_{jk}) = \varphi(r_{mi}x_{ii}, r_{mi}x_{jk}) = \varphi(z_{mi}, 0) = 0$ for all $i, j, k, m \in \{1, 2\}$. Similarly, we may infer that $r_{mj}\varphi(x_{ii}, x_{jk}) = 0$ for all $r_{mj} \in R_{mj}$ and $i, j, k, m \in \{1, 2\}$. Combining these relation, we get $R\varphi(x_{ii}, x_{jk}) = (0)$. By hypothesis (II), we get $\varphi(x_{ii}, x_{jk}) = 0$. Hence, we conclude that $\varphi(x_{ii}, x_{jk}) = 0$ for all $j \neq k$ and $i, j, k \in \{1, 2\}$. Analogously, we obtain $\varphi(x_{jk}, x_{ii}) = 0$ for all $j \neq k$ and $i, j, k \in \{1, 2\}$. $\square$

Lemma 5. $\varphi(x_{12}, y_{12}) = 0.$

Proof. Clearly, $e_1\varphi(x_{12}, y_{12}) = \varphi(e_1x_{12}, e_1y_{12}) = \varphi(x_{12}, y_{12})$ and $\varphi(x_{12}, y_{12})e_1 = \varphi(x_{12}e_1, y_{12} e_1) = \varphi(0, 0) = 0$. It implies that $\varphi(x_{12}, y_{12}) \in R_{12}$. Therefore, $\varphi(x_{12}, y_{12})a_{11} = 0$ and $\varphi(x_{12}, y_{12})a_{12} = 0$ for all $a_{11} \in R_{11}, a_{12} \in R_{12}$. Now for any $a_{21} \in R_{21}$, we have

$$\varphi(x_{12}, y_{12})a_{21} = \varphi(x_{12}a_{21}, y_{12}a_{21}) = \varphi(x_{12}(a_{21} + y_{12}a_{21}), e_1(a_{21} + y_{12}a_{21}))$$
$$= \varphi(x_{12}, e_1)(a_{21} + y_{12}a_{21}) = 0 \quad (\text{using Lemma } 4).$$

In the similar way, we can show that $\varphi(x_{12}, y_{12})a_{22} = 0$ for all $a_{22} \in R_{22}$. Combining all these relations, we get $\varphi(x_{12}, y_{12})R = (0)$. Hence, $\varphi(x_{12}, y_{12}) = 0$ by condition (I). $\square$

Lemma 6. $\varphi(x_{11}, y_{11}) = 0.$

Proof. Under the influence of Lemma 3, it is easy to see that $\varphi(x_{11}, y_{11}) \in R_{11}$. For any $a_{12} \in R_{12}$, we have $\varphi(x_{11}, y_{11})a_{12} = \varphi(x_{11}a_{12}, y_{11}a_{12}) = \varphi(y_{12}, z_{12}) = 0$ by Lemma 5. That means

$$\varphi(x_{12}, y_{12})R_{12} = (0). \tag{18}$$

Since $\varphi(x_{11}, y_{11}) \in R_{11}$, so $\varphi(x_{11}, y_{11}) = e_1\varphi(x_{11}, y_{11})e_1$. From Eq. (18), we get $\varphi(x_{11}, y_{11})R_{12} = e_1\varphi(x_{11}, y_{11})e_1R(1 - e_1) = (0)$. By condition (III), we obtain $e_1\varphi(x_{11}, y_{11})e_1 = 0$ and hence $\varphi(x_{11}, y_{11}) = 0$. $\square$

Lemma 7. $\varphi(x_{11} + x_{12}, y_{11} + y_{12}) = 0.$

Proof. For any $a_{11} \in R_{11}$ and $a_{12} \in R_{12}$ we see that $\varphi(x_{11} + x_{12}, y_{11} + y_{12})a_{11} = \varphi(x_{11}a_{11}, y_{11} a_{11}) = 0$ by Lemma 6, and $\varphi(x_{11} + x_{12}, y_{11} + y_{12})a_{12} = \varphi(x_{11}a_{12}, y_{11}a_{12}) = 0$ by Lemma 5. By repeating same arguments and utilization of Lemma 5, 6 we get $\varphi(x_{11} + x_{12}, y_{11} + y_{12})a_{21} = 0$ for all $a_{21} \in R_{21}$ and $\varphi(x_{11} + x_{12}, y_{11} + y_{12})a_{22} = 0$ for all $a_{22} \in R_{22}$. Add up all these equations in order to find $\varphi(x_{11} + x_{12}, y_{11} + y_{12})R = (0)$. Hence, $\varphi(x_{11} + x_{12}, y_{11} + y_{12}) = 0$ by hypothesis (I). $\square$

Proof of Theorem 2: By Lemma 7, $\varphi(u, v) = 0$ for all $u, v \in e_1R$. For any $x, y, r \in R$, we have $e_1r\varphi(x, y) = \varphi(e_1rx, e_1ry) = 0$. Since e_1 was arbitrary member chosen from the family $\{e_\lambda : \lambda \in \Lambda\}$, so we must have $e_\lambda R\varphi(x, y) = (0)$ for all $\lambda \in \Lambda$. By our hypothesis (II), we find that $\varphi(x, y) = 0$ for all $x, y \in R$. $\square$

REFERENCES

[1] Beidar K. I., Martindale III W. S., Mikhalev A. V. Rings with generalized identities. Marcel Dekker Inc., New York, 1996.

[2] Bergen J. *Derivations in prime rings*. Canad. Math. Bull. 1983, **26**(3), 267–270. doi:10.4153/CMB-1983-042-2

[3] Daif M. N. *When is a multiplicative derivation additive?* Internat. J. Math. Math. Sci. 1991, **14**(3), 615–618. doi:10.1155/S0161171291000844

[4] Ferreira B. L. M., Guzzo Jr. H., Ferreira J. C. M. *Additivity of Jordan elementary maps on standard rings*. Alg. Disc. Math. 2014, **18**(2), 203–233.

[5] Jacobson N. Structure of rings. Colloq. Pub. (Amer. Math. Soc.), Rhode Island, Vol. 37, 1964.

[6] Ling Z., Lu F. *Jordan maps of nest algebras*. Linear Algebra Appl. 2004, **387** (1), 361–368. doi:10.1016/j.laa.2004.02.031

[7] Lu F. *Jordan derivable maps of prime rings*. Commun. Algebra 2010, **38** (12), 4430–4440. doi:10.1080/00927870903366884

[8] Martindale III W. S. *When are multiplicative mappings additive?* Proc. Amer. Math. Soc. 1969, **21**(3), 695–698. doi:10.2307/2036449

[9] Peisheng J. *Jordan maps on triangular algebras*. Linear Algebra Appl. 2007, **426** (1), 190–198. doi:10.1016/j.laa.2007.04.009

[10] Rodrigues R. L., Guzzo Jr. H., Ferreira J. C. M. *When is a multiplicative derivation additive in alternative rings?* Comm. Algebra 2016, **44** (6), 2561–2566. doi:10.1080/00927872.2015.1065835

[11] Wang Y. *The additivity of multiplicative maps on rings*. Comm. Algebra 2009, **37** (7), 2351–2356. doi: 10.1080/00927870802623369

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Нехай R — деяке кільце і M — деякий R-бімодуль. Відображення $d : R \to M$ (не обов'язково адитивне) називається мультиплікативним диференціюванням кільця R, якщо $d(xy) = d(x)y + xd(y)$ для всіх $x, y \in R$. У цій статті ми намагаємось встановити адитивність d при деяких додаткових обмеженнях. Крім того ми вводимо мультиплікативне напівдиференціювання кільця і обговорюємо його адитивність.

Ключові слова і фрази: диференціювання, мультиплікативне диференціювання, мультиплікативне напівдиференціювання кільця, адитивність, розклад Пірса.