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ON A NEW APPLICATION OF QUASI POWER INCREASING SEQUENCES

In the present paper, absolute matrix summability of infinite series has been studied. A new theorem concerned with absolute matrix summability factors, which generalizes a known theorem dealing with absolute Riesz summability factors of infinite series, has been proved under weaker conditions by using quasi β-power increasing sequences. Also, a known result dealing with absolute Riesz summability has been given.

Key words and phrases: Riesz mean, almost increasing sequences, quasi power increasing sequences, Hölder inequality, Minkowski inequality, matrix transformation.

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INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \dots .$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A;\delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [9])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |A_n(s) - A_{n-1}(s)|^k < \infty. \tag{1}$$

In the special case for $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, we obtain the $|\bar{N}, p_n|_k$ summability (see [2]). Also, it should be noted that for $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, the $\varphi - |A;\delta|_k$ summability reduces to $|\bar{N}, p_n;\delta|_k$ summability (see [3]).

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1 KNOWN RESULT

A positive sequence (h_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants K and L such that $Kc_n \leq h_n \leq Lc_n$ (see [1]). By means of this sequence, Mazhar [7] has established following theorem.

Theorem 1. *If (X_n) is an almost increasing sequence and the conditions*

$$|\lambda_m| X_m = O(1) \quad as \quad m \to \infty, \tag{2}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{3}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty, \tag{4}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{5}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{6}$$

are satisfied, where (t_n) is the nth $(C,1)$ mean of the sequence (na_n), then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

2 MAIN RESULT

A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that $Kn^\beta \gamma_n \geq m^\beta \gamma_m$ holds for all $n \geq m \geq 1$ (see [6]). It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta \lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. One can find some applications of quasi power increasing sequences (see [4–6, 10]). The purpose of this paper is to obtain a theorem which generalizes Theorem 1 for $\varphi - |A; \delta|_k$ summability using quasi β-power increasing sequence. Before giving this theorem, let us introduce some further notations.

Let $A = (a_{nv})$ be a normal matrix, $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \tag{7}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots, \tag{8}$$

$\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{9}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{10}$$

Theorem 2. *Let $(\lambda_n) \in \mathcal{BV}$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \dots, \tag{11}$$

$$a_{n-1,v} \geq a_{nv}, \quad for \quad n \geq v+1, \tag{12}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{13}$$

$$\sum_{v=1}^{n-1} \frac{|\hat{a}_{n,v+1}|}{v} = O(a_{nn}), \tag{14}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k}|\Delta_v \hat{a}_{nv}| = O\left(\varphi_v^{\delta k-1}\right) \quad as \quad m \to \infty, \tag{15}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k}|\hat{a}_{n,v+1}| = O\left(\varphi_v^{\delta k}\right) \quad as \quad m \to \infty. \tag{16}$$

Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$ and $\varphi_n p_n = O(P_n)$. If conditions (2), (3) *of Theorem 1 and*

$$\sum_{n=1}^{m} \varphi_n^{\delta k}\frac{1}{n}|t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{17}$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1}|t_n|^k = O(X_m) \quad as \quad m \to \infty \tag{18}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |A;\delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

Lemma 1. *([4]). Under the conditions of Theorem 2, we have*

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{19}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty. \tag{20}$$

3 PROOF OF THEOREM 2

Let (I_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v$$

by (9) and (10). Now, using Abel's transformation,

$$\begin{aligned}\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\ &= \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v \left(\hat{a}_{nv}\right) \lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \lambda_{v+1} t_v + \frac{n+1}{n} a_{nn}\lambda_n t_n \\ &= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.\end{aligned}$$

To complete the proof of Theorem 2, by (1), we will prove that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \mid I_{n,r} \mid^k < \infty, \quad for \quad r = 1,2,3,4.$$

For $r = 1$, applying Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid I_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v| \, |t_v|\right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\right)^{k-1}.\end{aligned}$$

By (7) and (8), we have

$$\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v}.$$

Thus using (7), (11) and (12)

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \leq a_{nn}.$$

Hence,

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid I_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \, |\lambda_v|^k \, |t_v|^k\right) \\ &= O(1) \sum_{v=1}^{m} |\lambda_v|^k \, |t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\Delta_v(\hat{a}_{nv})| = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} |\lambda_v| |t_v|^k \\ &= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \varphi_r^{\delta k-1} |t_r|^k + O(1)|\lambda_m| \sum_{v=1}^{m} \varphi_v^{\delta k-1} |t_v|^k \\ &= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)|\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty,\end{aligned}$$

by (2), (13), (15), (18) and (20). For $r=2$, using Hölder's inequality, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid I_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k \right) \\
&= O(1) \sum_{v=1}^{m} |\Delta\lambda_v||t_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\hat{a}_{n,v+1}| = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} v|\Delta\lambda_v| \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{r=1}^{v} \varphi_r^{\delta k} \frac{1}{r} |t_r|^k + O(1) m |\Delta\lambda_m| \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v X_v |\Delta^2\lambda_v| + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1) m |\Delta\lambda_m| X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by using (3), (13), (16), (17), (19) and (20).

Again, for $r=3$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} |I_{n,3}|^k &\le \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|}{v} \right)^k \\
&\le \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} \frac{|\hat{a}_{n,v+1}|}{v} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|^k}{v} \right) \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \frac{|t_v|^k}{v} \sum_{n=v+1}^{m+1} \varphi_n^{\delta k} |\hat{a}_{n,v+1}| = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{1}{v} |\lambda_{v+1}||t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| \sum_{r=1}^{v} \varphi_r^{\delta k} \frac{1}{r} |t_r|^k + O(1) |\lambda_{m+1}| \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{1}{v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| X_{v+1} + O(1) |\lambda_{m+1}| X_{m+1} \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by using (2), (13), (14), (16), (17) and (20).

Finally, as in the process for $I_{n,1}$, by using Abel's transformation, we have

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \mid I_{n,4} \mid^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} a_{nn}^k |\lambda_n|^k |t_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k-1} |\lambda_n||t_n|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by using (2), (13), (18) and (20). This completes the proof of Theorem 2.

If we take (X_n) as an almost increasing sequence, $\varphi_n = \frac{P_n}{p_n}$, $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$ in Theorem 2, then we get Theorem 1. In this case the conditions (14), (17) and (18) reduce to the conditions (4), (5) and (6), respectively. Also, if we take (X_n) as an almost increasing sequence, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 2, then we get a theorem dealing with $|\bar{N}, p_n; \delta|_k$ summability (see [8]).

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У роботі досліджено абсолютну матричну сумовність нескінченних послідовностей. Нову теорему, яка стосується умов абсолютної матричної сумовності і узагальнює відому теорему про умови абсолютної сумовності Ріса для нескінчнних послідовностей доведено за слабших умов з використанням квазі-β-степеневих зростаючих послідовностей. Також, отримано один відомий результат, який стосується абсолютної сумовності Ріса.

Ключові слова і фрази: середнє за Рісом, майже зростаючі послідовності, квазі-степеневі зростаючі послідовності, нерівність Гельдера, нерівність Мінковського, матричні перетворення.