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CHARACTERIZATIONS OF REGULAR AND INTRA-REGULAR ORDERED Γ-SEMIHYPERGROUPS IN TERMS OF BI-Γ-HYPERIDEALS

The concept of Γ-semihypergroups is a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ-semigroups. In this paper, we study the notion of bi-Γ-hyperideals in ordered Γ-semihypergroups and investigate some properties of these bi-Γ-hyperideals. Also, we define and use the notion of regular ordered Γ-semihypergroups to examine some classical results and properties in ordered Γ-semihypergroups.

Key words and phrases: ordered Γ-semihypergroup, Γ-hyperideal, bi-Γ-hyperideal.

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1 INTRODUCTION

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation [24]. The notion of a Γ-semigroup was introduced by Sen and Saha [37] as a generalization of semigroups as well as of ternary semigroups. Since then, hundreds of papers have been written on this topic, see [6,7,16]. Many classical notions of semigroups have been extended to Γ-semigroups. Let $S = \{a, b, c, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ be two non-empty sets. Then, S is called a Γ-*semigroup* if there exists a mapping from $S \times \Gamma \times S$ to S, written as $(a, \gamma, b) \to a\gamma b$, satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all a, b, c in S and α, β in Γ. In this case by (S, Γ) we mean S is a Γ-semigroup. By an *ordered semigroup*, we mean an algebraic structure $(S, \cdot, \leq)$, which satisfies the following conditions: (1) $(S, \cdot)$ is a semigroup; (2) S is a partial ordered set by $\leq$; (3) If a and b are elements of S such that $a \leq b$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ for all $c \in S$. Ordered semigroups have been studied extensively by Kehayopulu and Tsingelis, for example, see [27–29]. The notions of an ordered Γ-groupoid and an ordered Γ-semigroup were defined by Sen and Seth in [38]. Many authors studied different aspects of ordered Γ-semigroups, for instance, Abbasi and Basar [1], Chinram and Tinpun [7,8], Dutta and Adhikari [16,17], Hila [22], Iampan [25], Kehayopulu [26], Kwon [31], Kwon and Lee [32,33], and many others. Recall from [38], that an *ordered* Γ-*semigroup* $(S, \Gamma, \leq)$ is a Γ-semigroup (S, Γ) together with an order relation $\leq$ such that $a \leq b$ implies that $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in S$ and $\gamma \in \Gamma$.

The concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. The concept of ordering hypergroups introduced by Chvalina [11] as a special class

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of hypergroups. Many authors studied different aspects of ordered semihypergroups, for instance, Davvaz et al. [15], Gu and Tang [19], Heidari and Davvaz [20], Tang et al. [39], and many others. Explicit study of ordered semihypergroups seems to have begun with Heidari and Davvaz [20] in 2011. Recall from [20], that an *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is compatible with the hyperoperation $\circ$, meaning that for any $x, y, z \in S$,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Recently, Davvaz et al. [4, 5, 13, 21, 23] studied the notion of Γ-semihypergroup as a generalization of a semigroup, a generalization of a semihypergroup and a generalization of a Γ-semigroup. They proved some results in this respect and presented many exmaples of Γ-semihypergroups. Many classical notions of semigroups and semihypergroups have been extended to Γ-semihypergroups. The notion of a Γ-hyperideal of a Γ-semihypergroup was introduced in [4]. Davvaz et al. [5] introduced the notion of Pawlak's approximations in Γ-semihypergroups. Abdullah et al. [2] studied M-hypersystems and N-hypersystems in a Γ-semihypergroup. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The concept of hyperstructure was first introduced by Marty [34] at the eighth Congress of Scandinavian Mathematicians in 1934. A comprehensive review of the theory of hyperstructures can be found in [9, 10, 12, 40]. Let S be a non-empty set and $P^*(S)$ be the family of all non-empty subsets of S. A mapping $\circ : S \times S \to P^*(S)$ is called a *hyperoperation* on S. A *hypergroupoid* is a set S together with a (binary) hyperoperation. In the above definition, if A and B are two non-empty subsets of S and $x \in S$, then we denote

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}}, \quad x \circ A = \{x\} \circ A \quad \text{and} \quad B \circ x = B \circ \{x\}.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for every x, y, z in S, $x \circ (y \circ z) = (x \circ y) \circ z$. That is,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

A non-empty subset K of a semihypergroup S is called a *subsemihypergroup* of S if $K \circ K \subseteq K$. A hypergroupoid $(S, \circ)$ is called a *quasihypergroup* if for every $x \in S$, $x \circ S = S = S \circ x$. This condition is called the reproduction axiom. The couple $(S, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. A non-empty subset K of S is a *subhypergroup* of S if $K \circ a = a \circ K = K$, for every $a \in K$. A hypergroup $(S, \circ)$ is called *commutative* if $x \circ y = y \circ x$, for every $x, y \in S$.

2 REVIEW: ORDERED Γ-SEMIHYPERGROUPS

The notion of a Γ-semihypergroup was introduced by Davvaz et al. [4,5,21]. In [20], Heidari and Davvaz introduced the concept of ordered semihypergroups, which is a generalization of ordered semigroups. In this section, we recall the notion of an ordered Γ-semihypergroup and then we present some definitions and properties which we will need in this paper. Throughout this paper, unless otherwise stated, S is always an ordered Γ-semihypergroup $(S, \Gamma, \leq)$.

Definition 1 ([4,5]). *Let S and Γ be two non-empty sets. Then, S is called a Γ-semihypergroup if every $\gamma \in \Gamma$ is a hyperoperation on S, i.e., $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$, we have $x\alpha(y\beta z) = (x\alpha y)\beta z$. If every $\gamma \in \Gamma$ is an operation, then S is a Γ-semigroup. Let A and B be two non-empty subsets of S. We define*

$$A\Gamma B = \bigcup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\} = \bigcup_{\gamma\in\Gamma} A\gamma B.$$

A Γ-semihypergroup S is called commutative if for all $x, y \in S$ and $\gamma \in \Gamma$, we have $x\gamma y = y\gamma x$. A Γ-semihypergroup S is called a Γ-hypergroup if for every $\gamma \in \Gamma$, (S, γ) is a hypergroup.

Now, we consider the notion of an ordered Γ-semihypergroup.

Definition 2 ([30]). *An algebraic hyperstructure $(S, \Gamma, \leq)$ is called an ordered Γ-semihypergroup if (S, Γ) is a Γ-semihypergroup and $(S, \leq)$ is a partially ordered set such that for any $x, y, z \in S$ and $\gamma \in \Gamma$, $x \leq y$ implies $z\gamma x \leq z\gamma y$ and $x\gamma z \leq y\gamma z$. Here, if A and B are two non-empty subsets of S, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.*

Let S be an ordered Γ-semihypergroup. By a sub Γ-semihypergroup of S we mean a non-empty subset A of S such that $a\gamma b \subseteq A$ for all $a, b \in A$ and $\gamma \in \Gamma$.

Example 1 ([30]). *Let $(S, \circ, \leq)$ be an ordered semihypergroup and Γ a non-empty set. We define $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then, $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup.*

Definition 3. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. A non-empty subset I of S is called a left Γ-hyperideal of S if it satisfies the following conditions:*

(1) $S\Gamma I \subseteq I$;

(2) When $x \in I$ and $y \in S$ such that $y \leq x$, imply that $y \in I$.

A right Γ-hyperideal of an ordered Γ-semihypergroup S is defined in a similar way. By *two-sided Γ-hyperideal* or simply *Γ-hyperideal*, we mean a non-empty subset of S which both left and right Γ-hyperideal of S. A Γ-hyperideal I of S is said to be *proper* if $I \neq S$.

Let K be a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. If H is a non-empty subset of K, then we define $(H]_K := \{k \in K \mid k \leq h \text{ for some } h \in H\}$. Note that if $K = S$, then we define $(H] := \{x \in S \mid x \leq h \text{ for some } h \in H\}$. For $H = \{h\}$, we write $(h]$ instead of $(\{h\}]$. Note that the condition (2) in Definition 3 is equivalent to $(I] \subseteq I$. If A and B are non-empty subsets of S, then we have

(1) $A \subseteq (A]$;

(2) $((A]] = (A]$;

(3) If $A \subseteq B$, then $(A] \subseteq (B]$;

(4) $(A]\Gamma(B] \subseteq (A\Gamma B]$;

(5) $((A]\Gamma(B]] = (A\Gamma B]$.

Lemma 1. *If I and J are Γ-hyperideals of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$, then $I \cap J$ is a Γ-hyperideal of S.*

Proof. Let $x \in I, y \in J$ and $\gamma \in \Gamma$. Then, $x\gamma y \subseteq I\Gamma J \subseteq I\Gamma S \subseteq I$ and $x\gamma y \subseteq I\Gamma J \subseteq S\Gamma J \subseteq J$. So, $x\gamma y \subseteq I \cap J$ and hence $\varnothing \neq I \cap J \subseteq S$. We have $(I \cap J)\Gamma S \subseteq I\Gamma S \subseteq I$ and $S\Gamma(I \cap J) \subseteq S\Gamma J \subseteq J$. Similarly, $(I \cap J)\Gamma S \subseteq J$ and $S\Gamma(I \cap J) \subseteq I$. So, we have $(I \cap J)\Gamma S \subseteq I \cap J$ and $S\Gamma(I \cap J) \subseteq I \cap J$. Now, let $x \in I \cap J, y \in S$ and $y \leq x$. Since I and J are Γ-hyperideals of S, we obtain $y \in I$ and $y \in J$. Thus, $y \in I \cap J$. This completes the proof. □

Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. A subset A of S is called *idempotent* if $A = (A\Gamma A]$.

Lemma 2. *The Γ-hyperideals of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$ are idempotent if and only if for any Γ-hyperideals I, J of S, we have $I \cap J = (I\Gamma J]$.*

Proof. The sufficiency is obvious. For the necessity, let I, J be Γ-hyperideals of S. We have $(I\Gamma J] \subseteq (I\Gamma S] \subseteq (I] = I$ and $(I\Gamma J] \subseteq (S\Gamma J] \subseteq (J] = J$. So, we have $(I\Gamma J] \subseteq I \cap J$. On the other hand, by Lemma 1, $I \cap J$ is a Γ-hyperideal of S. By assumption, we have $I \cap J = ((I \cap J)\Gamma(I \cap J)] \subseteq (I\Gamma J]$. This completes the proof. □

Theorem 1. *Let $(S, \Gamma, \leq)$ be a commutative ordered Γ-semihypergroup. If I is a Γ-hyperideal of S and A is a non-empty subset of S, then $(I : A) = \{x \in S \mid x\gamma a \subseteq I \text{ for all } a \in A \text{ and } \gamma \in \Gamma\}$ is a Γ-hyperideal of S.*

Proof. Suppose that $x \in (I : A)$, $s \in S$ and $\delta \in \Gamma$. Then, $x\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. We have $(s\delta x)\gamma a = s\delta(x\gamma a) \subseteq S\Gamma I \subseteq I$. So, we have $s\delta x \subseteq (I : A)$. In the similar way, we obtain $x\delta s \subseteq (I : A)$. Now, let $x \in (I : A)$, $y \in S$ and $y \leq x$. Then, $x\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. Also, we have $y\gamma a \leq x\gamma a$ for all $a \in A$ and $\gamma \in \Gamma$, by hypothesis. So, for any $u \in y\gamma a$, $u \leq v$ for some $v \in x\gamma a \subseteq I$. Since I is a Γ-hyperideal of S, it follows that $u \in I$. So, we have $y\gamma a \subseteq I$ for all $a \in A$ and $\gamma \in \Gamma$. Thus, we have $y \in (I : A)$. Therefore, $(I : A)$ is a Γ-hyperideal of S. □

3 BI-Γ-HYPERIDEALS

The study of ordered semihyperrings was first undertaken by Davvaz and Omidi [14]. In [35], Omidi, Davvaz and Corsini studied some properties of hyperideals in ordered Krasner hyperrings. The concept of a bi-ideal is a very interesting and important thing in semigroups and ordered semigroups. In 1952, Good and Hughes [18] introduced the notion of bi-ideals in semigroups. Recently, Davvaz et al. [4] introduced the notion of bi-Γ-hyperideal in Γ-semihypergroups (cf. [3]). In [36], Pibaljommee and Davvaz studied the properties of bi-hyperideals in ordered semihypergroups. The concept of bi-Γ-hyperideals of an ordered Γ-semihypergroup is a generalization of the concept of Γ-hyperideals (left Γ-hyperideals, right Γ-hyperideals) of an ordered Γ-semihypergroup. First, we define the concept of a bi-Γ-hyperideal in ordered Γ-semihypergroups.

Definition 4 ([30]). *A sub Γ-semihypergroup B of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is called a bi-Γ-hyperideal of S if the following conditions hold:*

(1) $B\Gamma S\Gamma B \subseteq B$;

(2) When $x \in B$ and $y \in S$ such that $y \leq x$, imply that $y \in B$.

The concept of bi-Γ-hyperideals of an ordered Γ-semihypergroup is a generalization of the concept of Γ-hyperideals (left Γ-hyperideals, right Γ-hyperideals) of an ordered Γ-semihypergroup. Obviously, every left (right) Γ-hyperideal of an ordered Γ-semihypergroup S is a bi-Γ-hyperideal of S, but the the following example shows that the converse is not true in general case. Indeed, If I is a left (right) Γ-hyperideal of S, then $I\Gamma I \subseteq S\Gamma I \subseteq I$. Hence, I is a sub Γ-semihypergroup of S.

Example 2. *Let $S = \{a,b,c,d,e,f\}$ and $\Gamma = \{\gamma,\beta\}$ be the sets of binary hyperoperations defined as follows.*

γ	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	$\{a,c\}$	a	a	$\{a,f\}$
d	a	b	$\{a,e\}$	a	a	$\{a,d\}$
e	a	b	$\{a,e\}$	a	a	$\{a,d\}$
f	a	b	$\{a,c\}$	a	a	$\{a,f\}$

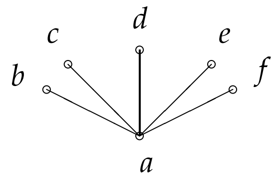
β	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	a	a	a	a
d	a	b	a	$\{a,d\}$	$\{a,e\}$	a
e	a	b	a	a	a	a
f	a	b	a	$\{a,f\}$	$\{a,c\}$	a

Then S is a Γ-semihypergroup [41]. We have $(S,\Gamma,\leq)$ is an ordered Γ-semihypergroup where the order relation $\leq$ is defined by:

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(a,e),(a,f),(b,b),(c,c),(d,d),(e,e),(f,f)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(a,b),(a,c),(a,d),(a,e),(a,f)\}.$$



Here,

(1) It is a routine matter to verify that $B_1 = \{a,b,c\}$ is a bi-Γ-hyperideal of S, but it is not a Γ-hyperideal of S.

(2) With a small amount of effort one can verify that $B_2 = \{a,b,c,f\}$ is a bi-Γ-hyperideal of S, but it is not a left Γ-hyperideal of S.

Lemma 3. *The intersection of any family of bi-Γ-hyperideals of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is a bi-Γ-hyperideal of S.*

Proof. Let $\{B_k \mid k \in \Lambda\}$ be a family of bi-Γ-hyperideals of S and $B = \bigcap_{k\in\Lambda} B_k$. It is easy to check that B is a sub Γ-semihypergroup of S. Now, let $x \in B\Gamma S\Gamma B$. Then, $x \in a\alpha s\beta b$ for some $a,b \in B$, $s \in S$ and $\alpha,\beta \in \Gamma$. Since each B_k is a bi-Γ-hyperideal of S, it follows that $a\alpha s\beta b \subseteq B_k\Gamma S\Gamma B_k \subseteq B_k$ for all $k \in \Lambda$. Then, $x \in B_k$ for all $k \in \Lambda$. So, we have $x \in \bigcap_{k\in\Lambda} B_k = B$. Since x was chosen arbitrarily, we have $B\Gamma S\Gamma B \subseteq B$. If $x \in B$ and $y \in S$ such that $y \leq x$, then $x \in B_k$ for all $k \in \Lambda$. Since each B_k is a bi-Γ-hyperideal of S, it follows that $y \in B_k$ for all $k \in \Lambda$. So, we have $y \in \bigcap_{k\in\Lambda} B_k = B$. Hence, B is a bi-Γ-hyperideal of S. ☐

Lemma 4. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. If B is a bi-Γ-hyperideal of S and C is a bi-Γ-hyperideal of B, such that $C = (C\Gamma C]$, then C is a bi-Γ-hyperideal of S.*

Proof. By assumption, we have that

$$C\Gamma C = (C\Gamma C]\Gamma(C\Gamma C] \subseteq (C\Gamma(C\Gamma C\Gamma C)] \subseteq (C\Gamma C] = C,$$

which shows that C is a sub Γ-semihypergroup of S. On the other hand, we have $B\Gamma S\Gamma B \subseteq B$ and $C\Gamma B\Gamma C \subseteq C$. Thus, we have

$$\begin{aligned} C\Gamma S\Gamma C &= (C\Gamma C]\Gamma S\Gamma(C\Gamma C] = (C\Gamma C]\Gamma(S]\Gamma(C\Gamma C] \\ &\subseteq (C\Gamma C\Gamma S]\Gamma(C\Gamma C] \subseteq (C\Gamma(C\Gamma S\Gamma C)\Gamma C] \\ &\subseteq (C\Gamma(B\Gamma S\Gamma B)\Gamma C] \subseteq (C\Gamma B\Gamma C] \subseteq (C]_B \subseteq C. \end{aligned}$$

Now, let $c \in C$ and $x \leq c$, where $x \in S$. Since B is a bi-Γ-hyperideal of S and $C \subseteq B$, we get $x \in B$. On the other hand, C is a bi-Γ-hyperideal of B. It follows that $x \in C$. This completes the proof. □

Let A be a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. We denote by $L_S(A)$ (resp. $R_S(A)$, $I_S(A)$) the left (resp. right, two-sided) Γ-hyperideal of S generated by A.

Lemma 5. *If A is a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$, then the following hold:*

(1) $L_S(A) = (A \cup S\Gamma A]$;

(2) $R_S(A) = (A \cup A\Gamma S]$;

(3) $I_S(A) = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$.

Proof. Since $A \subseteq L_S(A)$ and $S\Gamma A \subseteq L_S(A)$, it follows that $(A \cup S\Gamma A] \subseteq L_S(A)$. Clearly, $(A \cup S\Gamma A] \neq \varnothing$. We have

$$\begin{aligned} S\Gamma(A \cup S\Gamma A] &= (S]\Gamma(A \cup S\Gamma A] \subseteq (S\Gamma(A \cup S\Gamma A)] \\ &= (S\Gamma A \cup S\Gamma(S\Gamma A)] \subseteq (S\Gamma A] \subseteq (A \cup S\Gamma A]. \end{aligned}$$

Thus, $(A \cup S\Gamma A]$ is a left Γ-hyperideal of S containing A. This means that $L_S(A) \subseteq (A \cup S\Gamma A]$. This proves that (1) holds. The conditions (2) and (3) are proved similarly. □

Corollary 1. *Let a be an element of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then,*

(1) $L_S(a) = (a \cup S\Gamma a]$;

(2) $R_S(a) = (a \cup a\Gamma S]$;

(3) $I_S(a) = (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]$.

Let A be a non-empty subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. We define

$$\Theta = \{B \mid B \text{ is a bi-}\Gamma\text{-hyperideal of } S \text{ containing } A\}.$$

Since $S \in \Theta$, it follows that $\Theta \neq \varnothing$. We denote by $B_S(A)$ the bi-Γ-hyperideal of S generated by A. Clearly, $A \subseteq B_S(A) = \bigcap_{B \in \Theta} B$. By Lemma 3, $B_S(A)$ is a bi-Γ-hyperideal of S.

Lemma 6. *Let A be a non-empty subset of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. Then,*

$$B_S(A) = (A \cup A\Gamma A \cup A\Gamma S\Gamma A].$$

Proof. Set $B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$. Clearly, $B \neq \varnothing$. We have

$$\begin{aligned} B\Gamma B &= (A \cup A\Gamma A \cup A\Gamma S\Gamma A]\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A] \\ &\subseteq ((A \cup A\Gamma A \cup A\Gamma S\Gamma A)\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A)] \\ &\subseteq (A\Gamma S\Gamma A] \subseteq (A \cup A\Gamma A \cup A\Gamma S\Gamma A]. \end{aligned}$$

Hence, B is a sub Γ-semihypergroup of S. Now,

$$\begin{aligned} B\Gamma S\Gamma B &= (A \cup A\Gamma A \cup A\Gamma S\Gamma A]\Gamma S\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A] \\ &\subseteq ((A \cup A\Gamma A \cup A\Gamma S\Gamma A)\Gamma S\Gamma(A \cup A\Gamma A \cup A\Gamma S\Gamma A)] \\ &\subseteq (A\Gamma A \cup A\Gamma S\Gamma A] \subseteq (A \cup A\Gamma A \cup A\Gamma S\Gamma A]. \end{aligned}$$

Therefore, B is a bi-Γ-hyperideal of S, and hence $B_S(A) \subseteq (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$. Let C be a bi-Γ-hyperideal of S containing A. Then, $A\Gamma A \subseteq C$ and $A\Gamma S\Gamma A \subseteq C\Gamma S\Gamma C \subseteq C$. Thus, we have $B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A] \subseteq (C] = C$. Hence, B is the smallest bi-Γ-hyperideal of S containing A. Therefore, $B_S(A) = B = (A \cup A\Gamma A \cup A\Gamma S\Gamma A]$. □

Corollary 2. *Let a be an element of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$. Then,*

$$B_S(a) = (a \cup a\Gamma a \cup a\Gamma S\Gamma a].$$

4 MAIN RESULTS

The concepts of regular (resp. intra-regular) ordered Γ-semihypergroups generalize the corresponding concepts of regular (resp. intra-regular) Γ-semihypergroups as each regular (resp. intra-regular) Γ-semihypergroup endowed with the order $\leq := \{(a,b) \mid a = b\}$ is a regular (resp. intra-regular) ordered Γ-semihypergroup. In this section, we introduce the notion of regular ordered Γ-semihypergroups and investigate some related results. We characterize regular ordered Γ-semihypergroups in terms of bi-Γ-hyperideals, left Γ-hyperideals and right Γ-hyperideals of ordered Γ-semihypergroups. In this paper, some well known results of ordered semihypergroups in case of ordered Γ-semihypergroups are examined.

Definition 5. *An ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is called* regular *if for every $a \in S$ there exist $x \in S$, $\alpha,\beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. This is equivalent to saying that $a \in (a\Gamma S\Gamma a]$, for every $a \in S$ or $A \subseteq (A\Gamma S\Gamma A]$, for every $A \subseteq S$.*

Example 3. *Let $S = \{a,b,c,d,e\}$ and $\Gamma = \{\gamma,\beta\}$ be the sets of binary hyperoperations defined as follows.*

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,e\}$	c	$\{c,d\}$	e
b	$\{b,e\}$	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

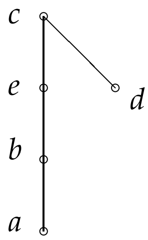
β	a	b	c	d	e
a	$\{b,e\}$	e	c	$\{c,d\}$	e
b	e	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

Then S is a Γ-semihypergroup [42]. We have $(S,\Gamma,\leq)$ is an ordered Γ-semihypergroup where the order relation $\leq$ is defined by:

$$\leq:=\{(a,a),(a,b),(a,c),(a,e),(b,b),(b,c),(b,e),(c,c),(d,c),(d,d),(e,c),(e,e)\}.$$

The covering relation and the figure of S are given by:

$$\prec=\{(a,b),(b,e),(d,c),(e,c)\}.$$



We can easily verify that S is a regular ordered Γ-semihypergroup.

Lemma 7. *Every Γ-hyperideal I of a regular ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is a regular sub Γ-semihypergroup of S.*

Proof. Let $a \in I$. Since S is a regular ordered Γ-semihypergroup, there exist $x \in S$, $\alpha,\beta,\gamma,\delta \in \Gamma$ such that $a \leq a\alpha x\beta a \leq a\alpha x\beta a\gamma x\delta a = a\alpha(x\beta a\gamma x)\delta a$. Since I is a Γ-hyperideal of S, it follows that $x\beta a\gamma x \subseteq S\Gamma I\Gamma S \subseteq I$. Thus, $a \leq t$ for some $t \in a\alpha(x\beta a\gamma x)\delta a \subseteq a\Gamma I\Gamma a$. So, we have $a \in (a\Gamma I\Gamma a]_I$. Therefore, I is a regular sub Γ-semihypergroup of S. □

Theorem 2. *If I and J are regular Γ-hyperideals of an ordered Γ-semihypergroup $(S,\Gamma,\leq)$, then $I\cap J$ is also a regular Γ-hyperideal of S.*

Proof. Let I and J are regular Γ-hyperideals of S. By Lemma 1, $I\cap J$ is a Γ-hyperideal of S. By Lemma 7, I and J are regular sub Γ-semihypergroups of S. Now, let $a \in I\cap J$. Then, $a \leq a\alpha x\beta a$ and $a \leq a\gamma y\delta a$ for some $x,y \in S$ and $\alpha,\beta,\gamma,\delta \in \Gamma$. So, we have $a \leq a\alpha x\beta a \leq (a\alpha x\beta a)\mu s\lambda(a\gamma y\delta a) = a\alpha(x\beta a\mu s\lambda a\gamma y)\delta a$. Since I and J are Γ-hyperideals of S, we obtain $x\beta a\mu s\lambda a\gamma y \subseteq I\cap J$. Thus, we have $a \leq t$ for some $t \in a\alpha(x\beta a\mu s\lambda a\gamma y)\delta a \subseteq a\Gamma(I\cap J)\Gamma a$ which implies that $a \in (a\Gamma(I\cap J)\Gamma a]_I$. Hence, there exists $z \in I\cap J$ such that $a \leq a\alpha z\delta a$. Therefore, $I\cap J$ is a regular sub Γ-semihypergroup of S. □

We now prove the following theorem which is the crucial theorem in the establishment of our main theorems.

Theorem 3. *An ordered Γ-semihypergroup $(S,\Gamma,\leq)$ is regular if and only if for every right Γ-hyperideal R and every left Γ-hyperideal L of S, we have $R\cap L = (R\Gamma L]$.*

Proof. Let R be a right Γ-hyperideal and L a left Γ-hyperideal of S. As $R\Gamma L \subseteq S\Gamma L \subseteq L$ and $R\Gamma L \subseteq R\Gamma S \subseteq R$, we have $R\Gamma L \subseteq R\cap L$. So, $(R\Gamma L] \subseteq (R\cap L] \subseteq (R]\cap(L] \subseteq R\cap L$. Let S be regular; we need to prove that $R\cap L \subseteq (R\Gamma L]$. Since S is regular, we have

$$R\cap L \subseteq ((R\cap L)\Gamma S\Gamma(R\cap L)] \subseteq (R\Gamma S\Gamma(R\cap L)] \subseteq (R\Gamma S\Gamma L] \subseteq (R\Gamma L].$$

Conversely, suppose that $R \cap L = (R\Gamma L]$ for any right Γ-hyperideal R and any left Γ-hyperideal L of S. Let $a \in S$. Since $a \in R_S(a)$ and $a \in L_S(a)$, it follows that $a \in R_S(a) \cap L_S(a)$. By hypothesis, we have that

$$\begin{aligned} a \in (R_S(a)\Gamma L_S(a)] &= ((a \cup a\Gamma S]\Gamma(a \cup S\Gamma a]] \\ &\subseteq (a\Gamma a \cup a\Gamma S\Gamma a \cup a\Gamma S\Gamma S\Gamma a] \subseteq (a\Gamma a \cup a\Gamma S\Gamma a]. \end{aligned}$$

Hence, $a \leq t$ for some $t \in a\Gamma a \cup a\Gamma S\Gamma a$. If $u \in a\Gamma S\Gamma a$, then $a \leq a\alpha x\beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$. Thus, we have $a \in (a\Gamma S\Gamma a]$. Therefore, S is a regular ordered Γ-semihypergroup. If $u \in a\Gamma a$, then $a \leq a\alpha a \leq a\alpha(a\beta a)$. So, we have $a \in (a\Gamma S\Gamma a]$. Therefore, S is regular. □

Now, we obtain the following corollaries.

Corollary 3. *If $(S, \Gamma, \leq)$ is a regular ordered Γ-semihypergroup, then $S = (S\Gamma S]$.*

Corollary 4. *An ordered Γ-semihypergroup S is called fully Γ-hyperidempotent if every Γ-hyperideal of S is idempotent. If S is a regular ordered Γ-semihypergroup, then S is fully Γ-hyperidempotent.*

Theorem 4. *Let $(S, \Gamma, \leq)$ be a regular ordered Γ-semihypergroup. Then, B is a bi-Γ-hyperideal of S if and only if there exists a right Γ-hyperideal R and a left Γ-hyperideal L of S such that $B = (R\Gamma L]$.*

Proof. Let S be a regular ordered Γ-semihypergroup and B a bi-Γ-hyperideal of S. First, we show that $(B\Gamma S]$ is a right Γ-hyperideal of S. Let $y \in S$ and $x \in (B\Gamma S]$. Then, there exist $b \in (B\Gamma S]$, $c \in B$, $s \in S$ and $\alpha \in \Gamma$ such that $x \leq b \leq c\alpha s$. Since S is an ordered Γ-semihypergroup, it follows that $x\beta y \leq b\beta y \leq b \leq (c\alpha s)\beta y \subseteq B\Gamma S$, where $\beta \in \Gamma$. Hence, $x\beta y \subseteq (B\Gamma S]$. If $y \leq x$, then $y \leq x \leq b$, and so $y \in (B\Gamma S]$. Therefore, $(B\Gamma S]$ is a right Γ-hyperideal of S. Similarly, we can prove that $(S\Gamma B]$ is a left Γ-hyperideal of S. Now, we prove that $B = ((B\Gamma S]\Gamma(S\Gamma B]]$. Since S is regular, it follows that $B \subseteq (B\Gamma S\Gamma B]$, for every $B \subseteq S$. Since B is a bi-Γ-hyperideal of S, it follows that $B\Gamma S\Gamma B \subseteq B$. So, we have $(B\Gamma S\Gamma B] \subseteq (B] = B$. Hence, $B = (B\Gamma S\Gamma B]$. By Corollary 3, we have $S = (S\Gamma S]$. Hence,

$$\begin{aligned} B &= (B\Gamma S\Gamma B] = (B\Gamma(S\Gamma S]\Gamma B] = ((B]\Gamma((S\Gamma S]]\Gamma B] = ((B\Gamma S\Gamma S]\Gamma B] \\ &= ((B\Gamma S\Gamma S]\Gamma(B]] = ((B\Gamma S\Gamma S)\Gamma B] = ((B\Gamma S]\Gamma(S\Gamma B]]. \end{aligned}$$

Conversely, suppose that R is a right Γ-hyperideal and L a left Γ-hyperideal of S such that $B = (R\Gamma L]$. We prove that $(R\Gamma L]$ is a bi-Γ-hyperideal of S. We have

$$(R\Gamma L]\Gamma(R\Gamma L] \subseteq ((R\Gamma L)\Gamma(R\Gamma L)] = ((R\Gamma L\Gamma R)\Gamma L] \subseteq ((R\Gamma S\Gamma R)\Gamma L] \subseteq (R\Gamma L].$$

Then, $(R\Gamma L]$ is a sub Γ-semihypergroup of S. Also, we have

$$\begin{aligned} (R\Gamma L]\Gamma S\Gamma(R\Gamma L] &= (R\Gamma L]\Gamma(S]\Gamma(R\Gamma L] \subseteq ((R\Gamma L)\Gamma S]\Gamma(R\Gamma L] \subseteq ((R\Gamma L)\Gamma S\Gamma(R\Gamma L)] \\ &\subseteq (R\Gamma(L\Gamma S)\Gamma R\Gamma L] \subseteq ((R\Gamma S)\Gamma R\Gamma L] \subseteq (R\Gamma R\Gamma L] \subseteq (R\Gamma S\Gamma L] \subseteq (R\Gamma L]. \end{aligned}$$

Now, suppose that $y \in S$ and $x \in (R\Gamma L]$ such that $y \leq x$. Since $x \in (R\Gamma L]$, it follows that $x \leq a$ for some $a \in R\Gamma L$. Since $y \leq x$ and $x \leq a$, we get $y \leq a$. So, we have $y \in (R\Gamma L]$. Therefore, $(R\Gamma L]$ is a bi-Γ-hyperideal of S. □

Theorem 5. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is regular if and only if for every right Γ-hyperideal R, every left Γ-hyperideal L and every bi-Γ-hyperideal B of S, we have $R \cap B \cap L \subseteq (R\Gamma B\Gamma L]$.*

Proof. Let R be right Γ-hyperideal, L a left Γ-hyperideal and B a bi-Γ-hyperideal of S. By hypothesis, we have

$$\begin{aligned} R \cap B \cap L &\subseteq ((R \cap B \cap L)\Gamma S\Gamma(R \cap B \cap L)] \\ &\subseteq ((R \cap B \cap L)\Gamma S\Gamma(R \cap B \cap L)\Gamma S\Gamma(R \cap B \cap L)\Gamma S\Gamma(R \cap B \cap L)] \\ &\subseteq (R\Gamma S\Gamma B\Gamma S\Gamma B\Gamma S\Gamma L] = ((R\Gamma S)\Gamma(B\Gamma S\Gamma B)\Gamma(S\Gamma L)] \subseteq (R\Gamma B\Gamma L]. \end{aligned}$$

Conversely, suppose that $R \cap B \cap L \subseteq (R\Gamma B\Gamma L]$ for every right Γ-hyperideal R, every left Γ-hyperideal L and every bi-Γ-hyperideal B of S. Since S is a bi-Γ-hyperideal of S, we have $R \cap L = R \cap S \cap L \subseteq (R\Gamma S\Gamma L] \subseteq (R\Gamma L]$. By Theorem 3, S is regular. □

Definition 6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. An element $a \in S$ is said to be intra-regular if there exist $x, y \in S$, $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq x\alpha a\beta a\gamma y$. An ordered Γ-semihypergroup S is called intra-regular if all elements of S are intra-regular.*

Equivalent definitions*:*

(1) $a \in (S\Gamma a\Gamma a\Gamma S]$*, for all* $a \in S$*.*

(2) $A \subseteq (S\Gamma A\Gamma A\Gamma S]$*, for all* $A \subseteq S$*.*

Example 4. *Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined as follows.*

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,c\}$	c	$\{d,e\}$	e
b	$\{b,c\}$	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

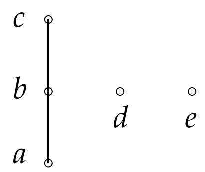
β	a	b	c	d	e
a	$\{b,c\}$	c	c	$\{d,e\}$	e
b	c	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

Then S is a Γ-semihypergroup [41]. We have $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup where the order relation $\leq$ is defined by:

$$\leq := \{(a,a), (a,b), (a,c), (b,b), (b,c), (c,c), (d,d), (e,e)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(a,b), (b,c)\}.$$



Then, by routine calculations, $(S, \Gamma, \leq)$ is intra-regular.

Theorem 6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, S is intra-regular if and only if for every right Γ-hyperideal R and every left Γ-hyperideal L of S, we have*

$$R \cap L \subseteq (L\Gamma R].$$

Proof. Let R be a right Γ-hyperideal and L a left Γ-hyperideal of S. Let S be intra-regular; we need to prove that $R \cap L \subseteq (L\Gamma R]$. Since S is intra-regular, we have

$$R \cap L \subseteq (S\Gamma(R \cap L)\Gamma(R \cap L)\Gamma S] \subseteq (S\Gamma L\Gamma R\Gamma S] \subseteq (L\Gamma R].$$

Conversely, suppose that $R \cap L \subseteq (L\Gamma R]$ for any right Γ-hyperideal R and any left Γ-hyperideal L of S. Let $a \in S$. Since $a \in R_S(a)$ and $a \in L_S(a)$, it follows that $a \in R_S(a) \cap L_S(a)$. By hypothesis, we have

$$\begin{aligned} a \in (L_S(a)\Gamma R_S(a)] &= ((a \cup S\Gamma a]\Gamma(a \cup a\Gamma S]] \\ &\subseteq (a\Gamma a \cup S\Gamma a\Gamma a \cup a\Gamma a\Gamma S \cup S\Gamma a\Gamma a\Gamma S]. \end{aligned}$$

Hence, $a \leq u$ for some $u \in a\Gamma a \cup S\Gamma a\Gamma a \cup a\Gamma a\Gamma S \cup S\Gamma a\Gamma a\Gamma S$. If $u \in S\Gamma a\Gamma a\Gamma S$, then $a \leq x\alpha a\beta a\gamma y$ for some $x, y \in S$, $\alpha, \beta, \gamma \in \Gamma$. Thus, we have $a \in (S\Gamma a\Gamma a\Gamma S]$. Therefore, S is intra-regular. If $u \in a\Gamma a$, then $a \leq a\alpha a \leq a\alpha(a\beta a) \leq a\alpha a\beta a\gamma a$. So, we have $a \in (S\Gamma a\Gamma a\Gamma S]$. Hence, S is intra-regular. If $u \in S\Gamma a\Gamma a$, then $a \leq x\alpha a\beta a \leq x\alpha(x\gamma a\delta a)\beta a$ for some $x \in S$, $\alpha, \beta, \gamma, \delta \in \Gamma$. So, we have $a \leq s\gamma a\delta a\beta a$. Hence, $a \in (S\Gamma a\Gamma a\Gamma S]$. If $u \in a\Gamma a\Gamma S$, in a similar way, we obtain $a \in (S\Gamma a\Gamma a\Gamma S]$. Therefore, S is intra-regular. □

Corollary 5. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, the following statements are equivalent:*

(1) *S is regular and intra-regular.*

(2) $(R\Gamma L] = R \cap L \subseteq (L\Gamma R]$ *for every right Γ-hyperideal R and every left Γ-hyperideal L of S.*

Proof. It is immediately followed by Theorem 3 and Theorem 6. □

Theorem 7. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is intra-regular if and only if for every right Γ-hyperideal R, every left Γ-hyperideal L and every bi-Γ-hyperideal B of S, we have $R \cap B \cap L \subseteq (L\Gamma B\Gamma R]$.*

Proof. The proof is similar to the proof of Theorem 5. □

By routine verification we have the following theorem.

Theorem 8. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is both regular and intra-regular if and only if for every right Γ-hyperideal R, every left Γ-hyperideal L and every bi-Γ-hyperideal B of S, we have $R \cap B \cap L \subseteq (B\Gamma R\Gamma L]$.*

Our main aim in the following is to introduce and study the notion of simple ordered Γ-semihypergroups. Also, we characterize this type of ordered Γ-semihypergroups in terms of Γ-hyperideals.

Definition 7. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is said to be left (resp. right) simple if S has no proper left (resp. right) Γ-hyperideals. S is called a simple ordered Γ-semihypergroup if it does not contain proper Γ-hyperideals, i.e., for any Γ-hyperideal $I \neq \varnothing$ of S, we have $I = S$.*

Lemma 8. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, the following assertions hold:*

(1) *S is left simple if and only if $(S\Gamma a] = S$, for all $a \in S$.*

(2) *S is right simple if and only if $(a\Gamma S] = S$, for all $a \in S$.*

Proof. (1): Suppose that S is a left simple ordered Γ-semihypergroup and $a \in S$. We have

$$S\Gamma(S\Gamma a] = (S]\Gamma(S\Gamma a] \subseteq (S\Gamma(S\Gamma a)] = ((S\Gamma S)\Gamma a)] \subseteq (S\Gamma a].$$

Now, suppose that $x \in (S\Gamma a]$ and $y \in S$ such that $y \leq x$. Since $x \in (S\Gamma a]$, it follows that $x \leq u$ for some $u \in S\Gamma a$. Since $y \leq x$ and $x \leq u$, we get $y \leq u$. So, we have $y \in (S\Gamma a]$. Hence, $(S\Gamma a]$ is a left hyperideal of S. Since S is a left simple ordered Γ-semihypergroup, we have $(S\Gamma a] = S$.

Conversely, suppose that $(S\Gamma a] = S$ for all $a \in S$. Let L be a left hyperideal of S and $x \in L$. By assumption, we have $(S\Gamma x] = S$. If $s \in S$, then $s \in (S\Gamma x]$. So, $s \leq v$ for some $v \in S\Gamma x \subseteq L$. Since L is a left Γ-hyperideal of S, we have $s \in L$, and so $L = S$. Therefore, S is a left simple ordered Γ-semihypergroup.

(2): The proof is similar to the proof of (1). □

Theorem 9. *If $(S, \Gamma, \leq)$ is a left (right) simple ordered Γ-semihypergroup, then S is a simple ordered Γ-semihypergroup.*

Proof. It is straightforward. □

Theorem 10. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is left and right simple if and only if for every $a \in S$, we have $(S\Gamma a\Gamma S] = S$.*

Proof. Let S be left and right simple and $a \in S$. By Lemma 8, $a \in (S\Gamma a]$ and $a \in (a\Gamma S]$. We have

$$a \in (a\Gamma S] \subseteq ((S\Gamma a]\Gamma S] \subseteq (S\Gamma a\Gamma S],$$

and so $S \subseteq (S\Gamma a\Gamma S]$. Thus, $(S\Gamma a\Gamma S] = S$.

Conversely, suppose that $(S\Gamma a\Gamma S] = S$ for every $a \in S$. Let I be a Γ-hyperideal of S such that $I \subsetneq S$. Let $x \in I$. By assumption, we have $s \leq s\mu x\lambda s$ for every $s \in S$ and $\mu, \lambda \in \Gamma$. We have

$$s\mu x\lambda s \subseteq S\Gamma I\Gamma S \subseteq (S\Gamma I\Gamma S] \subseteq (I] = I.$$

Then, $S \subseteq I$, a contradiction. Therefore, S has no proper left and right Γ-hyperideals. This completes the proof. □

In what follows, we characterize simple ordered Γ-semihypergroups in terms of bi-Γ-hyperideals.

Theorem 11. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is left and right simple if and only if S does not contain proper bi-Γ-hyperideals.*

Proof. Suppose that S is a left and right simple ordered Γ-semihypergroup and B a bi-Γ-hyperideal of S. We claim that $S \subseteq B$. Consider $s \in S$ and $x \in B$. Since S is left simple, we get $S = (x \cup S\Gamma x]$. We can consider the following two cases:
Case 1. If $s \leq x$, then we have $s \in B$.
Case 2. Let $s \in (u\gamma x]$ for some $u \in S$ and $\gamma \in \Gamma$. By hypothesis, S is a right simple ordered Γ-semihypergroup. Then, we have $S = (x \cup x\Gamma S]$. Since $u \in S$, we have $u \leq x$ or $u \in (x\delta w]$ for some $w \in S$ and $\delta \in \Gamma$. By Lemma 8, we have $S = (x\Gamma S] = (S\Gamma x]$, and so $x \in (x\Gamma S] = (x\Gamma(S\Gamma x]] \subseteq (x\Gamma S\Gamma x]$. Then, S is a regular ordered Γ-semihypergroup. Thus, there exists $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x \in (x\alpha a\beta x]$. If $u \leq x$, then we have

$$(u\gamma x] \subseteq (x\gamma x] \subseteq (x\gamma x\alpha a\beta x] \subseteq (B\Gamma S\Gamma B] \subseteq B,$$

and so $s \in B$. If $u \in (x\delta w]$, then we have

$$(u\gamma x] \subseteq (x\delta w\gamma x] \subseteq (B\Gamma S\Gamma B] \subseteq B,$$

and so $s \in B$. Therefore, $S \subseteq B$.

Conversely, suppose that S does not contain proper bi-Γ-hyperideals. Let L be a left Γ-hyperideal of S. Then, L is a bi-Γ-hyperideal of S. By assumption, we have $S = L$. Therefore, S is a left simple ordered Γ-semihypergroup. Similarly, we can show that S is a right simple ordered Γ-semihypergroup. □

In the following, we study some properties of bi-Γ-hyperideals and minimal bi-Γ-hyperideals in ordered Γ-semihypergroups.

Definition 8. *An ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is said to be B-simple if S does not contain any proper bi-Γ-hyperideals. A bi-Γ-hyperideal C of S is called a minimal bi-Γ-hyperideal of S if C does not properly contain any bi-Γ-hyperideal of S.*

Theorem 12. *Let B be a bi-Γ-hyperideal of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, $(u\Gamma B\Gamma v]$ is a bi-Γ-hyperideal of S for every $u, v \in S$. In particular, $(u\Gamma S\Gamma v]$ is a bi-Γ-hyperideal of S for every $u, v \in S$.*

Proof. The proof is similar to the proof of Theorem 2.2 in [8]. □

Corollary 6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. Then, S is B-simple if and only if $(u\Gamma S\Gamma u] = S$ for all $u \in S$.*

Proof. The necessity is obvious. For the sufficiency, let $(u\Gamma S\Gamma u] = S$ for all $u \in S$. We have

$$(u\Gamma S\Gamma u] \subseteq (S\Gamma u] \subseteq S \text{ and } (u\Gamma S\Gamma u] \subseteq (u\Gamma S] \subseteq S.$$

By assumption, we have $(S\Gamma u] = S$ and $(u\Gamma S] = S$ for all $u \in S$. Now, let B is a bi-Γ-hyperideal of S and $b \in B$. Then, $(S\Gamma b] = S = (b\Gamma S]$. So, we have

$$S = (b\Gamma S] = (b\Gamma(b\Gamma S]] \subseteq (b\Gamma S\Gamma b] \subseteq (B\Gamma S\Gamma B] \subseteq (B] \subseteq B.$$

This completes the proof. □

Corollary 7. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup. If C is a minimal bi-Γ-hyperideal of S and B a bi-Γ-hyperideal of S, then $C = (c\Gamma B\Gamma d]$ for every $c, d \in C$.*

Proof. By Theorem 12, $(c\Gamma B\Gamma d]$ is a bi-Γ-hyperideal of S. Since C is a minimal bi-Γ-hyperideal of S and $(c\Gamma B\Gamma d] \subseteq (C\Gamma B\Gamma C] \subseteq (C\Gamma S\Gamma C] \subseteq (C] \subseteq C$, we obtain $C = (c\Gamma B\Gamma d]$. □

At the end of the paper, we prove the following theorem.

Theorem 13. *Let B be a bi-Γ-hyperideal of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then, B is a minimal bi-Γ-hyperideal of S if and only if B is B-simple.*

Proof. Let B be a minimal bi-Γ-hyperideal of S. Then, B is a sub Γ-semihypergroup of S. Now, let C be a bi-Γ-hyperideal of B. Then, $C\Gamma B\Gamma C \subseteq C$. Put $K = (C\Gamma B\Gamma C]_C$. Then, $\varnothing \neq K \subseteq C \subseteq B$. Now, we prove that K is a bi-Γ-hyperideal of S. Let $k_1, k_2 \in K$, $x \in S$ and $\gamma, \delta \in \Gamma$. Then, $k_1 \leq c_1\alpha_1 b_1\beta_1 c_1'$ and $k_2 \leq c_2\alpha_2 b_2\beta_2 c_2'$ for some $c_1, c_1', c_2, c_2' \in C$, $b_1, b_2 \in B$ and $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \Gamma$. So, we have

$$k_1\gamma k_2 \leq c_1\alpha_1(b_1\beta_1 c_1'\gamma c_2\alpha_2 b_2)\beta_2 c_2'$$

and

$$k_1\gamma x\delta k_2 \leq c_1\alpha_1(b_1\beta_1 c_1'\gamma x\delta c_2\alpha_2 b_2)\beta_2 c_2'.$$

Since $b_1\beta_1 c_1'\gamma c_2\alpha_2 b_2 \subseteq B\Gamma S\Gamma B \subseteq B$, it follows that $k_1\gamma k_2 \subseteq K\Gamma K \subseteq C\Gamma C \subseteq C$. So, $k_1\gamma k_2 \subseteq (C\Gamma B\Gamma C]_C = K$. Hence, K is a sub Γ-semihypergroup of S. Since $b_1\beta_1 c_1'\gamma x\delta c_2\alpha_2 b_2 \subseteq B\Gamma S\Gamma B \subseteq B$, we get

$$c_1\alpha_1(b_1\beta_1 c_1'\gamma x\delta c_2\alpha_2 b_2)\beta_2 c_2' \subseteq C\Gamma B\Gamma C \subseteq C.$$

Since C is a bi-Γ-hyperideal of B and $k_1\gamma x\delta k_2 \subseteq K\Gamma S\Gamma K \subseteq B\Gamma S\Gamma B \subseteq B$, we obtain $k_1\gamma x\delta k_2 \subseteq C$. So, we have $k_1\gamma x\delta k_2 \subseteq (C\Gamma B\Gamma C]_C = K$. Therefore, $K\Gamma S\Gamma K \subseteq K$. Now, let $y \in (K]$. Then, $y \leq k$ for some $k \in K$. Since $k \in K$, there exist $c, c' \in C$, $b \in B$ and $\mu, \lambda \in \Gamma$ such that $k \leq c\mu b\lambda c'$. Since $c\mu b\lambda c' \subseteq C\Gamma B\Gamma C \subseteq C \subseteq B$ and B is a bi-Γ-hyperideal of S, we get $k \in B$. Since B is a bi-Γ-hyperideal of S, we have $y \in B$. So, $y \leq z$ for some $z \in c\mu b\lambda c' \subseteq C\Gamma B\Gamma C \subseteq C$. Since C is a bi-Γ-hyperideal of B, we have $y \in C$. So, we have $y \in (C\Gamma B\Gamma C]_C = K$. Therefore, K is a bi-Γ-hyperideal of S. Since B is a minimal bi-Γ-hyperideal of S, it follows that $K = B$. So, we have $C = B$. Therefore, B is B-simple.

Conversely, assume that B is B-simple. Let C be a bi-Γ-hyperideal of S such that $C \subseteq B$. Then, $B \cap C \neq \varnothing$. Let $c \in B \cap C$. By Theorem 12, $(c\Gamma B\Gamma c]$ is a bi-Γ-hyperideal of B. Since B is B-simple, we obtain $(c\Gamma B\Gamma c] = B$. Now, we have

$$B = (c\Gamma B\Gamma c] \subseteq (C\Gamma B\Gamma C] \subseteq (C\Gamma S\Gamma C] \subseteq (C] = C.$$

Hence, $C = B$. Therefore, B is a minimal bi-Γ-hyperideal of S. □

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Поняття Γ-напівгіпергруп є узагальненням напівгруп, узагальненням напівгіпергруп і узагальненням Γ-напівгруп. У даній роботі досліджується поняття бі-Γ-гіперідалів у впорядкованих Γ-напівгіпергрупах і досліджуються деякі властивості цих бі-Γ-гіперідеалів. Також ми визначаємо і використовуємо поняття регулярно впорядкованих Γ-напівгіпергруп для вивчення деяких класичних результатів і властивостей у впорядкованих Γ-напівгіпергрупах.

Ключові слова і фрази: упорядковані Γ-напівгіпергрупи, Γ-гіперідеали, bi-Γ-гіперідеали.