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ALGEBRAIC BASIS OF THE ALGEBRA OF BLOCK-SYMMETRIC POLYNOMIALS ON $\ell_1 \oplus \ell_\infty$

We concider so called block-symmetric polynomials on sequence spaces $\ell_1 \oplus \ell_\infty, \ell_1 \oplus c, \ell_1 \oplus c_0$, that is, polynomials which are symmetric with respect to permutations of elements of the sequences. It is proved that every continuous block-symmetric polynomials on $\ell_1 \oplus \ell_\infty$ can be uniquelly represented as an algebraic combination of some special block-symmetric polynomials, which form an algebraic basis. It is interesting to note that the algebra of block-symmetric polynomials is infinite-generated while ℓ_∞ admits no symmetric polynomials. Algebraic bases of the algebras of block-symmetric polynomials on $\ell_1 \oplus \ell_\infty$ and $\ell_1 \oplus c_0$ are described.

Key words and phrases: symmetric polynomials, block-symmetric polynomials, algebraic basis, topological algebra.

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1 INRODUCTION

Algebras of polynomials and analytic functions on a Banach space X which are invariant with respect to a group or semigroup of linear operators acting on X were studied by many authors (see e.g. [1, 3–5, 9]). In oder to study spectra of such algebras it is impotant to figure out with their algebraic bases (if exist). Let S_∞ be the group of all permutations of the set of natural numbers $\mathbb{N}$. That is, S_∞ consists of all bijections of $\mathbb{N}$ to itself. Let S^0_∞ be the subgroup in S_∞ of all finite permutations. If X is a sequence Banach space and for each $x = (x_1, x_2, \ldots, x_n, \ldots) \in X$, $\sigma(x)\colon = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots) \in X$, $\sigma \in S_\infty$, then we can concider functions which are invariants with respect to the operators $\sigma(x)$. A function $f : X \to \mathbb{C}$ is called *symetric* if $f(\sigma(x)) = f(x)$ for every $x \in X$ and $\sigma \in S_\infty$. If it is true for all $\sigma \in S^0_\infty$ then f is called *finitely symmetric*. In [11] Nemirovskii and Semenov described algebraic bases of algebra of continuous symmetric polynomials on real spaces ℓ_p, where $1 \le p < \infty$. Their results were generalized by Gonzalez et al. [7] for real separable rearrangement-invariant sequence spaces. Also, in [7] it is proved that for $\ell_p, 1 \le p < \infty$, finitely symmetric polynomials are symmetric and c_0 does not admit finitely symmetric polynomials. In [8] it is proved that there are no symmetric polynomials on ℓ_∞ but we have a lot of finitely symmetric polynomials. It is not difficult to check that every symmetric (and finitely symmetric) polynomial on c can be generated by the following one

$$L(x) = \lim_{n\to\infty} x_n.$$

In [9, 10] were concidered *block-symmetric polynomials*, wich also are called MacMahon Polynomials on Banach spaces. The block-symmetric polynomials can be defined by the following

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way. Let $X_1, \ldots, X_m$ be sequence spaces. Then every $x \in X_1 \times \ldots \times X_m$ can be represented by $x = (x^1, \ldots, x^m)$, where $x^j \in X_j$. For any $\sigma \in S_\infty$ we can define $\sigma(x) = (\sigma(x^1), \ldots, \sigma(x^m))$ and a polynomial $P : X_1 \times \ldots X_m$ is block-symmetric if $P(\sigma(x)) = P(x)$ for every $\sigma \in S_\infty$. In [10] algebra of block-symetric analytic functions on $\ell_1 \times \ell_1$ is investigated. In [9] constructed an algebraic basis of block-symmetric polynomials on $\underbrace{\ell_p \times \ldots \times \ell_p}_{n} \simeq \ell_p(\mathbb{C}^n)$. In this paper we construct an algebraic basis on the algebra of all block-symmetric polynomials on $\ell_1 \times \ell_\infty$. It is interesting to note that the algebra of block-symmetric polynomials is infinite-generated while ℓ_∞ admits no symmetric polynomials. Also, we concider block-symmetric polynomials on $\ell_1 \times c_0$ and $\ell_1 \times c$.

2 Main Results

Let us denote by $\ell_1 \oplus \ell_\infty$ the space with elements $\begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right)$, where $(x_1, x_2, \ldots, x_n, \ldots) \in \ell_1$, $(y_1, y_2, \ldots, y_n, \ldots) \in \ell_\infty$. The space $\ell_1 \oplus \ell_\infty$ with norm

$$||(x,y)||_{\ell_1 \oplus \ell_\infty} = \sum_{i=1}^{\infty} |x_i| + \sup_{i \geq 1} |y_i|$$

is a Banach space.

A polynomial P on the space $\ell_1 \oplus \ell_\infty$ is called block-symmetric (or vector-symmetric) if

$$P\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right) = P\left(\begin{pmatrix} x_{\sigma(1)} \\ y_{\sigma(1)} \end{pmatrix}, \ldots, \begin{pmatrix} x_{\sigma(m)} \\ y_{\sigma(m)} \end{pmatrix}, \ldots \right),$$

for every permutation σ on the set of natural numbers $\mathbb{N}$, where $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbb{C}^2$.

Let us denote by $\mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$ the algebra of block-symmetric polynomials on $\ell_1 \oplus \ell_\infty$; by $\mathcal{H}_{bvs}(\ell_1 \oplus \ell_\infty)$ the algebra of block-symmetric analytic functions of bounded type on $\ell_1 \oplus \ell_\infty$.

In [9] it was proved that polynomials $H^{k_1, \ldots, k_n}(x) = \sum\limits_{j=1}^{\infty} \prod\limits_{\substack{s=1 \\ k_s > 0}}^{n} (x_j^s)^{k_s}$, where $x = (x_1, x_2, \ldots) \in \ell_1(\mathbb{C}^n)$, $x_j = (x_j^1, \ldots, x_j^n) \in \mathbb{C}^n$ form an algebraic basis of the algebra $\mathcal{P}_s(\ell_1(\mathbb{C}^n))$.

For a multi-index $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n$ let $|k| = k_1 + k_2 + \ldots + k_n$. For an arbitrary nonempty finite set $M \in \mathbb{Z}_+^n$ let us define a mapping $\pi_M : c_{00}(\mathbb{C}^n) \to \mathbb{C}^{|M|}$, where $|M|$ is the cardinality of M, by

$$\pi_M(x) = (H^{k_1, \ldots, k_n}(x))_{(k_1, \ldots, k_n) \in M}.$$

In [9] it was proved the following theorem.

Theorem 1 ([9]). *Let M be a finite nonempty subset of $\mathbb{Z}_+^n$ such that $|k| \geq 1$ for every $k \in M$.*

1. *There exists $m \in \mathbb{N}$, such that for every $\xi = (\xi_{(k_1, \ldots, k_n)})_{(k_1, \ldots, k_n) \in M} \in \mathbb{C}^{|M|}$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\pi_M(x_\xi) = \xi$, where $c_{00}^{(m)}(\mathbb{C}^n)$ is the space of all sequences $x = (x_1, \ldots, x_m, 0, \ldots)$, $x_1, \ldots, x_m \in \mathbb{C}^n$;*

2. *There exists a constant $\rho_M > 0$ such that if $||\xi||_\infty < 1$, then $||x_\xi||_p \leq \rho_M$ for every $p \in [1, +\infty)$, where $||\xi||_\infty = \max\limits_{k \in M} |\xi_k|$.*

Let us denote by $(\ell_1 \oplus \ell_\infty)^{(m)}$ the space of all sequences

$$\begin{pmatrix} x \\ y \end{pmatrix}_m = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ldots \right),$$

where $(x_1, \ldots, x_m, 0 \ldots) \in \ell_1$, $(y_1, \ldots, y_m, 0 \ldots) \in \ell_\infty$. Clearly, that $c_{00}^{(m)}(\mathbb{C}^n) = (\ell_1 \oplus \ell_\infty)^{(m)}$.

For an arbitrary nonempty finite set $M \in \mathbb{Z}_+^2$ let us define a mapping $\pi_M : \ell_1 \oplus \ell_\infty \longrightarrow \mathbb{C}^{|M|}$ by

$$\pi_M((x,y)) = (H^{k_1,k_2}(x,y))_{(k_1,k_2)\in M}.$$

Corollary 1. *Let M be a finite nonempty subset of $\mathbb{Z}_+^2$ such that $k_1 + k_2 \geq 1$ for every $(k_1, k_2) \in M$.*

1. *There exists $m \in \mathbb{N}$, such that for every $\xi = (\xi_{(k_1,k_2)})_{(k_1,k_2)\in M} \in \mathbb{C}^{|M|}$ there exists $(x,y)_\xi \in (\ell_1 \oplus \ell_\infty)^{(m)}$ such that $\pi_M((x,y)_\xi) = \xi$;*
2. *There exists a constant $\rho_M > 0$ such that if $||\xi||_\infty < 1$, then $||(x,y)_\xi||_{\ell_1\oplus\ell_\infty} \leq \rho_M$.*

For elements $\begin{pmatrix} x \\ y \end{pmatrix}_m, \begin{pmatrix} z \\ t \end{pmatrix}_m \in \ell_1 \oplus \ell_\infty$, let

$$\begin{pmatrix} x \\ y \end{pmatrix}_m \oplus \begin{pmatrix} z \\ t \end{pmatrix}_m = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ldots \right).$$

For $(x,y)^1, (x,y)^2, \ldots, (x,y)^r \in \ell_1 \oplus \ell_\infty$, let

$$\bigoplus_{j=1}^r (x,y)^j = (x,y)^1 \oplus (x,y)^2 \oplus \ldots \oplus (x,y)^r.$$

Obviously that

$$\left|\left| \bigoplus_{j=1}^r (x,y)^j \right|\right|_{\ell_1\oplus\ell_\infty} \leq \sum_{j=1}^r \left|\left| (x,y)^j \right|\right|_{\ell_1\oplus\ell_\infty}.$$

Also note that for every $(k_1, k_2) \in \mathbb{Z}_+^2$, such that $k_1 + k_2 \geq 1$,

$$H^{k_1,k_2}\left(\bigoplus_{j=1}^r (x,y)^j \right) = \sum_{j=1}^r H^{k_1,k_2}((x,y)^j). \tag{1}$$

For $N \in \mathbb{N}$ let M_N be a finite nonempty subset $\mathbb{Z}_+^2$ such that $1 \leq k_1 + k_2 \leq N$ for every $(k_1, k_2) \in M_N$.

By Corollary 1, for $M = M_N$ there exists $\rho = \rho_M$, such that $\pi_{M_N}(V_\rho)$ contains the open unit ball of the space $\mathbb{C}^{|M|}$ with norm $||\xi||_\infty$, where

$$V_\rho = \left\{ (x,y) \in \ell_1 \oplus \ell_\infty : ||(x,y)||_{\ell_1\oplus\ell_\infty} \leq \rho \right\}.$$

Proposition 1. *Let $q(\xi_{(l_1,l_2)})_{(l_1,l_2)\in M_N}$ be a polynomial on $\mathbb{C}^{|M_N|}$. If q is bounded on $\pi_M(V_\rho)$, then q does not depend on $\xi_{(0,k)}$, $k \in \mathbb{N}$.*

Proof. Let $(0,k) \in \mathbb{Z}_+^2, k \in \mathbb{N}$. Let $K = \pi_{M_N}(V_\rho)$, $K_1 = \pi_{M_N\setminus\{(0,k)\}}(V_\rho)$ and $\eta : K \to K_1$ be an orthogonal projection, defined by

$$\eta : (\xi_{(l_1,l_2)})_{(l_1,l_2)\in M_N} \mapsto (\xi_{(l_1,l_2)})_{(l_1,l_2)\in M_N\setminus\{(0,k)\}}.$$

Let us show that for every ball

$$B(u,r) = \left\{\xi \in \mathbb{C}^{|M_N\setminus\{(0,k)\}|} : ||\xi - u||_\infty < r\right\}$$

centered at $u = (u_{(l_1,l_2)})_{(l_1,l_2)\in M_N\setminus\{(0,k)\}} \in \mathbb{C}^{|M_N\setminus\{(0,k)\}|}$ and of radius $r > 0$ such that $B(u,r) \subset \pi_{M_N\setminus\{(0,k)\}}(V_\rho)$, the set $\eta^{-1}(B(u,r))$ is unbounded. Since $u \in \pi_{M_N\setminus\{(0,k)\}}(V_\rho)$, there exists $(x,y)_u \in V_\rho$ such that $\pi_{M_N}((x,y)_u) = u$ by Corollary 1. For $m \in \mathbb{N}$, we set $(x,y)_m = \oplus_{j=1}^m \left(\frac{1}{j^{\frac{1}{|k|}}}\right) a_k$, where $a_k = \frac{1}{k^{\frac{1}{k}}}\left(\begin{pmatrix} 0 \\ \alpha_{k,0}\end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \alpha_{k,k-1}\end{pmatrix}, \begin{pmatrix} 0 \\ 0\end{pmatrix} \ldots\right)$, $\alpha_{m,j} = (\sqrt[m]{-1})_j$, $0 \le j \le m-1$.

Choose ε such that

$$0 < \varepsilon < \min\left\{1, \frac{\rho - ||(x,y)_u||_{\ell_1\oplus\ell_\infty}}{||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k})}, \frac{r}{||a_k||_1^N\zeta(s-1+\frac{1}{k})}\right\},$$

where $\zeta(\cdot)$ is the Riemann zeta-function.

Let $(x,y)_{m,\varepsilon} = (\varepsilon(x,y)_m) \oplus (x,y)_u$. Let us show that $(x,y)_{m,\varepsilon} \in V_\rho$.

$$||(x,y)_m||_{\ell_1\oplus\ell_\infty} = \sum_{j=1}^m \left|\left|\frac{1}{j^{\frac{1}{k}}}a_k\right|\right|_{\ell_1\oplus\ell_\infty} = \sum_{j=1}^m \frac{1}{j^{\frac{1}{k}}}||a_k||_{\ell_1\oplus\ell_\infty} = ||a_k||_{\ell_1\oplus\ell_\infty}\sum_{j=1}^m \frac{1}{j^{\frac{1}{k}}} < ||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k}).$$

Therefore, $||(x,y)_m||_{\ell_1\oplus\ell_\infty} < ||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k})$. Then

$$||(x,y)_{m,\varepsilon}||_{\ell_1\oplus\ell_\infty} \le \varepsilon||(x,y)_m||_{\ell_1\oplus\ell_\infty} + ||(x,y)_u||_{\ell_1\oplus\ell_\infty} < ||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k}) + ||(x,y)_u||_{\ell_1\oplus\ell_\infty}.$$

Since $\varepsilon < \frac{\rho - ||(x,y)_u||_{\ell_1\oplus\ell_\infty}}{||a_k||_{\ell_1\oplus\ell_\infty}\zeta(\frac{1}{k})}$, it follows that $||(x,y)_{m,\varepsilon}||_{\ell_1\oplus\ell_\infty} < \rho$. Hence, $(x,y)_{m,\varepsilon} \in V_\rho$.

Note that for arbitrary $(l_1,l_2) \in \mathbb{Z}_+^2$ such that $l_1 + l_2 \ge 1$, by equality (1),

$$H^{l_1,l_2}((x,y)_m) = \sum_{j=1}^m \frac{1}{j^{\frac{l_1+l_2}{k}}} H^{l_1,l_2}(a_k) = H^{l_1,l_2}(a_k)\sum_{j=1}^m \frac{1}{j^{\frac{l_1+l_2}{k}}},$$

$$\begin{aligned} H^{l_1,l_2}((x,y)_{m,\varepsilon}) &= \varepsilon^{l_1+l_2}H^{l_1,l_2}((x,y)_m) + H^{l_1,l_2}((x,y)_u) \\ &= \varepsilon^{l_1+l_2}H^{l_1,l_2}(a_k)\sum_{j=1}^m \frac{1}{j^{\frac{l_1+l_2}{k}}} + H^{l_1,l_2}((x,y)_u). \end{aligned} \tag{2}$$

Let us show that $\pi_{M_N\setminus\{(0,k)\}}((x,y)_{m,\varepsilon}) \in B(u,r)$. For $(l_1,l_2) \in M_N \setminus \{(0,k)\}$, such that $l_2 \not\equiv 0 \mod k$, $H^{l_1,l_2}(a_k) = 0$ (see [9, Prop. 3]) and therefore, by (2), $H^{l_1,l_2}((x,y)_u) = u_{(l_1,l_2)}$.

Let $(l_1,l_2) \in M_N \setminus \{(0,k)\}$ be such that $l_1 = 0$ and $l_2 \equiv 0 \mod k$. Then $l = l_2 = s\cdot k$, $s \ge 1$, $s \in \mathbb{N}$. Hence

$$\begin{aligned} \left|H^{0,l}((x,y)_{m,\varepsilon}) - u_{(0,l)}\right| &< \varepsilon^l|H^{0,l}(a_k)|\sum_{j=1}^m \frac{1}{j^{\frac{l}{k}}} < \varepsilon^l|H^{0,l}(a_k)|\sum_{j=1}^m \frac{1}{j^{s-1+\frac{1}{k}}} \\ &< \varepsilon^l|H^{0,l}(a_k)|\zeta(s-1+\frac{1}{k}). \end{aligned}$$

Since $||H^{0,l}|| \le 1$ (see [9, Prop. 2]), $|H^{0,l}(a_k)| \le ||a_k||_1^l$. Since $\varepsilon < 1$, and $\varepsilon^l < \varepsilon$, so

$$\varepsilon^l |H^{0,l}(a_k)| \zeta(s-1+\frac{1}{k}) < \varepsilon ||a_k||_1^l \zeta(s-1+\frac{1}{k}).$$

From the inequality $\varepsilon < \frac{r}{||a_k||_1^N \zeta(s-1+\frac{1}{k})}$, it follows that $\left| H^{0,l}((x,y)_{m,\varepsilon}) - u_{(0,l)} \right| < r$ and therefore $\pi_{M_N \setminus \{(0,k)\}}((x,y)_{m,\varepsilon}) \in B(u,r)$.

By [9, Prop. 3], $H^{0,k}(a_k) = 1$. Then

$$H^{0,k}((x,y)_{m,\varepsilon}) = \varepsilon^k \sum_{j=1}^{m} \frac{1}{j} + H^{(0,k)}((x,y)_u) \longrightarrow \infty$$

as $m \to \infty$. Hence, $\eta^{-1}(B(u,r))$ is unbounded. By [9, Lemma 11], q does not depend on $\xi_{(0,k)}$. □

Theorem 2. *Polynomials*

$$H^{k_1,k_2}(x,y) = \sum_{i=1}^{\infty} x_i^{k_1} y_i^{k_2},$$

form an algebraic basis of the algebra $\mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$, where $k_1, k_2 \in \mathbb{N}, k_1 \ge 1, k_2 \ge 0$.

Proof. In [9] it was proved that polynomials $H^{k_1,k_2}(x,y) = \sum\limits_{i=1}^{\infty} x_i^{k_1} y_i^{k_2}$, where $k_1, k_2 \in \mathbb{N}$, $k_1 \ge 0$, $k_2 \ge 0$ form an algebraic basis of the algebra $\mathcal{P}_{vs}(\ell_1 \oplus \ell_1)$. Thus they are algebraically independed. Let us show that $H^{k_1,k_2}(x,y) = \sum\limits_{i=1}^{\infty} x_i^{k_1} y_i^{k_2}$, where $k_1, k_2 \in \mathbb{N}, k_1 \ge 1, k_2 \ge 0$ are algebraically independed on $\ell_1 \oplus \ell_\infty$. Suppose the opposite. Then there exists $Q \ne 0$ such that $Q(H^{1,0}(x,y), H^{2,0}(x,y), H^{1,1}(x,y), \ldots, H^{k_1,k_2}(x,y)) = 0$. Let Q_0 be the restriction of Q on $\ell_1 \oplus \ell_1$. Then $Q_0(H^{1,0}(x,y), H^{2,0}(x,y), H^{1,1}(x,y), \ldots, H^{k_1,k_2}(x,y)) = 0$, where $Q_0 \ne 0$. But it cntradicts algebraically independed of polynomials H^{k_1,k_2} on $\ell_1 \oplus \ell_1$, where $k_1, k_2 \in \mathbb{N}$, $k_1 \ge 0$, $k_2 \ge 0$. So, polynomials H^{k_1,k_2}, $k_1, k_2 \in \mathbb{N}, k_1 \ge 1, k_2 \ge 0$ are algebraically independed.

Let us prove that $H^{k_1,k_2}(x,y)$ are continuous on $\ell_1 \oplus \ell_\infty$. Indead,

$$\left| H^{k_1,k_2}(x,y) \right| = \left| \sum_{i=1}^{\infty} x_i^{k_1} y_i^{k_2} \right| \le \sum_{i=1}^{\infty} |x_i|^{k_1} |y_i|^{k_1}.$$

Since $||(x,y)||_{\ell_1 \oplus \ell_\infty} = \sum\limits_{i=1}^{\infty} |x_i| + \sup\limits_{i \ge 1} |y_i| \le 1$ then $\sum\limits_{i=1}^{\infty} |x_i| \le 1$ and $\sup\limits_{i \ge 1} |y_i| \le 1$.

Moreover $\sum\limits_{i=1}^{\infty} |x_i|^{k_1} |y_i|^{k_2} \le \sum\limits_{i=1}^{\infty} |x_i|^{k_1} \cdot \left(\sup\limits_{i \ge 1} |y_i| \right)^{k_2}$.

Hence

$$\left|\left| H^{k_1,k_2} \right|\right| = \sup_{||(x,y)|| \le 1} \left| H^{k_1,k_2}(x,y) \right| \le \sup_{||(x,y)|| \le 1} \left(\sum_{i=1}^{\infty} |x_i|^{k_1} \cdot \left(\sup_{i \ge 1} |y_i| \right)^{k_2} \right) \le 1.$$

Therefore $H^{k_1,k_2}(x,y)$ are bounded and so continuous on $\ell_1 \oplus \ell_\infty$.

Let us prove that every continuous block-symmetric polynomial $P \in \mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$ can be represented as an algebraic combination of polynomials $H^{k_1,k_2}(x,y)$, $k_1, k_2 \in \mathbb{N}, k_1 \ge 1, k_2 \ge 0$.

Let $\tilde{P}$ be restriction of P on $\ell_1 \oplus \ell_1$. For polynomial $\tilde{P}$ there exists a unique polynomial $q : \mathbb{C}^{M_N} \to \mathbb{C}$ such that $\tilde{P} = q \circ \pi_{M_N}$. Since $\tilde{P}$ is continuous, $\tilde{P}$ is bounded on V_ρ, so q is bounded on $\pi_{M_N}(V_\rho)$.

By Proposition 1, a polynomial q does not depend on $\xi_{(0,k)}, k \in \mathbb{N}$.

Since polynomials H^{k_1,k_2}, where $(k_1,k_2) \in M_N \setminus \{(0,k)\}$ are well-defined and continuous on $\ell_1 \oplus \ell_\infty$, then $P = q \circ \pi_{M_N \setminus \{(0,k)\}}$.

Therefore $H^{k_1,k_2}(x,y)$, $k_1,k_2 \in \mathbb{N}, k_1 \geq 1, k_2 \geq 0$ form an algebraic basis of the algebra $\mathcal{P}_{vs}(\ell_1 \oplus \ell_\infty)$. □

Note that there are finitely symmetric polynomials on $\ell_1 \oplus \ell_\infty$ which are not symmetric. For example, let U be a free ultrafilter on $\mathbb{N}$. Then polynomials of the form

$$P_U(x,y) = \lim_U y_n \quad \text{and} \quad Q_{U,k}(x,y) = \lim_U \frac{\sum\limits_{n=1}^{m} y_n^k}{m}$$

are finitely symmetric but not symmetric (see [8]).

Since, $\ell_1 \oplus c_0 \subset \ell_1 \oplus \ell_\infty$, we can concider the algebra of block-symmetric polynomials on $\ell_1 \oplus c_0$, $\mathcal{P}_{vs}(\ell_1 \oplus c_0)$.

Proposition 2. *The restriction $H_0^{k_1,k_2}$ of polynomials H^{k_1,k_2}, $k_1 \in \mathbb{Z}_+, k_2 \in \mathbb{N}$ onto $\ell_1 \oplus c_0$ form an algebraic basis in $\mathcal{P}_{vs}(\ell_1 \oplus c_0)$.*

Proof. Since $\ell_1 \oplus \ell_1 \subset \ell_1 \oplus c_0 \subset \ell_1 \oplus \ell_\infty$ and the restriction of H^{k_1,k_2} onto $\ell_1 \oplus \ell_1$ are algebraically independed, so $H_0^{k_1,k_2}$ are algebraically independed. Let P be a symmetric polynomial on $\ell_1 \oplus c_0$ and $\tilde{P}$ its Aron-Berner extension (see [2]) to the second dual $(\ell_1 \oplus c_0)'' = \ell'_\infty \oplus \ell_\infty$. It is known that the map $P \mapsto \tilde{P}$ is an algebra homomorphism and $\tilde{P}$ is symmetric on $(\ell_1 \oplus c_0)''$ with respect to extension of operators $\sigma(x,y), \sigma \in S_\infty$ (see [6]). Let $\tilde{P}_1$ be the restriction of $\tilde{P}$ to $\ell_1 \oplus \ell_\infty = (\ell_1 \oplus c_0)''$. Then $\tilde{P}_1$ is symmetric and according to Theorem 2 can be represented by

$$\tilde{P}_1 = \sum_{l_1|k^1|+\ldots+l_r|k^r|=0}^{\infty} a_{k^1,\ldots,k^r,l_1,\ldots,l_r} \left(H^{k^1}\right)^{l_1} \ldots \left(H^{k^r}\right)^{l_r},$$

where $k^j = (k_1^j, k_2^j), |k^j| = k_1^j + k_2^j$.

So P is the restriction of $\tilde{P}_1$ to $\ell_1 \oplus c_0$ and have the representation

$$P = \sum_{l_1|k^1|+\ldots+l_r|k^r|=0}^{\infty} a_{k^1,\ldots,k^r,l_1,\ldots,l_r} \left(H_0^{k^1}\right)^{l_1} \ldots \left(H_0^{k^r}\right)^{l_r},$$

where $k^j = (k_1^j, k_2^j), |k^j| = k_1^j + k_2^j$.

Hence, $H_0^{k_1,k_2}$, $k_1 \in \mathbb{Z}_+, k_2 \in \mathbb{N}$ form an algebraic basis in $\mathcal{P}_{vs}(\ell_1 \oplus c_0)$. □

Note that $\ell_1 \oplus c$ admits a block-symmetric polynomial

$$L(x,y) = \lim_{n \to \infty} y_n$$

wich can not be obtained by an algebraic combination of H^{k_1,k_2}, $k_1 \in \mathbb{Z}_+, k_2 \in \mathbb{N}$.

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Кравців В.В. *Алгебраїчний базис алгебри блочно-симетриних поліномів на $\ell_1 \oplus \ell_\infty$* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 89–95.

В робті розглянуто так звані блочно-симетричні поліноми на просторах послідовностей $\ell_1 \oplus \ell_\infty, \ell_1 \oplus c, \ell_1 \oplus c_0$, а саме, поліноми які є симетричними відносно перестановок елементів послідовностей. Доведено, що кожен неперервний блочно-симетричний поліном на $\ell_1 \oplus \ell_\infty$ може бути єдиним чином поданий як алгебраїчна комбінація деяких спеціальних блочно-симетричних поліномів, які утворюють алгебраїчний базис. Цікаво зауважити, що алгебра блочно-симетричних поліномів є нескінченно породжена, при цьому на ℓ_∞ не існує симетричних поліномів. У статті описано алгебраїчні базиси алгебр блочно-симетричних поліномів на $\ell_1 \oplus \ell_\infty$ та $\ell_1 \oplus c_0$.

Ключові слова і фрази: симетричні поліноми, блочно-симетрині поліноми, алгебраїчний базис, топологічна алгебра.