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COUPLED FIXED POINT RESULTS ON METRIC SPACES DEFINED BY BINARY OPERATIONS

In parallel with the various generalizations of the Banach fixed point theorem in metric spaces, this theory is also transported to some different types of spaces including ultra metric spaces, fuzzy metric spaces, uniform spaces, partial metric spaces, b-metric spaces etc. In this context, first we define a binary normed operation on nonnegative real numbers and give some examples. Then we recall the concept of T-metric space and some important and fundamental properties of it. A T-metric space is a 3-tuple $(X, T, \diamond)$, where X is a nonempty set, $\diamond$ is a binary normed operation and T is a T-metric on X. Since the triangular inequality of T-metric depends on a binary operation, which includes the sum as a special case, a T-metric space is a real generalization of ordinary metric space. As main results, we present three coupled fixed point theorems for bivariate mappings satisfying some certain contractive inequalities on a complete T-metric space. It is easily seen that not only existence but also uniqueness of coupled fixed point guaranteed in these theorems. Also, we provide some suitable examples that illustrate our results.

Key words and phrases: binary normed operation, T-metric space, coupled fixed point.

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1 INTRODUCTION

It is well known that the Banach contraction principle is a fundamental result in metrical fixed point theory. After this classical result, many authors have extended, generalized and improved this theorem using different contractive conditions (see [1, 3, 4, 6]). On the other hand, fixed and common fixed point results in different types of spaces including ultra metric spaces, fuzzy metric space, uniform space, partial metric space, b-metric space etc, have been developed (see [2, 5, 8, 9, 12]). An interesting generalization of metric space named as T-metric space has been recently introduced by [11] (see also [10]). Briefly, the concept of T-metric space is based on the fact that the triangle inequality in the metric definition depends on a binary operation.

This study was organized as follows: first, we recall the definition of T-metric and some properties of it. Finally, we prove some coupled fixed point theorems for single valued mappings in complete T-metric spaces satisfying different contractive type condition.

Here we will emphasize the concept of ultra metric because of it will be mentioned in the next. Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \ \forall x, y, z \in X,$$

then d is called an ultra metric on X and the pair (X, d) is called an ultra metric space. An ultra metric space (X, d) is said to be spherically complete if every shrinking collection of balls (that is, every nested decreasing sequence of balls) in X has a nonempty intersection.

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2 A BINARY NORMED OPERATION AND T-METRIC SPACES

In this section, we define a binary normed operation and give some examples.

A *binary normed* operation is a mapping $\diamond : [0,\infty) \times [0,\infty) \to [0,\infty)$ which satisfies the following conditions:

(i) $\diamond$ is associative and commutative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in [0,\infty)$,

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,\infty)$.

Some typical examples of $\diamond$ are as follows: let $a, b \in [0,\infty)$

(a) $a \diamond_1 b = \max\{a, b\}$,

(b) $a \diamond_2 b = \sqrt{a^2 + b^2}$,

(c) $a \diamond_3 b = a + b$,

(d) $a \diamond_4 b = ab + a + b$,

(e) $a \diamond_5 b = (\sqrt{a} + \sqrt{b})^2$.

Straightforward calculations lead to the following relations among normed binary operations given above

$$a \diamond_1 b \leq a \diamond_2 b \leq a \diamond_3 b \leq a \diamond_4 b$$

and

$$a \diamond_3 b \leq a \diamond_5 b.$$

The following lemma defines a normed binary operation exploiting some properties of a self map on $[0,\infty)$.

Lemma 1. *Let $f : [0,\infty) \to [0,\infty)$ be any continuous, increasing and onto mapping. Let $\diamond : [0,\infty) \times [0,\infty) \to [0,\infty)$ be defined by*

$$a \diamond b = f^{-1}(f(a) + f(b))$$

for $a, b \in [0,\infty)$. Then $\diamond$ is a normed binary operation.

Proof. It follows immediately. $\square$

Example 1. *Let $f : [0,\infty) \to [0,\infty)$ defined by $f(x) = e^x - 1$. Obviously f is a continuous and increasing map. Therefore by Lemma 1, $a \diamond b = \ln(e^a + e^b - 1)$ defines a normed binary operation.*

We have the following simple observations about a normed binary operation.

Lemma 2. *The following statements hold for any normed binary operation.*
i) If $r, r' \geq 0$, then $r \leq r \diamond r'$.
ii) For $\delta \in (0, r)$, there exists $\delta' \in (0, r)$ such that $\delta' \diamond \delta < r$.
iii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta \diamond \delta < \varepsilon$.

Proof. i) Since $r' \geq 0$, using properties (iii) and (iv) of a normed binary operation $\diamond$, we have $r \diamond r' \geq r \diamond 0 = r$.

ii) If we assume that every $\delta' > 0$ gives $\delta' \diamond \delta \geq r$. In particular, if we set $\delta' = \frac{1}{n}$, we get $\frac{1}{n} \diamond \delta \geq r$ which on taking the limit as $n \to \infty$ implies that $0 \diamond \delta \geq r$ which is a contradiction. Hence, by part (i) of this lemma we obtain $\delta' \leq \delta' \diamond \delta < r$.

iii) Assume the contrary, i.e., for all $\delta > 0$, $\delta \diamond \delta \geq \varepsilon$. For $\delta = \frac{1}{n}$ we have $\frac{1}{n} \diamond \frac{1}{n} \geq \varepsilon$ which on taking the limit as $n \to \infty$ gives $0 \geq \varepsilon$, which is a contradiction. Hence iii) follows. $\square$

Now, we recall the concept of T-metric.

Definition 1 ([10]). *Let X be a nonempty set. A T-metric on X is a function $T : X^2 \to \mathbb{R}$ that satisfies the following conditions, for each $x, y, z \in X$,*

1. *$T(x,y) \geq 0$ and $T(x,y) = 0$ if and only if $x = y$,*
2. *$T(x,y) = T(y,x)$,*
3. *$T(x,y) \leq T(x,z) \diamond T(y,z)$.*

The 3-tuple $(X, T, \diamond)$ is called a T-metric space.

Example 2 ([11]). *i) Every ordinary metric d is a T-metric with $a \diamond b = a + b$.*

ii) Every ultra metric d is a T-metric with $a \diamond b = \max\{a, b\}$.

iii) Let $X = \mathbb{R}$ and $T(x,y) = \sqrt{|x-y|}$ for all $x, y \in \mathbb{R}$. If we take $a \diamond b = \sqrt{a^2 + b^2}$, then we have

$$T(x,y) = \sqrt{|x-y|} \leq \sqrt{|x-z| + |z-y|} = \sqrt{\sqrt{|x-z|}^2 + \sqrt{|z-y|}^2} = T(x,z) \diamond T(z,y).$$

Therefore, the function T is a T-metric on X.

iv) Let $X = \mathbb{R}$ and $T(x,y) = (x-y)^2$ for every $x, y \in \mathbb{R}$. If we take $a \diamond b = (\sqrt{a} + \sqrt{b})^2$, then we get

$$T(x,y) = (x-y)^2 = |x-y|^2 \leq (|x-z| + |z-y|)^2 = \left(\sqrt{|x-z|^2} + \sqrt{|z-y|^2}\right)^2 = T(x,z) \diamond T(z,y).$$

Hence, the function T is a T-metric on X.

Remark 1 ([11]). *For a fixed $0 \leq \alpha \leq \frac{\pi}{4}$, if there exist β, γ such that $0 \leq \alpha \leq \beta + \gamma < \frac{\pi}{2}$, then*

$$\tan \alpha \leq \tan \beta + \tan \gamma + \tan \beta \tan \gamma.$$

Example 3 ([11]). *Let $X = [0,1]$ and $T(x,y) = \tan(\frac{\pi}{4}|x-y|)$ for every $x, y \in X$. If we take $a \diamond b = a + b + ab$, then by Remark 1 we obtain*

$$\begin{aligned} T(x,y) &= \tan(\frac{\pi}{4}|x-y|) \\ &\leq \tan(\frac{\pi}{4}|x-z|) + \tan(\frac{\pi}{4}|z-y|) + \tan(\frac{\pi}{4}|x-z|)\tan(\frac{\pi}{4}|z-y|) \\ &= T(x,z) \diamond T(z,y). \end{aligned}$$

So, the function T is a T-metric on X.

Let $(X, T, \diamond)$ be a T-metric space. For $r > 0$ define

$$B_T(x, r) = \{y \in X : T(x, y) < r\}.$$

Definition 2 ([11]). *Let $(X, T, \diamond)$ be a T-metric space $r > 0$ and $A \subset X$.*

1. *The set $B_T(x, r)$ is called the open ball of a center x and a radius r.*
2. *If for all $x \in A$ there exists $r > 0$ such that $B_T(x, r) \subset A$, then the subset A is called an open subset of X.*
3. *The subset A of X is said to be T-bounded if there exists $r > 0$ such that $T(x, y) < r$ for all $x, y \in A$.*
4. *A sequence $\{x_n\}$ in X converges to x if $T(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $T(x_n, x) < \varepsilon$ for all $n \geq n_0$.*
5. *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $T(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.*
6. *The T-metric space $(X, T, \diamond)$ is said to be complete if every Cauchy sequence is convergent.*

Let τ be the set of all open subsets of X, then τ is a topology on X (induced by the T-metric T). Note that if A and B are open subsets of X and $x \in A \cap B$, then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $B_T(x, \varepsilon_1) \subset A$ and $B_T(x, \varepsilon_2) \subset B$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$, then by Lemma 2 (iii), there exists $\delta > 0$ such that $\delta \diamond \delta < \varepsilon$. In this case, we have $B_T(x, \delta \diamond \delta) \subset B_T(x, \varepsilon_1) \cap B_T(x, \varepsilon_2) \subset A \cap B$, hence $A \cap B$ is open.

Lemma 3 ([11]). *Let $(X, T, \diamond)$ be a T-metric space. If $r > 0$, then the open ball $B_T(x, r)$ with a center $x \in X$ and a radius r is an open set.*

Lemma 4 ([11]). *Let $(X, T, \diamond)$ be a T-metric space. If a sequence $\{x_n\}$ in X converges to x, then x is unique.*

Lemma 5 ([11]). *Let $(X, T, \diamond)$ be a T-metric space. Then every convergent sequence $\{x_n\}$ in X is a Cauchy sequence.*

Definition 3 ([11]). *Let $(X, T, \diamond)$ be a T-metric space. T is said to be continuous if*

$$\lim_{n\to\infty} T(x_n, y_n) = T(x, y),$$

whenever

$$\lim_{n\to\infty} T(x_n, x) = \lim_{n\to\infty} T(y_n, y) = 0.$$

Lemma 6. *Let $(X, T, \diamond)$ be a T-metric space. Then T is a continuous function.*

Proof. Assume that $\lim_{n\to\infty} T(x_n, x) = \lim_{n\to\infty} T(y_n, y) = 0$. By the triangular inequality we have

$$T(x_n, y_n) \leq T(x_n, x) \diamond T(x, y) \diamond T(y, y_n).$$

Hence we get

$$\limsup_{n\to\infty} T(x_n, y_n) \leq T(x, y).$$

Similarly, we obtain

$$T(x,y) \leq T(x,x_n) \diamond T(x_n,y_n) \diamond T(y_n,y)$$

and so

$$T(x,y) \leq \lim_{n\to\infty} \inf T(x_n,y_n).$$

Therefore

$$\lim_{n\to\infty} T(x_n,y_n) = T(x,y).$$

□

Henceforth, we assume that $\diamond$ is a binary operation on $[0,\infty) \times [0,\infty)$ such that
i) $\alpha(a \diamond b) = \alpha a \diamond \alpha b$ for every $\alpha \in \mathbb{R}^+$,
ii) there exists $h \geq 0$ such that $\underbrace{1 \diamond 1 \diamond \cdots \diamond 1}_{n} \leq n^h$.

Example 4. *Let $a \diamond b = \max\{a,b\}$, $a \diamond b = \sqrt{a^2+b^2}$, $a \diamond b = a+b$ and $a \diamond b = (\sqrt{a}+\sqrt{b})^2$. We take $h \geq 0, h \geq \dfrac{1}{2}, h \geq 1$ and $h \geq 2$ respectively in (ii). But if $a \diamond b = a+b+ab$, then is not necessary that $\diamond$ satisfies the above conditions.*

3 COUPLED FIXED POINT THEOREMS IN T-METRIC SPACES

Now, we remember the concept of a coupled fixed point on a T-metric space.

Definition 4 ([7]). *Let X be a nonempty set and $F : X \times X \to X$ be a function. An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the map F if $F(x,y) = x$ and $F(y,x) = y$.*

Example 5. *Let $X = \mathbb{R}$. Define a map F on $X \times X$ by $F(x,y) = xy^2$. It is easy to see that $(1,-1) \in X \times X$ is a coupled fixed point of the mapping F.*

Theorem 1. *Let $(X,T,\diamond)$ be a complete T-metric space. Suppose that the map $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$*

$$T(F(x,y),F(u,v)) \leq kT(x,u) \diamond lT(y,v), \tag{1}$$

where k,l are nonnegative constants with $k \diamond l < 1$. Then F has a unique coupled fixed point.

Proof. Choose $x_0,y_0 \in X$ and set $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. We can define sequences $\{x_n\}$ and $\{y_n\}$ by $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$. By (1) we have

$$\begin{aligned} T(x_n,x_{n+1}) &= T(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \\ &\leq kT(x_{n-1},x_n) \diamond lT(y_{n-1},y_n). \end{aligned} \tag{2}$$

Similarly

$$\begin{aligned} T(y_n,y_{n+1}) &= T(F(y_{n-1},x_{n-1}),F(y_n,x_n)) \\ &\leq kT(y_{n-1},y_n) \diamond lT(x_{n-1},x_n). \end{aligned} \tag{3}$$

Letting

$$d_n = T(x_n,x_{n+1}) \diamond T(y_n,y_{n+1}), \tag{4}$$

we get

$$\begin{aligned} d_n &= T(x_n, x_{n+1}) \diamond T(y_n, y_{n+1}) \\ &\leq kT(x_{n-1}, x_n) \diamond lT(y_{n-1}, y_n) \diamond kT(y_{n-1}, y_n) \diamond lT(x_{n-1}, x_n) \\ &= (k \diamond l)[T(x_{n-1}, x_n) \diamond T(y_{n-1}, y_n)] \\ &= (k \diamond l) d_{n-1}. \end{aligned} \tag{5}$$

Consequently, if we set $\delta = k \diamond l$, then for each $n \in \mathbb{N}$ we obtain

$$d_n \leq \delta d_{n-1} \leq \delta^2 d_{n-2} \leq \cdots \leq \delta^n d_0. \tag{6}$$

If $d_0 = 0$ then $T(x_0, x_1) \diamond T(y_0, y_1) = 0$. Hence, we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$, i.e., (x_0, y_0) is a coupled fixed point of F. Now suppose that $d_0 > 0$. For each $n > m$ we have

$$T(x_n, x_m) \leq T(x_n, x_{n-1}) \diamond T(x_{n-1}, x_{n-2}) \diamond \cdots \diamond T(x_{m+1}, x_m).$$

In the same manner, we get

$$T(y_n, y_m) \leq T(y_n, y_{n-1}) \diamond T(y_{n-1}, y_{n-2}) \diamond \cdots \diamond T(y_{m+1}, y_m).$$

Thus

$$\begin{aligned} T(x_n, x_m) &\leq T(x_n, x_m) \diamond T(y_n, y_m) \\ &\leq d_{n-1} \diamond d_{n-2} \diamond \cdots \diamond d_m \\ &\leq (\delta^{n-1} \diamond \delta^{n-2} \diamond \cdots \diamond \delta^m) d_0 \\ &\leq \delta^m d_0 \underbrace{(1 \diamond \cdots \diamond 1)}_{n-m} \\ &\leq \delta^m d_0 \underbrace{(1 \diamond \cdots \diamond 1)}_{n} \\ &\leq \delta^m d_0 n^h \longrightarrow 0. \end{aligned}$$

Hence for $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for all $n > m \geq n_0$ we get $T(x_n, x_m) < \varepsilon$. Similarly, we can get $T(y_n, y_m) < \varepsilon$. It follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy and by the completeness of X, $\{x_n\}$ and $\{y_n\}$ converge to u^* and v^* in X respectively. Thus

$$\lim_{n \to \infty} T(x_n, u^*) = \lim_{n \to \infty} T(y_n, v^*) = 0. \tag{7}$$

Using the triangular inequality and (1) we get

$$\begin{aligned} T(F(u^*, v^*), u^*) &\leq T(F(u^*, v^*), x_{n+1}) \diamond T(x_{n+1}, u^*) \\ &= T(F(u^*, v^*), F(x_n, y_n)) \diamond T(x_{n+1}, u^*) \\ &\leq kT(x_n, u^*) \diamond lT(y_n, v^*) \diamond T(x_{n+1}, u^*). \end{aligned}$$

Letting $n \to \infty$, then from (7), we obtain $T(F(u^*, v^*), u^*)) = 0$ and so $F(u^*, v^*) = u^*$. In the same maner, we have $F(v^*, u^*) = v^*$; i.e., (u^*, v^*) is a coupled fixed point of F. Now, if (u', v') is another coupled fixed point of F we get

$$T(u', u^*) = T(F(u', v'), F(u^*, v^*)) \leq kT(u', u^*) \diamond lT(v', v^*)$$

and

$$T(v', v^*) = T(F(v', u'), F(v^*, u^*)) \leq kT(v', v^*) \diamond lT(u', u^*).$$

Then

$$T(u', u^*) \diamond T(v', v^*) \leq (k \diamond l)[T(u', u^*) \diamond T(v', v^*)].$$

As $k \diamond l < 1$, we have $T(u', u^*) \diamond T(v', v^*) = 0$ and so $u' = u^*$ and $v' = v^*$. The proof of Theorem 1 is completed. □

Example 6. *Let $X = \mathbb{R}$ and $T(x,y) = \sqrt{|x-y|}$ for all $x,y \in \mathbb{R}$. If we take $a \diamond b = \sqrt{a^2+b^2}$, then the function T is a T-metric on X. Let $F(x,y) = \dfrac{x+2y}{5} - 1$ for all $x,y \in X$. For all $x,y,u,v \in X$, we obtain*

$$\begin{aligned} T(F(x,y),F(u,v)) &= \sqrt{\frac{|(x-u)+2(y-v)|}{5}} \\ &\leq \frac{1}{\sqrt{5}}(\sqrt{|x-u|+2|y-v|}) \\ &= \frac{1}{\sqrt{5}}T(x,u) \diamond \frac{\sqrt{2}}{\sqrt{5}}T(y,v). \end{aligned}$$

Hence for $k = \dfrac{1}{\sqrt{5}}$ and $l = \dfrac{\sqrt{2}}{\sqrt{5}}$, we get $k \diamond l < 1$. It follows that all conditions of Theorem 1 hold, and $(-\frac{5}{2}, -\frac{5}{2}) \in X \times X$ is the unique coupled fixed point of the mapping F.

Example 7. *Let $X = \mathbb{R}$ and $T(x,y) = (x-y)^2$ for all $x,y \in \mathbb{R}$. If we take $a \diamond b = (\sqrt{a}+\sqrt{b})^2$, then the function T is a T-metric on X. Let $F(x,y) = \dfrac{x+2y}{5} - 1$ for all $x,y \in X$. For all $x,y,u,v \in X$, we obtain*

$$\begin{aligned} T(F(x,y),F(u,v)) &= \Big(\frac{x-u}{5} + 2\frac{y-v}{5}\Big)^2 \\ &\leq 2\Big(\frac{x-u}{5}\Big)^2 + 4\Big(\frac{y-v}{5}\Big)^2 \\ &= \frac{2}{25}(|x-u|)^2 + \frac{8}{25}(|y-v|)^2 \\ &\leq \Big(\frac{\sqrt{2}}{5}|x-u| + \frac{2\sqrt{2}}{5}|y-v|\Big)^2 \\ &= \frac{2}{25}T(x,u) \diamond \frac{8}{25}T(y,v). \end{aligned}$$

Hence for $k = \dfrac{2}{25}$ and $l = \dfrac{8}{25}$, we get $k \diamond l = \frac{18}{25} < 1$. It follows that the all conditions of Theorem 1 hold, and $\Big(-\frac{5}{2}, -\frac{5}{2}\Big) \in X \times X$ is the unique coupled fixed point of the mapping F.

Theorem 2. *Let $(X, T, \diamond)$ be a complete T-metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$*

$$T(F(x,y),F(u,v)) \leq kT(F(x,y),x) \diamond lT(F(u,v),u), \tag{8}$$

where k, l are nonnegative constants with $k \diamond l < 1$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. We can define sequences $\{x_n\}$ and $\{y_n\}$ by $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$. By (8), we have

$$\begin{aligned} T(x_n,x_{n+1}) &= T(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \\ &\leq kT(F(x_{n-1},y_{n-1}),x_{n-1}) \diamond lT(F(x_n,y_n),x_n) \\ &= kT(x_n,x_{n-1}) \diamond lT(x_{n+1},x_n). \end{aligned}$$

If $T(x_{n+1},x_n) \geq T(x_n,x_{n-1})$ then

$$\begin{aligned} T(x_n,x_{n+1}) &\leq kT(x_n,x_{n-1}) \diamond lT(x_{n+1},x_n) \\ &\leq (k\diamond l)T(x_{n+1},x_n) \\ &< T(x_{n+1},x_n), \end{aligned}$$

which is a contradiction. Hence

$$T(x_n,x_{n+1}) \leq (k\diamond l)T(x_{n-1},x_n) = \delta T(x_{n-1},x_n).$$

Similarly

$$T(y_n,y_{n+1}) \leq (k\diamond l)T(y_{n-1},y_n) = \delta T(y_{n-1},y_n).$$

So, if $m > n$

$$\begin{aligned} T(x_n,x_m) &\leq T(x_n,x_{n+1}) \diamond T(x_{n+1},x_{n+2}) \diamond \cdots \diamond T(x_{m-1},x_m) \\ &\leq \delta^n T(x_0,x_1) \diamond \delta^{n+1} T(x_0,x_1) \diamond \cdots \diamond \delta^{m-1} T(x_0,x_1) \\ &= \delta^n T(x_0,x_1)(1\diamond\delta\diamond\delta^2\diamond\cdots\diamond\delta^{m-n-1}) \\ &\leq \delta^n T(x_0,x_1)(\underbrace{1\diamond 1\diamond 1\diamond\cdots\diamond 1}_{m-n}) \\ &\leq \delta^n T(x_0,x_1)(\underbrace{1\diamond 1\diamond 1\diamond\cdots\diamond 1}_{m}) \\ &\leq \delta^n T(x_0,x_1)m^h. \end{aligned}$$

It is easy to see that for all $m > n$ there exists $s > 0$ such that $m \leq n^s$. Thus

$$T(x_n,x_m) \leq \delta^n T(x_0,x_1)n^{hs} \to 0.$$

Hence for $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$ we get $T(x_n,x_m) < \varepsilon$. Similarly, we can get $T(y_n,y_m) < \varepsilon$. It follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy and by the completeness of X, $\{x_n\}$ and $\{y_n\}$ converge to u^* and v^* in X respectively. Thus

$$\lim_{n\to\infty} T(x_n,u^*) = \lim_{n\to\infty} T(y_n,v^*) = 0. \tag{9}$$

Applying the triangular inequality and (8) we get

$$\begin{aligned} T(F(u^*,v^*),u^*) &\leq T(F(u^*,v^*),x_{n+1}) \diamond T(x_{n+1},u^*) \\ &= T(F(u^*,v^*),F(x_n,y_n) \diamond T(x_{n+1},u^*) \\ &\leq kT(F(u^*,v^*),u^*) \diamond lT(F(x_n,y_n),x_n) \diamond T(x_{n+1},u^*). \end{aligned}$$

Letting $n \to \infty$ and from (9) we obtain $T(F(u^*,v^*),u^*)) \leq kT(F(u^*,v^*),u^*))$ which implies that $T(F(u^*,v^*),u^*)) = 0$ and so $F(u^*,v^*) = u^*$. In the similar manner, we have $F(v^*,u^*) = v^*$, i.e; (u^*,v^*) is a coupled fixed point of F. Now, if (u',v') is another coupled fixed point of F, then

$$\begin{aligned} T(u',u^*) &= T(F(u',v'),F(u^*,v^*)) \\ &\leq kT(F(u',v'),u') \diamond lT(F(u^*,v^*),u^*) \\ &= kT(u',u') \diamond lT(u^*,u^*) = 0. \end{aligned}$$

This implies that $T(u',u^*) = 0$ and so $u' = u^*$. Siimilarly $v' = v^*$. The proof of Theorem 2 is completed. □

Theorem 3. *Let $(X, T, \diamond)$ be a complete T-metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$*

$$T(F(x,y), F(u,v)) \leq kT(F(x,y), u) \diamond lT(F(u,v), x), \tag{10}$$

where k, l are nonnegative constants with $k \diamond l \diamond l < 1$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. We can define sequences $\{x_n\}$ and $\{y_n\}$ by $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. By (10), we have

$$\begin{aligned}
T(x_n, x_{n+1}) &= T(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\leq kT(F(x_{n-1}, y_{n-1}), x_n) \diamond lT(F(x_n, y_n), x_{n-1}) \\
&= kT(x_n, x_n) \diamond lT(x_{n+1}, x_{n-1}) \\
&= lT(x_{n+1}, x_{n-1}) \\
&\leq lT(x_{n+1}, x_n) \diamond T(x_n, x_{n-1}).
\end{aligned}$$

If $T(x_{n+1}, x_n) \geq T(x_n, x_{n-1})$ then

$$\begin{aligned}
T(x_n, x_{n+1}) &\leq lT(x_{n+1}, x_n) \diamond lT(x_{n+1}, x_n) \\
&\leq (l \diamond l)T(x_{n+1}, x_n) \\
&\leq (k \diamond l \diamond l)T(x_{n+1}, x_n) \\
&< T(x_{n+1}, x_n).
\end{aligned}$$

which is contradiction. Hence

$$T(x_n, x_{n+1}) \leq (l \diamond l)T(x_{n-1}, x_n) = \delta T(x_{n-1}, x_n),$$

Similarly

$$T(y_n, y_{n+1}) \leq (l \diamond l)T(y_{n-1}, y_n) = \delta T(y_{n-1}, y_n),$$

where $\delta = l \diamond l \leq k \diamond l \diamond l < 1$. So, if $m > n$,

$$\begin{aligned}
T(x_n, x_m) &\leq T(x_n, x_{n+1}) \diamond T(x_{n+1}, x_{n+2}) \diamond \cdots \diamond T(x_{m-1}, x_m) \\
&\leq \delta^n T(x_0, x_1) \diamond \delta^{n+1} T(x_0, x_1) \diamond \cdots \diamond \delta^{m-1} T(x_0, x_1) \\
&= \delta^n T(x_0, x_1)(1 \diamond \delta \diamond \delta^2 \diamond \cdots \diamond \delta^{m-n-1}) \\
&\leq \delta^n T(x_0, x_1)(\underbrace{1 \diamond 1 \diamond 1 \diamond \cdots \diamond 1}_{m-n}) \\
&\leq \delta^n T(x_0, x_1)(\underbrace{1 \diamond 1 \diamond 1 \diamond \cdots \diamond 1}_{m}) \\
&\leq \delta^n T(x_0, x_1) m^h.
\end{aligned}$$

It is easy to see that for all $m > n$ there exists $s > 0$ such that $m \leq n^s$. Thus

$$T(x_n, x_m) \leq \delta^n T(x_0, x_1) n^{hs} \to 0.$$

Hence for $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$ we get $T(x_n, x_m) < \varepsilon$. Similarly, we can get $T(y_n, y_m) < \varepsilon$. It follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy and by the completeness of X, $\{x_n\}$ and $\{y_n\}$ converge to u^* and v^* in X respectively. Thus

$$\lim_{n \to \infty} T(x_n, u^*) = \lim_{n \to \infty} T(y_n, v^*) = 0. \tag{11}$$

Using the triangular inequality and (10) we get

$$\begin{aligned} T(F(u^*,v^*),u^*) &\leq T(F(u^*,v^*),x_{n+1}) \diamond T(x_{n+1},u^*) \\ &= T(F(u^*,v^*),F(x_n,y_n) \diamond T(x_{n+1},u^*) \\ &\leq kT(F(u^*,v^*),x_n) \diamond lT(F(x_n,y_n),u^*) \diamond T(x_{n+1},u^*). \end{aligned}$$

Letting $n \to \infty$, then from (11) we obtain $T(F(u^*,v^*),u^*)) \leq kT(F(u^*,v^*),u^*))$. This implies that $T(F(u^*,v^*),u^*)) = 0$ and so $F(u^*,v^*) = u^*$. In the similar manner, we have $F(v^*,u^*) = v^*$; i.e., (u^*,v^*) is a coupled fixed point of F. Now, if (u',v') is another coupled fixed point of F, then

$$\begin{aligned} T(u',u^*) &= T(F(u',v'),F(u^*,v^*)) \\ &\leq kT(F(u',v'),u^*) \diamond lT(F(u^*,v^*),u') \\ &= kT(u',u^*) \diamond lT(u^*,u') \\ &= (k \diamond l)T(u',u^*) \\ &\leq (k \diamond l \diamond l)T(u',u^*) \\ &< T(u',u^*). \end{aligned}$$

This implies that $T(u',u^*) = 0$ and so $u' = u^*$. Similarly $v' = v^*$. The proof of Theorem 3 is completed. ☐

If we set $a \diamond b = a + b$ and $T(x,y) = d(x,y)$ in Theorem 1 we have

Corollary 1. *Let (X,d) be a complete metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$*

$$d(F(x,y),F(u,v)) \leq kd(x,u) + ld(y,v),$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

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Карамі А., Шакері Р., Седхі С., Алтун І. *Результати про пари нерухомих точок на метричних просторах, визначених бінарними операціями* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 313–323.

Паралельно до різних узагальнень теореми Банаха про нерухому точку в метричних просторах, ця теорія є застосовною до різних типів просторів, зокрема, таких як ультраметричні простори, нечіткі метричні простори, рівномірні простори, частково метричні простори, b-метричні простори та ін. У цьому контексті спочатку ми визначаємо бінарну нормовану операцію на невід'ємних дійсних числах і даємо кілька прикладів. Тоді ми згадуємо поняття T-метричного простору та його важливі і фундаментальні властивості. T-метричний простір — це набір $(X, T, \diamond)$, де X є непорожньою множиною, $\diamond$ — бінарною нормованою операцією і T є деякою T-метрикою на X. Оскільки нерівність трикутника для T-метрики залежить від бінарної операції, для якої частковим випадком є сума, T-метричний простір є справжнім узагальненням звичайного метричного простору. Головними результатами, які ми представляємо, є три теореми для пар нерухомих точок для двохвимірних відображень, що задовольняють деякі нерівності стиску в повних T-метричних просторах. Легко бачити, що не тільки існування, але і єдиність пари нерухомих точок гарантується цими теоремами. Також ми представляємо деякі придатні приклади, що ілюструють наші результати.

Ключові слова і фрази: бінарна нормована операція, T-метричний простір, пара нерухомих точок.