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UNCONVENTIONAL ANALOGS OF SINGLE-PARAMETRIC METHOD OF ITERATIONAL AGGREGATION

When we solve practical problems that arise, for example, in mathematical economics, in the theory of Markov processes, it is often necessary to use the decomposition of operator equations using methods of iterative aggregation. In the studies of these methods for the linear equation $x = Ax + b$ the most frequent are the conditions of positiveness of the operator A, constant b and the aggregation functions, and also the implementation of the inequality $\rho(A) < 1$ for the spectral radius $\rho(A)$ of the operator A.

In this article for an approximate solution of a system composed of the equation $x = Ax + b$ represented in the form $x = A_1x + A_2x + b$, where $b \in E$, E is a Banach space, A_1, A_2 are linear continuous operators that act from E to E and the auxiliary equation $y = \lambda y - (\varphi, A_2x) - (\varphi, b)$ with a real variable y, where (φ, x) is the value of the linear functional $\varphi \in E^*$ on the elements $x \in E$, E^* is conjugation with space E, an iterative process is constructed and investigated

$$x^{(n+1)} = Ax^{(n)} + b + \frac{\sum\limits_{i=1}^{m} A_1^i x^{(n)}}{(\varphi, x^{(n)}) \sum\limits_{i=0}^{m} \lambda^i}(y^{(n)} - y^{(n+1)}) \quad (m < \infty),$$
$$y^{(n+1)} = \lambda y^{(n+1)} - (\varphi, A_2x^{(n)}) - (\varphi, b).$$

The conditions are established under which the sequences $x^{(n)}, y^{(n)}$, constructed with the help of these formulas, converge to x^*, y^* as a component of solving the system constructed from equations $x = A_1x + A_2x + b$ and the equation $y = \lambda y - (\varphi, A_2x) - (\varphi, b)$ not slower than the rate of convergence of the geometric progression with the denominator less than 1. In this case, it is required that the operator A be a compressive and constant by sign, and that the space E is semi-ordered. The application of the proposed algorithm to systems of linear algebraic equations is also shown.

Key words and phrases: aggregating functional, decomposition, iterative aggregation.

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INTRODUCTION

Actuality of the investigation of iterative aggregation methods connected with necessity of solving big dimensional problems with the aid of multiprocessor computable technical devices using decompositional algorithms for corresponding mathematical models. Multiparametric iterative aggregation has appeared to be an effective in mathematical economy, in investigation of Markov processes etc. (see [1–3, 6, 7, 13]) due to ability to make an acceptable results even

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in circumstances when convergence conditions of algorithms is unknown (see [5, p. 158]). The simplest single parametric method of iterative aggregation for equation

$$x = Ax + b \tag{1}$$

in [5, p. 155–158] was described by formula

$$x^{(n+1)} = \frac{(\varphi, b)}{(\varphi, x^{(n)} - Ax^{(n)})} Ax^{(n)} + b, \tag{2}$$

where (φ, x) are values of linear functional φ on elements x of Banach space E, $A : E \to E$. Instead of (2) we can consider

$$x^{(n+1)} = \frac{(\varphi, x^{(n+1)})}{(\varphi, x^{(n)})} Ax^{(n)} + b. \tag{3}$$

In [4, 8–12] it is launched method of algorithm (3) convergency investigation and its multi-parametric generalization under conditions of not semi ordered space E and inequality $\rho(A) < 1$ of spectral radius $\rho(A)$ of operator A does not demand.

1 Main suggestions

Let us suppose that equation (1) can be considered in the form

$$x = A_1 x + A_2 x + b, \tag{4}$$

where $b \in E$, E is a Banach space, A_1, A_2 are linear continuous operators that act from E to E. Let us denote by (φ, x) values of linear functional $\varphi \in E^*$ on elements $x \in E$, E^* is the adjoint space to E, A_1^* is the adjoint operator to A_1, E' is a set of real numbers. Let us consider the system formed by equation (4) and additional equation

$$y = \lambda y - (\varphi, A_2 x) - (\varphi, b) \tag{5}$$

with the real unknown y. Let us define a norm of $\{x, y\}$ $(x \in E, y \in E')$ by formula

$$||x, y|| = \sqrt{||x||^2 + |y|^2},$$

where $||x||$ is a norm of element $x \in E$, $|y|$ is an absolute value of number $y \in E'$. We denote by ε a set of pairs $\{x, y\}$ $(x \in E, y \in E')$ that satisfy the equation

$$(\varphi, x) + y = 0. \tag{6}$$

Theorem 1. *Let the following conditions hold*
1) pair (x^*, y^*) is the solution of system (4), (5) *in $\widetilde{E} = E \times E'$;*
2) the following equality takes place

$$A_1^* \varphi = \lambda \varphi, \quad \lambda \in E', \lambda \neq 1. \tag{7}$$

Then $(x^*, y^*) \in \varepsilon$.

Proof. From the condition 2) and the equalities (4), (5) for $x = x^*, y = y^*$ it follows that

$$\begin{aligned}(\varphi, x^*) + y^* &= (\varphi, A_1 x^*) + (\varphi, A_2 x^*) + (\varphi, b) + \lambda y^* - (\varphi, A_2 x^*) - (\varphi, b) \\ &= (A_1^* \varphi, x^*) + \lambda y^* = \lambda[(\varphi, x^*) + y^*].\end{aligned}$$

Since $\lambda \neq 1$, then we obtain that (x^*, y^*) satisfies (6). □

Theorem 2. *Let us consider operator $a(x)w$ which is continuous by $x \in E$ and linear and continuous by $w \in E'$. Let us suppose that equality*

$$(\varphi, a(x)) = \lambda, \quad \lambda \in E', \lambda \neq 1 \tag{8}$$

takes place and condition 2) of Theorem 1 holds. If $\{x, y\} \in \varepsilon$, $x \in E$, $y \in E'$, then for pair $\{u, v\}$, which is the solution of system

$$u = A_1 x + A_2 x + b + a(x)(y - v), \tag{9}$$

$$v = \lambda v - (\varphi, A_2 x) - (\varphi, b), \tag{10}$$

we can state that $\{u, v\} \in \varepsilon$.

Proof. Let us prove that (u, v) satisfies (6). Really,

$$\begin{aligned}(\varphi, u) + v = (\varphi, A_1 x) + (\varphi, A_2 x) + (\varphi, b) + (\varphi, a(x))y - (\varphi, a(x))v \\ + \lambda v - (\varphi, A_2 x) - (\varphi, b) = \lambda[(\varphi, x) + y].\end{aligned}$$

Therefore $(u, v) \in \varepsilon$. □

Theorem 3. *If the condition 2) of Theorem 1 takes place, then the operator*

$$a(x) = \frac{\sum\limits_{i=1}^{m} A_1^i x}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} \quad (m < \infty) \tag{11}$$

satisfies equality (8).

Proof. Using (7) we obtain from (11) following:

$$\begin{aligned}(\varphi, a(x)) &= \frac{\sum\limits_{i=1}^{m} (\varphi, A_1^i x)}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} = \frac{\sum\limits_{i=1}^{m} (A_1^* \varphi, A_1^{i-1} x)}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} \\ &= \frac{\lambda \sum\limits_{i=1}^{m} (\varphi, A_1^{i-1} x)}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} = \frac{\lambda \sum\limits_{i=0}^{m-1} \lambda^i (\varphi, x)}{(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda^i} = \lambda.\end{aligned}$$

The theorem is proved. □

2 ITERATIVE FORMULAS AND SUFFICIENT CONDITIONS OF CONVERGENCY

Let us construct sequence $\{x^{(n)}\}$, $\{y^{(n)}\}$ with starting approximation $(x^{(0)}, y^{(0)}) \in \varepsilon$ by formulas

$$x^{(n+1)} = Ax^{(n)} + b + \frac{\sum\limits_{i=1}^{m} A_1^i x^{(n)}}{(\varphi, x^{(n)}) \sum\limits_{i=0}^{m} \lambda^i}(y^{(n)} - y^{(n+1)}) \quad (m < \infty), \tag{12}$$

$$y^{(n+1)} = \lambda y^{(n+1)} - (\varphi, A_2 x^{(n)}) - (\varphi, b), \tag{13}$$

where $x \in E, y \in E', \lambda \in E', \lambda \neq 1$. From (5) and (13) we get

$$y^{(n+1)} - y^* = -\frac{1}{1-\lambda}(\varphi, A_2(x^{(n)} - x^*)).$$

From the Theorems 1 and 2 we obtain equality

$$y^{(n)} - y^* = -(\varphi, x^{(n)} - x^*).$$

From (12), (13) and (11) we get

$$\begin{aligned} x^{(n+1)} - x^* &= A(x^{(n)} - x^*) - a(x^{(n)})(\varphi, x^{(n)} - x^*) + a(x^{(n)})\frac{(\varphi, A_2(x^{(n)} - x^*))}{1-\lambda} \\ &= A(x^{(n)} - x^*) - \frac{a(x^{(n)})}{1-\lambda}(\varphi(I - A)(x^{(n)} - x^*)), \end{aligned}$$

or

$$x^{(n+1)} - x^* = A(x^{(n)} - x^*) - \frac{\sum\limits_{i=1}^{m} A_1^i x^{(n)}}{(1-\lambda)(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda}(\varphi, (I - A)(x^{(n)} - x^*)), \tag{14}$$

where I is the identity operator.

Theorem 4. *Let the conditions of Theorems 1–3 take place. If for $(x, y) \in \varepsilon$, $w = x - x^*$ and operator $H_1(x)w$ defined by the formula*

$$H_1(x)w = Aw = \frac{\sum\limits_{i=1}^{m} A_1^i x}{(1-\lambda)(\varphi, x) \sum\limits_{i=0}^{m-1} \lambda_i}(\varphi, (I - A)w),$$

the inequality

$$||H_1(x)|| \leqslant q_1 \tag{15}$$

holds for $q_1 < 1$, then every sequence of $\{x^{(n)}\}$, $\{y^{(n)}\}$, constructed by formulas (12), (13)*, converges respectively to x^*, y^*, as a components of solution of system* (4), (5)*, not slowly then geometry progression with multiplier q_1.*

Proof. It is sufficient to use formulas (14), inequality (15) and condition $q_1 < 1$. □

3 APPLICATION TO A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Let us consider case when A_1, A_2 are the squared matrices of order N, $N < \infty$. For $(x, y) \in \varepsilon, w \in E'$ let us define operator $H_2(x)w$ by the formula

$$H_2(x)w = [A - \frac{\sum_{i=1}^{m} A^i x}{(1-\lambda)\varphi^T x \sum_{i=0}^{m-1} \lambda_i}\varphi^T(I - A)]w,$$

where notation $\varphi^T x$ used instead of (φ, x), φ^T is a line vector, x is a row vector, T is the transposition symbol, $\lambda \in E', \lambda \neq 1$.

Theorem 5. *If for matrices A_1, A_2 conditions of theorems 1 — 3 take place and inequalities $||H_2(x)|| \leqslant q_2 < 1$ hold, then sequences $\{x^{(n)}\}$, $\{y^{(n)}\}$, constructed by formulas* (12), (13) *converge to x^* and y^* respectively as a components of solution of system* (4), (5) *not slowly then geometry progression with multiplier q_2.*

Proof. The theorem is a partial case of Theorem 4. □

4 EXPANSION ON CASE $m = \infty$

Let us change formula (12) as follows

$$x^{(n+1)} = Ax^{(n)} + b + \frac{A_1(I - A_1)^{-1}x^{(n)}}{(\varphi, x^{(n)})}(1 - \lambda), \tag{16}$$

where $\lambda \in E', \lambda \neq 1, x \in E$, and consider iterative process, which describes pair of formulas (16) and (13) with starting approximation $\{x^{(0)}, y^{(0)}\} \in \varepsilon$. Let us restrict ourselves to the situation, when $\lambda < 1$.

For $\{x, y\} \in \varepsilon, w = x - x^*$ let us define operator $H_3(x)w$ by the formula

$$H_3(x)w = Aw - (1 - \lambda)\frac{A_1(I - A_1)^{-1}x}{(\varphi, x)}(\varphi, (I - A)w). \tag{17}$$

Theorem 6. *Let the conditions of Theorems 1–3 take place and for operator $H_3(x)w$, defined by the formula* (17)*, following inequality holds*

$$||H_3(x)|| \leqslant q_3 < 1. \tag{18}$$

Then sequences $\{x^{(n)}\}$, $\{y^{(n)}\}$, constructed with the help of formulas (13), (16)*, converge to x^* and y^* respectively as a components of solution of system* (4), (5) *not slowly then geometry progression with multiplier q_3.*

Proof. The proof of the theorem can be obtained by notions (17), (18). □

Theorem 6 is an analogue of Theorem 4. Using similar way we can obtain analogue of Theorem 5 for systems of linear algebraic equations.

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При розв'язанні практичних завдань, що виникають, наприклад, в математичній економіці, в терії марківських процесів, часто доводиться використовувати декомпозицію операторних рівнянь за допомогою методів ітеративного агрегування. В дослідженнях цих методів для лінійного рівняння $x = Ax + b$ найчастішими є вимоги додатності оператора A, вільного члена b та агрегуючих функціоналів, а також виконання нерівності $\rho(A) < 1$ для спектрального радіуса $\rho(A)$ оператора A.

В статті для наближеного розв'язання системи, складеної з рівняння $x = Ax + b$, представленого у вигляді $x = A_1x + A_2x + b$, де $b \in E$, E — банахів простір, A_1, A_2 — лінійні неперервні оператори, що діють з E в E, і допоміжного рівняння $y = \lambda y - (\varphi, A_2x) - (\varphi, b)$ з дійсним невідомим y, де (φ, x) — значення лінійного функціоналу $\varphi \in E^*$ на елементах $x \in E$, E^* — спряжений з E простір, побудовано і досліджено ітеративний процес

$$x^{(n+1)} = Ax^{(n)} + b + \frac{\sum\limits_{i=1}^{m} A_1^i x^{(n)}}{(\varphi, x^{(n)}) \sum\limits_{i=0}^{m} \lambda^i}(y^{(n)} - y^{(n+1)}) \quad (m < \infty),$$
$$y^{(n+1)} = \lambda y^{(n+1)} - (\varphi, A_2x^{(n)}) - (\varphi, b).$$

Встановлено умови, при виконанні яких послідовності $x^{(n)}, y^{(n)}$, побудовані з допомогою цих формул, збігаються відповідно до x^*, y^* як компонент розв'язку системи, складеної з рівняння $x = A_1x + A_2x + b$ та рівняння $y = \lambda y - (\varphi, A_2x) - (\varphi, b)$, не повільніше від швидкості збіжності геометричної прогресії зі знаменником, меншим від одиниці. При цьому вимагається, щоб оператор A був стискуючим та знакосталим, а простір E напівупорядкованим. Показано також застосування запропонованого алгоритму до систем лінійних алгебраїчних рівнянь.

Ключові слова і фрази: декомпозиція, ітеративне агрегування, агрегуючі функціонали.