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SIGNLESS LAPLACIAN DETERMINATIONS OF SOME GRAPHS WITH INDEPENDENT EDGES

Let G be a simple undirected graph. Then the signless Laplacian matrix of G is defined as $D_G + A_G$ in which D_G and A_G denote the degree matrix and the adjacency matrix of G, respectively. The graph G is said to be determined by its signless Laplacian spectrum (DQS, for short), if any graph having the same signless Laplacian spectrum as G is isomorphic to G. We show that $G \sqcup rK_2$ is determined by its signless Laplacian spectra under certain conditions, where r and K_2 denote a natural number and the complete graph on two vertices, respectively. Applying these results, some DQS graphs with independent edges are obtained.

Key words and phrases: spectral characterization, signless Laplacian spectrum, cospectral graphs, union of graphs.

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Introduction

All graphs considered here are simple and undirected. All notions on graphs that are not defined here can be found in [13, 16]. Let G be a simple graph with the vertex set $V = V(G) = \{v_1, \ldots, v_n\}$ and the edge set $E = E(G)$. Denote by d_i the degree of the vertex v_i. The *adjacency matrix* A_G of G is a square matrix of order n, whose (i, j)-entry is 1 if v_i and v_j are adjacent in G and 0 otherwise. The *degree matrix* D_G of G is a diagonal matrix of order n defined as $D_G = \operatorname{diag}(d_1, \ldots, d_n)$. The matrices $L_G = D_G - A_G$ and $Q_G = D_G + A_G$ are called the *Laplacian matrix* and the *signless Laplacian matrix* of G, respectively. The multiset of eigenvalues of Q_G (resp. L_G, A_G) is called the *Q-spectrum* (resp. *L-spectrum*, *A-spectrum*) of G. For any bipartite graph, its Q-spectrum coincides with its L-spectrum. Two graphs are Q-cospectral (resp. L-cospectral, A-cospectral) if they have the same Q-spectrum (resp. L-spectrum, A-spectrum). A graph G is said to be DQS (resp. DLS, DAS) if there is no other non-isomorphic graph Q-cospectral (resp. L-cospectral, A-cospectral) with G. Let us denote the Q-spectrum of G by $\operatorname{Spec}_Q(G) = \{[q_1]^{m_1}, [q_2]^{m_2}, \ldots, [q_n]^{m_n}\}$, where m_i denotes the multiplicity of q_i and $q_1 \geq q_2 \geq \ldots \geq q_n$.

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The *join* of two graphs G and H is a graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H. We denote the join of two graphs G and H by $G\nabla H$. The complement of a graph G is denoted by $\overline{G}$. For two disjoint graphs G and H, let $G \sqcup H$ denotes the disjoint union of G and H, and rG denotes the disjoint union of r copies of G, i.e., $rG = \underbrace{G \sqcup \ldots \sqcup G}_{r-\text{times}}$.

Let G be a connected graph with n vertices and m edges. Then G is called *unicyclic* (resp. *bicyclic*) if $m = n$ (resp. $m = n + 1$). If G is a unicyclic graph containing an odd (resp. even) cycle, then G is called *odd unicyclic* (resp. *even unicyclic*).

Let C_n, P_n, K_n be the cycle, the path and the complete graph of order n, respectively. $K_{s,t}$ the complete bipartite graph with s vertices in one part and t in the other.

Let us remind that the *coalescence* [21] of two graphs G_1 with distinguished vertex v_1 and G_2 with distinguished vertex v_2, is formed by identifying vertices v_1 and v_2 that is, the vertices v_1 and v_2 are replaced by a single vertex v adjacent to the same vertices in G_1 as v_1 and the same vertices in G_2 as v_2. If it is not necessary v_1 or v_2 may not be specified.

The *friendship graph* F_n is a graph with $2n + 1$ vertices and $3n$ edges obtained by the coalescence of n copies of C_3 with a common vertex as the distinguished vertex; in fact, F_n is nothing but $K_1\nabla nK_2$.

The *lollipop graph*, denoted by $H_{n,p}$, is the coalescence of a cycle C_p with arbitrary distinguished vertex and a path P_{n-p} with a pendent vertex as the distinguished vertex; for example $H_{11,6}$ is depicted in Figure 1 (b). We denote by $T(a, b, c)$ the *T-shape* tree obtained by identifying the end vertices of three paths P_{a+2}, P_{b+2} and P_{c+2}. In fact, $T(a, b, c)$ is a tree with one and only one vertex v of degree 3 such that $T(a, b, c) - \{v\} = P_{a+1} \sqcup P_{b+1} \sqcup P_{c+1}$; for example $T(0, 1, 1)$ is depicted in Figure 1 (a).

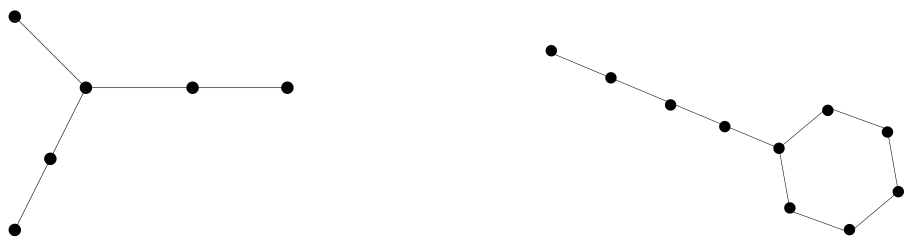

Figure 1: (a) The T-shape tree $T(0, 1, 1)$ (b) The lollipop graph $H_{11,6}$

A *kite graph* $Ki_{n,w}$ is a graph obtained from a clique K_w and a path P_{n-w} is the coalescence of K_w with an arbitrary distinguished vertex and a path P_{n-w+1} with a pendent vertex as the distinguished vertex. A tree is called *starlike* if it has exactly one vertex of degree greater than two. We denote by $U_{r,n-r}$ the graph obtained by attaching $n - r$ pendent vertices to a vertex of C_r. In fact, $U_{r,n-r}$ is the coalescence of $K_{1,n-r-1}$ and P_{n-w+1} where distinguished vertices are the vertex of degree $n - r$ and a pendent vertex, respectively. A graph is a *cactus*, or a *treelike graph*, if any pair of its cycles has at most one common vertex [35]. If all cycles of the cactus G have exactly one common vertex, then G is called a *bundle* [12]. Let $S(n, c)$ be the bundle with n vertices and c cycles of length 3 depicted in Figure 2, where $n \geq 2c + 1$ and $c \geq 0$. By the definition, it follows that $S(n, c) = K_1\nabla(cK_2 \sqcup (n - 2c - 1)K_1)$. In fact $S(n, c)$ is the coalescence of F_c and $K_{1,n-2c-1}$ where the distinguished vertices are the vertex of the degree $2c$ and the vertex of the degree $n - 2c - 1$, respectively.

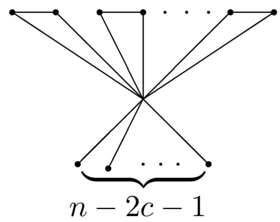

Figure 2: The bundle $S(n,c)$

Let G be a graph with n vertices, H be a graph with m vertices. The *corona* of G and H, denoted by $G \circ H$, is the graph with $n+mn$ vertices obtained from G and n copies of H by joining the i-th vertex of G to each vertex in the i-th copy of H $(i \in \{1,\ldots,n\})$; for example $C_4 \circ 2K_1$ is depicted in Figure 3.

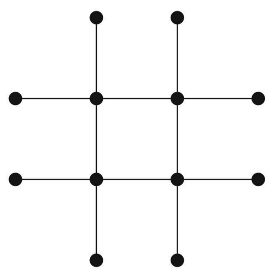

Figure 3: $C_4 \circ 2K_1$

A *complete split graph* $CS(n,\alpha)$, is a graph on n vertices consisting of a clique on $n-\alpha$ vertices and an independent set on the remaining α $(1 \le \alpha \le n-1)$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. The *dumbbell graph*, denoted by $D_{p,k,q}$, is a bicyclic graph obtained from two cycles C_p, C_q and a path P_{k+2} by identifying each pendant vertex of P_{k+2} with a vertex of a cycle, respectively. The *theta graph*, denoted by $\Theta_{r,s,t}$, is the graph formed by joining two given vertices via three disjoint paths P_r, P_s and P_t, respectively, see Figure 4.

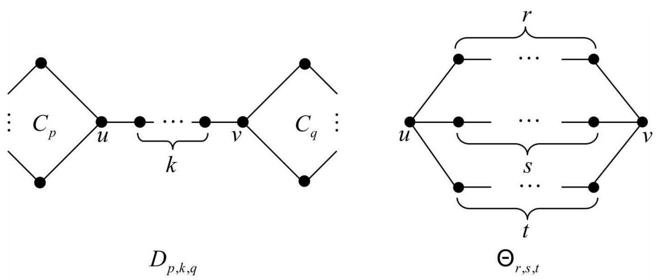

Figure 4: The graphs $D_{p,k,q}$ and $\Theta_{r,s,t}$

The problem "which graphs are determined by their spectrum?" was posed by Günthard and Primas [24] more than 60 years ago in the context of Hückel's theory in chemistry. In the most recent years mathematicians have devoted their attention to this problem and many papers focusing on this topic are now appearing. In [36] van Dam and Haemers conjectured that almost all graphs are determined by their spectra. Nevertheless, the set of graphs that are known to be determined by their spectra is too small. So, discovering infinite classes of graphs that are determined by their spectra can be an interesting problem. Cvetković, Rowlinson and Simić in [17–20] discussed the development of a spectral theory of graphs based on the signless Laplacian matrix, and gave several reasons why it is superior to other graph matrices such as the adjacency and the Laplacian matrix. It is interesting to construct new DQS (DLS) graphs from known DQS (DLS) graphs. Up to now, only some graphs with special structures are shown to be *determined by their spectra* (DS, for short) (see [1–11, 15, 17, 19, 22, 23, 25–34, 38–41] and the references cited in them). About the background of the question "Which graphs are determined by their spectrum?", we refer to [36, 37]. For a DQS graph G, $G\nabla K_2$ is also DQS under some conditions [30]. A graph is DLS if and only if its complement is DLS. Hence we can obtain DLS graphs from known DLS graphs by adding independent edges. In [25] it was shown that $G \sqcup rK_1$ is DQS under certain conditions. In this paper, we investigate signless Laplacian spectral characterization of graphs with independent edges. For a DQS graph G, we show that $G \sqcup rK_2$ is DQS under certain conditions. Applying these results, some DQS graphs with independent edges are obtained.

1 PRELIMINARIES

In this section, we give some lemmas which are used to prove our main results.

Lemma 1 ([17, 19]). *Let G be a graph. For the adjacency matrix of G, the following can be deduced from the spectrum.*

(1) *The number of vertices.*

(2) *The number of edges.*

(3) *Whether G is regular.*

For the Laplacian matrix, the following follows from the spectrum:

(4) *The number of components.*

For the signless Laplacian matrix, the following follow from the spectrum:

(5) *The number of bipartite components, i.e., the multiplicity of the eigenvalue 0 of the signless Laplacian matrix is equal to the number of bipartite components.*

(6) *The sum of the squares of degrees of vertices.*

Lemma 2 ([17]). *Let G be a graph with n vertices, m edges, t triangles and the vertex degrees $d_1, d_2, \ldots, d_n$. If $T_k = \sum_{i=1}^{n} q_i(G)^k$, then we have*

$$T_0 = n, \quad T_1 = \sum_{i=1}^{n} d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6t + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

For a graph G, let $P_L(G)$ and $P_Q(G)$ denote the product of all nonzero eigenvalues of L_G and Q_G, respectively. Note that $P_L(K_2) = P_Q(K_2) = 2$. We assume that $P_L(G) = P_Q(G) = 1$ if G has no edges.

Lemma 3 ([16]). *For any connected bipartite graph G of order n, we have $P_Q(G) = P_L(G) = n\tau(G)$, where $\tau(G)$ is the number of spanning trees of G. Especially, if T is a tree of order n, then $P_Q(T) = P_L(T) = n$.*

Lemma 4 ([32]). *Let G be a graph with n vertices and m edges.*

(i) $\det(Q_G) = 4$ *if and only if G is an odd unicyclic graph.*

(ii) *If G is a non-bipartite connected graph and $m > n$, then* $\det(Q_G) \geq 16$, *with equality if and only if G is a non-bipartite bicyclic graph with C_4 as its induced subgraph.*

Lemma 5 ([16]). *Let e be any edge of a graph G of order n. Then*

$$q_1(G) \geq q_1(G-e) \geq q_2(G) \geq q_2(G-e) \geq \ldots \geq q_n(G) \geq q_n(G-e) \geq 0.$$

Lemma 6 ([21]). *Let H be a proper subgraph of a connected graph G. Then $q_1(G) > q_1(H)$.*

Lemma 7 ([21]). *Let G be a graph with n vertices and m edges. Then $q_1(G) \geq \frac{4m}{n}$, with equality if and only if G is regular.*

Lemma 8 ([17]). *For a graph G, $0 < q_1(G) < 4$ if and only if all components of G are paths.*

Lemma 9 ([36]). *A regular graph is* DQS *if and only if it is* DAS. *A regular graph G is* DAS (DQS) *if and only if $\overline{G}$ is* DAS (DQS).

Lemma 10 ([19]). *Let G be a k-regular graph of order n. Then G is* DAS *when $k \in \{0, 1, 2, n-3, n-2, n-1\}$.*

Lemma 11 ([15]). *Let G be a k-regular graph of order n. Then $G \nabla K_1$ is* DQS *for $k \in \{1, n-2\}$, for $k = 2$ and $n \geq 11$. For $k = n-3$, $G \nabla K_1$ is* DQS *if and only if $\overline{G}$ has no triangles.*

Lemma 12 ([30]). *Let G be a k-regular graph of order n. Then $G \nabla K_2$ is* DQS *for $k \in \{1, n-2\}$. For $k = n-3$, $G \nabla K_2$ is* DQS *if and only if $\overline{G}$ has no triangles.*

Lemma 13 ([25]). *The following hold for graphs with isolated vertices:*

(i) *Let T be a* DLS *tree of order n. Then $T \sqcup rK_1$ is* DLS. *If n is not divisible by 4, then $T \sqcup rK_1$ is* DQS.

(ii) *The graphs $\overline{P_n}$ and $\overline{P_n} \sqcup rK_1$ are* DQS.

(iii) *Let G be a graph obtained from K_n by deleting a matching. Then G and $G \sqcup rK_1$ are* DQS.

(iv) *A ($n-4$)-regular graph of order n is* DAS (DQS) *if and only if its complement is a 3-regular* DAS (DQS) *graph.*

(v) *Let G be a ($n-3$)-regular graph of order n. Then $G \sqcup rK_1$ is* DQS.

Now let us list some known families of DQS graphs.

Lemma 14. *The following graphs are* DQS.

(i) *The graphs* P_n, C_n, K_n, $K_{m,m}$, rK_n, $P_{n_1} \sqcup P_{n_2} \sqcup \ldots \sqcup P_{n_k}$ *and* $C_{n_1} \sqcup C_{n_2} \sqcup \ldots \sqcup C_{n_k}$*, [36].*

(ii) *Any wheel graph* $K_1 \nabla C_n$*, [26].*

(iii) *Every lollipop graph* $H_{n,p}$*, [41].*

(iv) *Every kite graph* $Ki_{n,n-1}$ *for* $n \geq 4$ *and* $n \neq 5$*, [23].*

(v) *The friendship graph* F_n*, [38].*

(vi) $(C_n \circ tK_1)$*, for* $n \notin \{32, 64\}$ *and* $t \in \{1, 2\}$*, [14,32].*

(vii) *The line graph of a* T*-shape tree* $T(a, b, c)$ *except* $T(t, t, 2t + 1)$ $(t > 1)$*, [39].*

(viii) *The starlike tree with maximum degree 4, [34].*

(ix) $U_{r,n-r}$ *for* $r \geq 3$*, [27].*

(x) $CS(n, \alpha)$ *when* $1 \leq \alpha \leq n - 1$ *and* $\alpha \neq 3$*, [22].*

(xi) *For* $n \geq 2c + 1$ *and* $c \geq 0$*,* $\overline{S(n, c)}$ *and* $S(n, c)$ *except for the case of* $c = 0$ *and* $n = 4$*, [29].*

(xii) $K_{1,n-1}$ *for* $n \neq 4$*, [29].*

(xiii) $G \nabla K_m$ *where* G *is an* $(n - 2)$*-regular graph on* n *vertices, and* $\overline{K_n} \nabla K_2$ *except for* $n = 3$*, [28].*

(xiv) *All dumbbell graphs different from* $D_{3q,0,q}$ *and all theta graphs, [40].*

It is easy to see that $K_{1,3}$ and $K_3 \sqcup K_1$ are Q-cospectral, i.e., $\mathrm{Spec}_Q(K_{1,3}) = \mathrm{Spec}_Q(K_3) = \{[4]^1, [1]^2, [0]^1\}$. Therefore, $S(n, c)$ is not DQS when $c = 0$ and $n = 4$, since $S(n, 0)$ is nothing but $K_{1,n-1}$.

2 MAIN RESULTS

We first investigate spectral characterizations of the union of a tree and several complete graphs K_2.

Theorem 1. *Let* T *be a* DLS *tree of order* n*. Then* $T \sqcup rK_2$ *is* DLS *for any positive integer* r*. Moreover, if* n *is odd and* $r = 1$*, then* $T \sqcup rK_2$ *is* DQS.

Proof. For $n, r \in \{1, 2\}$ see Lemma 13 (i) and Lemma 14 (i). So, one may suppose that $n, r \geq 3$. Let G be any graph L-cospectral with $T \sqcup rK_2$. By Lemma 1, G has $n + 2r$ vertices, $n - 1 + r$ edges and $r + 1$ components. So each component of G is a tree. Suppose that $G = G_0 \sqcup G_1 \sqcup \ldots \sqcup G_r$, where G_i is a tree with n_i vertices and $n_0 \geq n_1 \geq \ldots \geq n_r \geq 2$. For $n_i, n_r \in \{1\}$ see Lemma 13 (i) and Lemma 14 (i). Hence we consider $n, n_i, r \geq 2$. Since G is L-cospectral with $T \sqcup rK_2$, by Lemma 3, we get

$$n_0 n_1 \ldots n_r = P_L(G_0) \ldots P_L(G_r) = P_L(G_0 \sqcup \ldots \sqcup G_r) = P_L(G) = P_L(T) P_L(K_2)^r = n2^r.$$

We claim that $n_r = 2$. Suppose not and so $n_r \geq 3$. This means that $n_0 \geq n_1 \geq \ldots \geq n_r \geq 3$. Hence $n2^r = n_0 n_1 \ldots n_r \geq 3^{r+1}$ or $n(\frac{2}{3})^r \geq 3$. Now, if $r \to \infty$, then $0 \geq 3$, a contradiction. So, we must have $n_r = 2$. By a similar argument one can show that $n_1 = \ldots = n_{r-1} = 2$ and so $n_0 = n$. Hence $G = G_0 \sqcup rK_2$. Since G and $T \sqcup rK_2$ are L-cospectral, G_0 and T are L-cospectral. Since T is DLS, we have $G_0 = T$, and thus $G = T \sqcup rK_2$. Hence $T \sqcup rK_2$ is DLS.

Let H be any graph Q-cospectral with $T \sqcup rK_2$. By Lemma 1, H has $n + 2r$ vertices, $n - 1 + r$ edges and $r + 1$ bipartite components. So one of the following holds:

(i) H has exactly $r + 1$ components, and each component of H is a tree.

(ii) H has $r + 1$ components which are trees, the other components of H are odd unicyclic.

In what follows we show that (ii) does not occur if n is odd and $r = 1$. If (ii) holds, then by Lemma 4, $P_Q(H)$ is divisible by 4 since H has a cycle of odd order as a component. Since T is a tree of order n, by Lemma 3, $P_Q(H) = P_Q(T)P_Q(K_2)^r = n2^r$ is divisible by 4, a contradiction. Therefore (i) must hold. In this case, H and $T \sqcup rK_2$ are both bipartite, and so they are also L-cospectral. By the previous part, $T \sqcup rK_2$ is DLS. So we have $H = T \sqcup rK_2$.

Hence $T \sqcup rK_2$ is DQS when n is odd and $r = 1$. □

Remark 1. *Some* DLS *trees are given in [25] and references therein. We can obtain some* DLS *(*DQS*) trees with independent edges from Theorem 1.*

Lemma 14 and Theorem 1 imply the following corollary.

Corollary 1. *For an odd positive integer n, we have the following*

(i) *Let T be a starlike tree of order n and with maximum degree 4. Then $T \sqcup K_2$ is* DQS*.*

(ii) *$P_n \sqcup K_2$ is* DQS*.*

(iii) *For $n \neq 4$, $K_{1,n-1} \sqcup K_2$ is* DQS*.*

(iv) *Let $\mathcal{L}$ be the line graph of a T-shape tree $T(a,b,c)$ except $T(t,t,2t+1)$ $(t > 1)$. Then $\mathcal{L} \sqcup K_2$ is* DQS *if $a + b + c - 3$ is odd.*

Theorem 2. *Let G be a* DQS *odd unicyclic graph of order $n \geq 7$. Then $G \sqcup rK_2$ is* DQS *for any positive integer r.*

Proof. Let H be any graph Q-cospectral with $G \sqcup rK_2$. By Lemma 1(5), 0 is not an eigenvalue of G since it is an odd unicyclic. So by Lemma 4, we have $4 = \det(Q_G) = P_Q(G)$. Moreover,

$$P_Q(H) = P_Q(G \sqcup rK_2) = P_Q(G)P_Q(K_2)^r = \det(Q_G)2^r = 4 \cdot 2^r = 2^{r+2}.$$

By Lemma 1, H has $n + 2r$ vertices, $n + r$ edges and r bipartite components. So one of the following holds:

(i) H has exactly r components each of which is a tree.

(ii) H has r components which are trees, the other components of H are odd unicyclic.

We claim that (i) does not hold, otherwise, we may assume that $H = H_1 \sqcup \ldots \sqcup H_r$, where H_i is a tree with n_i vertices and $n_1 \geq \ldots \geq n_r \geq 1$. It follows from Lemma 3 that

$$n_1 \ldots n_r = P_Q(H_1) \ldots P_Q(H_r) = P_Q(H) = 4 \cdot 2^r = 2^{r+2}.$$

So $n_1 \ldots n_r = 2^{r+2}$, $n_1 \leq 8$. Since G contains a cycle, say C, by Lemma 7 we have

$$q_1(H) = q_1(G) \geq q_1(C) = 4. \tag{1}$$

Let $\Delta(H)$ be the maximum degree of H. If $\Delta(H) \leq 2$, then all components of H are paths, hence by Lemma 8, $q_1(H) < 4$, contradicting Eq. (1). So $\Delta(H) \geq 3$. From $n_1 \leq 8$ and $n_1 \ldots n_r = 4 \cdot 2^r = 2^{(r+2)}$, we may assume that $H_1 = K_{1,7}$, $H_2 = \ldots = H_r = K_2$. Since $H = K_{1,7} \sqcup (r-1)K_2$ has $n + 2r$ vertices, we get $n = 6$, a contradiction to $n \geq 7$.
If (ii) holds, then we may assume that $H = U_1 \sqcup \ldots \sqcup U_c \sqcup H_1 \sqcup \ldots \sqcup H_r$, where U_i is odd unicyclic, H_i is a tree with n_i vertices. By Lemmas 3 and 4, $4 \cdot 2^r = P_Q(H) = 4^c n_1 \ldots n_r$. So $c = 1$, $H_1 = \ldots = H_r = K_2$. Since $H = U_1 \sqcup rK_2$ and $G \sqcup rK_2$ are Q-cospectral, U_1 and G are Q-cospectral. Since G is DQS, we have $U_1 = G$, $H = G \sqcup rK_2$. □

Remark 2. *Note that $C_4 \sqcup 2P_3$ and $C_6 \sqcup 2K_2$ are Q-cospectral, i.e.,* $\mathrm{Spec}_Q(C_4 \sqcup 2P_3) = \mathrm{Spec}_Q(C_6 \sqcup 2K_2) = \{[4]^1, [3]^2, [2]^2, [1]^2, [0]^3\}$. *It follows that the condition "odd unicyclic of order $n \geq 7$" is essential in Theorem 2.*

Remark 3. *Some* DQS *unicyclic graphs are given in [25] and references therein. We can obtain some* DQS *graphs with independent edges from Theorem 2.*

Theorem 3. *Let G be a* DQS *graph of order $n \geq 5$. If G is non-bipartite bicyclic graph with C_4 as its induced subgraph, then $G \sqcup rK_2$ is* DQS *for any positive integer r.*

Proof. Let H be any graph Q-cospectral with $G \sqcup rK_2$. By Lemma 4, we have

$$P_Q(H) = P_Q(G \sqcup rK_2) = P_Q(G) P_Q(K_2)^r = P_Q(G) 2^r.$$

By Lemma 1(5), 0 is not an eigenvalue of G since it is non-bipartite. So by Lemma 4, we have $16 = \det(G_Q) = P_Q(G)$ and thus $P_Q(H) = 16 \cdot 2^r$.

By Lemma 1, H has $n + 2r$ vertices, $n + 1 + r$ edges and r bipartite components. So H has at least $r - 1$ components which are trees. Suppose that $H_1, H_2, \ldots, H_r$ are r bipartite components of H, where $H_2, \ldots, H_r$ are trees. If H_1 contains an even cycle, then by Lemmas 4 and 5, we have $P_Q(H) \geq P_Q(H_1) \geq 16$, and $P_Q(H) = 16 \cdot (2^{r-1}) = 2^{r-3}$ if and only if $H = C_4 \sqcup (r-1)K_2$. By $P_Q(H) = 16 \cdot (2^{r-1}) = 2^{r-3}$, we have $H = C_4 \sqcup (r-1)K_2$. Since H has $n + 2r$ vertices, we get $n = 2$, a contradiction (G contains C_4). Hence $H_1, H_2, \ldots, H_r$ are trees. Since H has $n + 2r$ vertices, $n + 1 + r$ edges and r bipartite components, H has a non-bipartite component H_0 which is a bicyclic graph. Lemmas 4 and 5 imply that $P_Q(H) \geq P_Q(H_0) \geq 16$, and $P_Q(H) = 16 \cdot 2^r$ if and only if $H = H_0 \sqcup rK_2$ and H_0 contains C_4 as its induced subgraph. By $P_Q(H) = 16 \cdot 2^r$, we have $H = H_0 \sqcup rK_2$. Since H and $G \sqcup rK_2$ are Q-cospectral, H_0 and G are Q-cospectral. Taking into account that G is DQS, we conclude that $H_0 = G$ and $H = G \sqcup rK_2$. Hence $G \sqcup rK_2$ is DQS. □

Remark 4. *Some* DQS *bicyclic graphs are given in [25] and references therein. We can obtain* DQS *graphs with independent edges from Theorem 3.*

Lemma 15. *Let G be a connected graph. Then there is no subgraph of G with the Q-spectrum identical to* $\mathrm{Spec}_Q(G) \cup \left\{[2]^1\right\}$. *Moreover, if G is of order at least 3, then $q_1(G) \geq 3$.*

Proof. Suppose by the contrary that there is a subgraph of G, say G', such that $\mathrm{Spec}_Q(G') = \mathrm{Spec}_Q(G) \cup \left\{[2]^1\right\}$. But, in this case $|E(G')| = |E(G)| + 1$ and $|V(G')| = |V(G)| + 1$. Therefore there exists a vertex v of G' with the degree one such that $G' - v = G$. This means that G is a proper subgraph of the connected graph G' and so by Lemma 6, $q_1(G') > q_1(G)$, a contradiction. If G is a connected graph of order at least 3, it has K_3 or $K_{1,2}$ as its subgraph. Moreover, $\mathrm{Spec}_Q(K_3) = \{[4], [1]^2\}$ and $\mathrm{Spec}_Q(K_{1,2}) = \{[3], [1], [0]\}$. Therefore by Lemma 5, $q_1(G) \geq 3$. □

Theorem 4. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices which is* DQS. *Then for any positive integer r, $G \sqcup rK_2$ is* DQS.

Proof. Let H be a graph Q-cospectral with $G \sqcup rK_2$. Then by Lemmas 1 and 2, H has $n + 2r$ vertices, $n + 1 + r$ edges and exactly r bipartite components. We perform mathematical induction on r. Suppose that H is a graph Q-cospectral with $G \sqcup K_2$. Then

$$\mathrm{Spec}_Q(H) = \mathrm{Spec}_Q(G) \cup \mathrm{Spec}_Q(K_2) = \mathrm{Spec}_Q(G) \cup \left\{[2]^1, [0]^1\right\}.$$

Since G is a connected non-bipartite graph, by Lemma 1, it has not 0 as its signless Laplacian eigenvalue. Therefore, H has exactly one bipartite component. Therefore, by Lemma 15 we get $H = G \sqcup K_2$. Now, let the assertion holds for r; that is, if $\mathrm{Spec}_Q(G_1) = \mathrm{Spec}_Q(G) \cup \mathrm{Spec}_Q(rK_2)$, then $G_1 = G \sqcup rK_2$. We show that it follows from $\mathrm{Spec}_Q(K) = \mathrm{Spec}_Q(G) \cup \mathrm{Spec}_Q((r+1)K_2)$ that $K = G \sqcup (r+1)K_2$. Obviously, K has 2 vertices, one edge and one bipartite component more than G_1. So, we must have $K = G_1 \sqcup K_2$. Now, the inductive hypothesis holds the proof. □

Lemma 11 and Theorem 4 imply the following corollary.

Corollary 2. *For a k-regular graph G of order n, $(G \nabla K_1) \sqcup rK_2$ is* DQS *if either of the following conditions holds:*

(i) $k \in \{1, n-2\}$,

(ii) $k = 2$ *and* $n \geq 11$,

(iii) $k = n - 3$ *and* $\overline{G}$ *has no triangles.*

Lemma 12 and Theorem 4 imply the following corollary.

Corollary 3. *Let G be a k-regular graph of order n. Then $(G \nabla K_2) \sqcup rK_2$ is* DQS *for $k \in \{1, n-2\}$. For $k = n-3$, $(G \nabla K_2) \sqcup rK_2$ is* DQS *if $\overline{G}$ has no triangles.*

Lemma 13 and Theorem 4 imply the following corollary.

Corollary 4. *Let G be a non-bipartite graph obtained from K_n by deleting a matching. Then $G \sqcup rK_2$ is* DQS.

Remark 5. *Some* 3*-regular* DAS *graphs are given in [25] and references therein. We can obtain* DQS *graphs with independent edges from Corollary 4.*

Lemmas 9 and 10 and Theorem 4 imply the following corollary.

Corollary 5. *Let G be a k-regular connected non-bipartite graph of order n. Then $G \sqcup rK_2$ is* DQS *if either of the following holds*

(i) $k \in \{2, n-1, n-2, n-3\}$.

(ii) $k = n-4$ *and G is* DAS.

Lemma 14 and Theorem 4 imply the following corollary.

Corollary 6. *Let G be any of the following graphs. Then $G \sqcup rK_2$ is* DQS.

(i) *The graphs C_n (n is odd), K_n ($n \geq 4$).*

(ii) *The graphs $\overline{P_n}$ ($n \geq 5$).*

(iii) *The wheel graph $K_1 \nabla C_n$.*

(iv) *Every lollipop graph $H_{n,p}$ when p is odd and $n \geq 8$.*

(v) *The kite graph $Ki_{n,n-1}$ for $n \geq 4$ and $n \neq 5$.*

(vi) *The friendship graph F_n.*

(vii) $(C_n \circ tK_1)$, *when n is odd and $n \notin \{32, 64\}$ and $t \in \{1, 2\}$.*

(viii) $U_{r,n-r}$ *if $r (\geq 3)$ is odd and $n \geq 7$.*

(ix) $CS(n, \alpha)$ *when $1 \leq \alpha \leq n-1$ and $\alpha \neq 3$.*

(x) $S(n, c)$ *and its complement where $n \geq 2c+1$ and $c \geq 1$.*

(xi) $H \nabla K_m$ *where H is an $(n-2)$-regular graph on n vertices, and $\overline{K_n} \nabla K_2$ except for $n = 3$.*

(xii) *The dumbbell graphs $D_{p,k,q}$ (p or q is odd) different from $D_{3q,0,q}$ and all non-bipartite theta graphs $\Theta_{r,s,t}$.*

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Шарафдіні Р., Абдіан А.З. *Беззнакові лапласіанові визначення деяких графів з незалежними вершинами* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 185–196.

Нехай G простий ненапрямлений граф. Тоді беззнакова лапласіанова матриця G визначається як $D_G + A_G$, де D_G і A_G позначають матрицю степенів і матрицю суміжності графу G відповідно. Граф G називають визначеним своїм беззнаковим лапласіановим спектром (скорочення DQS), якщо будь-який граф, що має такий самий беззнаковий лапласіановий спектр як G, є ізоморфним до G. У роботі показано, що $G \sqcup rK_2$ визначений своїм беззнаковим лапласіановим спектром за певних умов, де r і K_2 позначають натуральне число і повний граф на двох вершинах відповідно. Застосовуючи ці результати ми отримали деякі DQS графи з незалежними вершинами.

Ключові слова і фрази: спектральна характеризація, беззнаковий лапласіановий спектр, коспектральні графи, об'єднання графів.