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THE GROWTH OF THE MAXIMAL TERM OF DIRICHLET SERIES

Let Λ be the class of nonnegative sequences (λ_n) increasing to $+\infty$, $A \in (-\infty, +\infty]$, L_A be the class of continuous functions increasing to $+\infty$ on a half-closed interval of the form $[A_0, A)$, and $F(s) = \sum a_n e^{s\lambda_n}$ be a Dirichlet series such that its maximum term $\mu(\sigma, F) = \max_n |a_n|e^{\sigma\lambda_n}$ is defined for every $\sigma \in (-\infty, A)$. It is proved that for all functions $\alpha \in L_{+\infty}$ and $\beta \in L_A$ the equality

$$\rho^*_{\alpha,\beta}(F) = \max_{(\eta_n)\in\Lambda} \overline{\lim_{n\to\infty}} \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}$$

holds, where $\rho^*_{\alpha,\beta}(F)$ is the generalized α, β-order of the function $\ln\mu(\sigma, F)$, i.e. $\rho^*_{\alpha,\beta}(F) = 0$ if the function $\mu(\sigma, F)$ is bounded on $(-\infty, A)$, and $\rho^*_{\alpha,\beta}(F) = \overline{\lim}_{\sigma\uparrow A}\, \alpha(\ln\mu(\sigma, F))/\beta(\sigma)$ if the function $\mu(\sigma, F)$ is unbounded on $(-\infty, A)$.

Key words and phrases: Dirichlet series, maximal term, central index, generalized order.

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1 INTRODUCTION

We denote by $\mathbb{N}_0$ the class of nonnegative integer numbers, and let Λ be the class of nonnegative increasing sequences $\lambda = (\lambda_n)_{n\in\mathbb{N}_0}$ tending to $+\infty$.

Let $\lambda \in \Lambda$. We consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \tag{1}$$

and set

$$\sigma^*(F) = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \ln\frac{1}{|a_n|}, \qquad E(F) = \{\sigma \in \mathbb{R} : |a_n|e^{\sigma\lambda_n} = o(1), n \to \infty\}.$$

It is easy to see that

$$\sigma^*(F) = \begin{cases} -\infty, & \text{if} \quad E(F) = \varnothing; \\ \sup E(F), & \text{if} \quad E(F) \neq \varnothing. \end{cases}$$

If $\sigma^*(F) > -\infty$, then for all $\sigma \in (-\infty; \sigma^*(F))$ we define the maximal term and central index of the series F respectively by

$$\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \in \mathbb{N}_0\}, \qquad \nu(\sigma, F) = \max\{n \in \mathbb{N}_0 : |a_n|e^{\sigma\lambda_n} = \mu(\sigma, F)\}.$$

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Let $A \in (-\infty, +\infty]$, and $\alpha : D_\alpha \to \mathbb{R}$ be a real function. We say that $\alpha \in L_A$ if two following conditions are fulfilled: (i) the domain D_α of α is a half-closed interval of the form $[A_0, A)$; (ii) the function α is continuous and increasing to $+\infty$ on D_α. If $\alpha \in L_A$ and $A \le x \le +\infty$, then we assume that $\alpha(x) = +\infty$.

For a given $A \in (-\infty, +\infty]$ and $\lambda \in \Lambda$ we denote by $\mathcal{D}^*_A(\lambda)$ the class of Dirichlet series of the form (1) such that $\sigma^*(F) \ge A$ and put $\mathcal{D}^*_A = \cup_{\lambda\in\Lambda}\mathcal{D}^*_A(\lambda)$.

Let $\alpha \in L_{+\infty}$, $\beta \in L_A$ and $F \in \mathcal{D}^*_A$. If the function $\mu(\sigma, F)$ is bounded on $(-\infty, A)$, we set $\rho^*_{\alpha,\beta}(F) = 0$; if the function $\mu(\sigma, F)$ is unbounded on $(-\infty, A)$, we put

$$\rho^*_{\alpha,\beta}(F) = \varlimsup_{\sigma\uparrow A} \frac{\alpha(\ln \mu(\sigma, F))}{\beta(\sigma)}.$$

Let p be a positive constant. Under some conditions on functions $\alpha, \beta \in L_{+\infty}$, Sheremeta [1] proved that

$$\rho^*_{\alpha,\beta}(F) = \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)} \tag{2}$$

for every Dirichlet series $F \in \mathcal{D}^*_{+\infty}$ of the form (1). Note that without additional conditions on functions $\alpha, \beta \in L_{+\infty}$ formula (1) is false in general (see e.g. [2, 3]).

The following theorem indicates a formula for calculating $\rho^*_{\alpha,\beta}(F)$ in the case of arbitrary $A \in (-\infty, +\infty]$, $\alpha \in L_{+\infty}$, $\beta \in L_A$, and $F \in \mathcal{D}^*_A$.

Theorem 1. *Let $A \in (-\infty, +\infty]$, $\alpha \in L_{+\infty}$, $\beta \in L_A$. Then for every Dirichlet series $F \in \mathcal{D}^*_A$ of the form* (1) *we have*

$$\rho^*_{\alpha,\beta}(F) = \max_{\eta\in\Lambda} \varlimsup_{n\to\infty} \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}.$$

It can easily be shown that Theorem 1 is equivalent to the following theorem.

Theorem 2. *Let $A \in (-\infty, +\infty]$, $\alpha \in L_{+\infty}$, $\beta \in L_A$. Then for every Dirichlet series $F \in \mathcal{D}^*_A$ of the form* (1) *we have*

$$\rho^*_{\alpha,\beta}(F) = \varlimsup_{n\to\infty} \sup_{x\in D_\alpha} \frac{\alpha(x)}{\beta\left(\frac{x}{\lambda_n} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}.$$

2 PROOF OF THEOREM 1

For a sequence $\eta \in \Lambda$ set $k(\eta) = \varlimsup_{n\to\infty} \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}$.

Consider a Dirichlet series $F \in \mathcal{D}^*$ of the form (1) and prove that $k(\eta) \le \rho^*_{\alpha,\beta}(F)$. If $\rho^*_{\alpha,\beta}(F) = +\infty$ it is trivial. Assume that $\rho^*_{\alpha,\beta}(F) < +\infty$, and let $\rho > \rho^*_{\alpha,\beta}(F)$ be a constant. Then

$$\ln \mu(\sigma) \le \alpha^{-1}(\rho\beta(\sigma)), \quad \sigma \in [\sigma_0, A).$$

Hence, for every $n \in \mathbb{N}_0$ we have $\ln |a_n| \le \alpha^{-1}(\rho\beta(\sigma)) - \lambda_n\sigma, \quad \sigma \in [\sigma_0, A)$. Therefore, using the notation $\sigma_n = \beta^{-1}\left(\frac{1}{\rho}\alpha(\eta_n)\right)$ for all $n \ge n_0$ we obtain

$$\ln |a_n| \le \alpha^{-1}(\rho\beta(\sigma_n)) - \lambda_n\sigma_n = \eta_n - \lambda_n\beta^{-1}\left(\frac{1}{\rho}\alpha(\eta_n)\right),$$

and this can also be written as

$$\rho \geq \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)}, \quad n \geq n_0.$$

This yields the inequality $k(\eta) \leq \rho$. Since $\rho > \rho^*_{\alpha,\beta}(F)$ is an arbitrary, we obtain $k(\eta) \leq \rho^*_{\alpha,\beta}(F)$.

It remains to show that there exists a sequence $\eta \in \Lambda$ such that $k(\eta) \geq \rho^*_{\alpha,\beta}(F)$. If $\rho^*_{\alpha,\beta}(F) = 0$, then $k(\eta) \geq \rho^*_{\alpha,\beta}(F)$ for every sequence $\eta \in \Lambda$ as is proved above. Let $\rho^*_{\alpha,\beta}(F) > 0$, and $(\rho_k)_{k\in\mathbb{N}_0}$ be a positive sequence that increase to $\rho^*_{\alpha,\beta}(F)$. Then it follows from the definition of $\rho^*_{\alpha,\beta}(F)$ that there exists a sequence $(\sigma_k)_{k\in\mathbb{N}_0}$ increasing to A such that the sequence $(\nu(\sigma_k, F))_{k\in\mathbb{N}_0}$ is also increasing and

$$\alpha(\ln\mu(\sigma_k)) \geq \rho_k\beta(\sigma_k), \quad k \in \mathbb{N}_0.$$

Let $n_k = \nu(\sigma_k, F)$, $k \in \mathbb{N}_0$. Consider a sequence $\eta \in \Lambda$ such that $\eta_{n_k} = \alpha^{-1}(\rho_k\beta(\sigma_k))$, $k \geq k_0$. Then for every $k \geq k_0$ we have

$$\ln|a_{n_k}| + \lambda_{n_k}\beta^{-1}\left(\frac{1}{\rho_k}\alpha(\eta_{n_k})\right) = \ln|a_{n_k}| + \lambda_{n_k}\sigma_k = \ln\mu(\sigma_k) \geq \alpha^{-1}(\rho_k\beta(\sigma_k)) = \eta_{n_k}.$$

This yields $\rho_k < \dfrac{\alpha(\eta_{n_k})}{\beta\left(\frac{\eta_{n_k}}{\lambda_{n_k}} + \frac{1}{\lambda_{n_k}}\ln\frac{1}{|a_{n_k}|}\right)}$ for all sufficiently large k.

Therefore, $k(\eta) \geq \overline{\lim}_{k\to\infty}\rho_k = \rho^*_{\alpha,\beta}(F)$. Theorem 1 is proved.

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Нехай Λ — клас невід'ємних зростаючих до $+\infty$ послідовностей (λ_n), $A \in (-\infty, +\infty]$, L_A — клас неперервних зростаючих до $+\infty$ функцій, заданих на напіввідкритому інтервалі вигляду $[A_0, A)$, а $F(s) = \sum a_n e^{s\lambda_n}$ — ряд Діріхле такий, що його максимальний член $\mu(\sigma, F) = \max_n |a_n|e^{\sigma\lambda_n}$ є визначеним для всіх $\sigma \in (-\infty, A)$. В роботі доведено, що для довільних функцій $\alpha \in L_{+\infty}$ і $\beta \in L_A$ правильна рівність

$$\rho^*_{\alpha,\beta}(F) = \max_{(\eta_n)\in\Lambda}\overline{\lim_{n\to\infty}}\frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)},$$

де $\rho^*_{\alpha,\beta}(F)$ — узагальнений α,β-порядок функції $\ln\mu(\sigma, F)$, тобто $\rho^*_{\alpha,\beta}(F) = 0$, якщо функція $\mu(\sigma, F)$ обмежена на $(-\infty, A)$, і $\rho^*_{\alpha,\beta}(F) = \overline{\lim}_{\sigma\uparrow A}\alpha(\ln\mu(\sigma, F))/\beta(\sigma)$, якщо функція $\mu(\sigma, F)$ необмежена на $(-\infty, A)$.

Ключові слова і фрази: ряд Діріхле, максимальний член, центральний індекс, узагальнений порядок.