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VASYLYSHYN T.V.

METRIC ON THE SPECTRUM OF THE ALGEBRA OF ENTIRE SYMMETRIC FUNCTIONS OF BOUNDED TYPE ON THE COMPLEX L_∞

It is known that every complex-valued homomorphism of the Fréchet algebra $H_{bs}(L_\infty)$ of all entire symmetric functions of bounded type on the complex Banach space L_∞ is a point-evaluation functional δ_x (defined by $\delta_x(f) = f(x)$ for $f \in H_{bs}(L_\infty)$) at some point $x \in L_\infty$. Therefore, the spectrum (the set of all continuous complex-valued homomorphisms) M_{bs} of the algebra $H_{bs}(L_\infty)$ is one-to-one with the quotient set $L_\infty/_\sim$, where an equivalence relation "$\sim$" on L_∞ is defined by $x \sim y \Leftrightarrow \delta_x = \delta_y$. Consequently, M_{bs} can be endowed with the quotient topology. On the other hand, M_{bs} has a natural representation as a set of sequences which endowed with the coordinate-wise addition and the quotient topology forms an Abelian topological group. We show that the topology on M_{bs} is metrizable and it is induced by the metric $d(\xi, \eta) = \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n - \eta_n|}$, where $\xi = \{\xi_n\}_{n=1}^\infty, \eta = \{\eta_n\}_{n=1}^\infty \in M_{bs}$.

Key words and phrases: symmetric function, spectrum of the algebra.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: taras.v.vasylyshyn@gmail.com

INTRODUCTION

Symmetric functions on Banach spaces were studied by a number of authors [1, 3–8, 10, 12, 13] (see also a survey [2]). In particular, symmetric polynomials and symmetric analytic functions on L_∞ (see definition below) were studied in [6, 12, 13].

Let L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm $\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]}|x(t)|$.

Let Ξ be the set of all measurable bijections of $[0,1]$ that preserve the measure. A function $f : L_\infty \to \mathbb{C}$ is called symmetric if $f(x \circ \sigma) = f(x)$ for every $x \in L_\infty$ and for every $\sigma \in \Xi$.

Let $H_{bs}(L_\infty)$ be the Fréchet algebra of all entire symmetric functions $f : L_\infty \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets. By [6, Theorem 4.3], polynomials $R_n : L_\infty \to \mathbb{C}$, $R_n(x) = \int_{[0,1]}(x(t))^n\,dt$ for $n \in \mathbb{N}$ form an algebraic basis in the algebra of all symmetric continuous polynomials on L_∞. Since every $f \in H_{bs}(L_\infty)$ can be described by its Taylor series of continuous symmetric homogeneous polynomials, it follows that f can be uniquely represented as

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x)\cdots R_n^{k_n}(x).$$

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Consequently, for every non-trivial continuous homomorphism $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$, taking into account $\varphi(1) = 1$, we have

$$\varphi(f) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \varphi(R_1)^{k_1} \cdots \varphi(R_n)^{k_n}.$$

Therefore, φ is completely determined by the sequence of its values on $R_n : (\varphi(R_1), \varphi(R_2), \ldots)$. By the continuity of φ, the sequence $\{\sqrt[n]{|\varphi(R_n)|}\}_{n=1}^{\infty}$ is bounded. On the other hand, we have the following

Theorem 1 ([6, Section 3]). *For every sequence $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty$, there exists $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$ and $\|x_\xi\|_\infty \le \frac{2}{M} \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|}$, where*

$$M = \prod_{n=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{1}{n+1}\right). \tag{1}$$

Hence, for every sequence $\xi = \{\xi_n\}_{n=1}^{\infty}$ such that $\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty$, there exists the point-evaluation functional $\varphi = \delta_{x_\xi}$ such that $\varphi(R_n) = \xi_n$ for every $n \in \mathbb{N}$. Since every such a functional is a continuous homomorphism, it follows that the spectrum (the set of all continuous complex-valued homomorphisms) of the algebra $H_{bs}(L_\infty)$, which we denote by M_{bs}, can be identified with the set of all sequences $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded.

Let $\nu : L_\infty \to M_{bs}$ be defined by

$$\nu(x) = (R_1(x), R_2(x), \ldots).$$

Let τ_∞ be the topology on L_∞, generated by $\|\cdot\|_\infty$. Let us define an equivalence relation on L_∞ by $x \sim y \Leftrightarrow \nu(x) = \nu(y)$. Let τ be the quotient topology on M_{bs} :

$$\tau = \{\nu(V) : V \in \tau_\infty\}.$$

Note that ν is a continuous open mapping.

The operation of coordinate-wise addition $+ : M_{bs}^2 \to M_{bs}$ is defined by

$$a + b = (a_1 + b_1, a_2 + b_2, \ldots)$$

for $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in M_{bs}$. In [13] it is shown that $(M_{bs}, +, \tau)$ is an Abelian topological group. In this work we show that (M_{bs}, τ) is a metrizable topological space. Also we explicitly construct the metric which induces τ.

1 THE MAIN RESULT

Let us denote $B(x, r)$ the open ball of radius r and center x in L_∞.

Proposition 1. *The identity element $0 = (0, 0, \ldots)$ of the topological group $(M_{bs}, +, \tau)$ has a countable local basis of neighborhoods.*

Proof. For $n \in \mathbb{N}$ let $U_n = \nu(B(0, \frac{1}{n}))$. Since ν is an open mapping, it follows that $U_n \in \tau$. Note that $0 \in U_n$. Thus, U_n is an open neighborhood of 0 for every $n \in \mathbb{N}$. Let us show that a family $\{U_n : n \in \mathbb{N}\}$ form a local basis of neighborhoods of 0. Let $W \subset M_{bs}$ be an arbitrary open neighborhood of 0. Then $\nu^{-1}(W)$ is open in L_∞ and $\nu^{-1}(W)$ contains 0. Therefore, there exists $r > 0$ such that $B(0, r) \subset \nu^{-1}(W)$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} < r$. Then $B(0, \frac{1}{n}) \subset B(0, r) \subset \nu^{-1}(W)$. Therefore, $\nu(B(0, \frac{1}{n})) \subset W$, i. e. $U_n \subset W$. □

We will use Birkhoff-Kakutani theorem.

Theorem 2 ([9, p.34]). *Let G be a Hausdorff topological group whose open sets at the identity element have a countable basis. Then G is metrizable and, moreover, there exists a metric which is right-invariant.*

Corollary 1. *There exists an invariant metric d on M_{bs} which induces topology τ.*

Proof. By [13, Corollary 1], $(M_{bs}, +, \tau)$ is an Abelian topological group. By [13, Theorem 2], τ is Hausdorff. By Proposition 1, the identity element of M_{bs} has a countable local basis. Therefore by Theorem 2 there exists a right-invariant metric d on M_{bs} which induces topology τ. Since $(M_{bs}, +, \tau)$ is Abelian, the metric d is also left-invariant. □

For $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots) \in M_{bs}$ let

$$d_I(a, b) = \sup_{n\in\mathbb{N}} \sqrt[n]{|a_n - b_n|}.$$

Note that analogical metric is defined on spaces of entire functions of one complex variable (where a role of sequences a and b play sequences of coefficients of the Taylor series of functions) and it is called Iyer metric (see e. g. [11]). Also note that a metric space (M_{bs}, d_I) is isometric to the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ of the exponential type such that $f(0) = 0$ with Iyer metric.

Let $V(a, r)$ be the open ball in M_{bs} of radius r and center $a \in M_{bs}$ with respect to the metric d_I.

Lemma 1. *Let $r > 0$ and $0 < \rho < \frac{Mr}{2}$, where M is defined by* (1)*. Then $V(0, \rho) \subset \nu(B(0, r))$.*

Proof. Let $a = (a_1, a_2, \ldots) \in V(0, \rho)$. Let us show that $a \in \nu(B(0, r))$. By Theorem 1, there exists $x_a \in L_\infty$ such that $\nu(x_a) = a$ and $\|x_a\|_\infty < \frac{2}{M} \sup_{n\in\mathbb{N}} \sqrt[n]{|a_n|}$. Since $a \in V(0, \rho)$, it follows that $d_I(0, a) < \rho$, i. e. $\sup_{n\in\mathbb{N}} \sqrt[n]{|a_n|} < \rho$. Thus, $\|x_a\|_\infty < \frac{2}{M}\rho$. Since $\rho < \frac{Mr}{2}$, it follows that $\|x_a\|_\infty < r$, i. e. $x_a \in B(0, r)$. Therefore $\nu(x_a) \in \nu(B(0, r))$, i. e. $a \in \nu(B(0, r))$. □

Theorem 3. *The metric d_I induces the topology τ.*

Proof. Since both metrics d_I and d (given by Corollary 1) are invariant with respect to translations (in the sense that $d(a + c, b + c) = d(a, b)$ for every $a, b, c \in M_{bs}$), it suffices to prove that every open neighborhood of 0 with respect to τ contains some open ball with center 0 with respect to d_I and vice versa.

Let $W \in \tau$ such that $0 \in W$. Then $\nu^{-1}(W)$ is the open neighborhood of 0 in L_∞. Therefore, there exists $r > 0$ such that $B(0, r) \subset \nu^{-1}(W)$. By Lemma 1, for $0 < \rho < \frac{2r}{M}$ we have $V(0, \rho) \subset \nu(B(0, r))$. Since $\nu(B(0, r)) \subset W$, it follows that $V(0, \rho) \subset W$.

Let us show that for every open ball $V(0, r)$ there exists $W \in \tau$ such that $0 \in W$ and $W \subset V(0, r)$. Set $W = \nu(B(0, r))$. Let us show that $W \subset V(0, r)$. It suffices to prove that $\nu(x) \in V(0, r)$ for every $x \in B(0, r)$. For $x \in B(0, r)$ we have $\|x\|_\infty < r$ and, consequently,

$$|R_n(x)| \le \|x\|_\infty^n < r^n.$$

Therefore

$$d_I(0, \nu(x)) = \sup_{n\in\mathbb{N}} \sqrt[n]{|R_n(x)|} < r.$$

Thus, $\nu(x) \in V(0, r)$. □

REFERENCES

[1] Aron R., Galindo P., Pinasco D., Zalduendo I. *Group-symmetric holomorphic functions on a Banach space*. Bull. London Math. Soc. 2016, **48** (5), 779–796. doi:10.1112/blms/bdw043

[2] Chernega I. *Symmetric Polynomials and Holomorphic Functions on infinite dimensional spaces*. Journal of Vasyl Stefanyk Precarpathian National University 2015, **2** (4), 23–49. doi:10.15330/jpnu.2.4.23-49

[3] Chernega I., Galindo P., Zagorodnyuk A. *Some algebras of symmetric analytic functions and their spectra*. Proc. Edinburgh Math. Soc. 2012, **55** (1), 125–142. doi:10.1017/S0013091509001655

[4] Chernega I., Galindo P., Zagorodnyuk A. *The convolution operation on the spectra of algebras of symmetric analytic functions*. J. Math. Anal. Appl. 2012, **395** (2), 569–577. doi:10.1016/j.jmaa.2012.04.087

[5] Chernega I., Galindo P., Zagorodnyuk A. *A multiplicative convolution on the spectra of algebras of symmetric analytic functions*. Revista Matemática Complutense 2014, **27** (2), 575–585. doi:10.1007/s13163-013-0128-0

[6] Galindo P., Vasylyshyn T., Zagorodnyuk A. *The algebra of symmetric analytic functions on L_∞*. Proc. Roy. Soc. Edinburgh Sect. A 2017, **147** (4), 743–761. doi:10.1017/S0308210516000287

[7] González M., Gonzalo R., Jaramillo J. A. *Symmetric polynomials on rearrangement invariant function spaces*. J. London Math. Soc. 1999, **59** (2), 681–697. doi:10.1112/S0024610799007164

[8] Kravtsiv V., Vasylyshyn T., Zagorodnyuk A. *On algebraic basis of the algebra of symmetric polynomials on $\ell_p(\mathbb{C}^n)$*. J. Funct. Spaces 2017, **2017**, Article ID 4947925, 8 pages. doi:10.1155/2017/4947925

[9] Montgomery D., Zippin L. Topological transformation groups. Interscience Publishers, New York, 1955.

[10] Nemirovskii A. S., Semenov S. M. *On polynomial approximation of functions on Hilbert space*. Mat. USSR Sbornik 1973, **21** (2), 255–277. doi:10.1070/SM1973v021n02ABEH002016

[11] Sisarcick W. C. *Metric spaces of entire functions*. Indian J. Pure Appl. Math. 1975, **6** (6), 628–636.

[12] Vasylyshyn T. *Symmetric continuous linear functionals on complex space $L_\infty[0,1]$*. Carpathian Math. Publ. 2014, **6** (1), 8–10. doi:10.15330/cmp.6.1.8-10.

[13] Vasylyshyn T. *Topology on the spectrum of the algebra of entire symmetric functions of bounded type on the complex L_∞*. Carpathian Math. Publ. 2017, **9** (1), 22–27. doi:10.15330/cmp.9.1.22-27

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Василишин Т.В. *Метрика на спектрі алгебри цілих симетричних функцій обмеженого типу на комплексному просторі L_∞* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 198–201.

Відомо, що кожен комплекснозначний гомоморфізм алгебри Фреше $H_{bs}(L_\infty)$ усіх цілих симетричних функцій обмеженого типу на комплексному банаховому просторі L_∞ є функціоналом обчислення значення в точці δ_x (визначеного як $\delta_x(f) = f(x)$ для $f \in H_{bs}(L_\infty)$) у деякій точці $x \in L_\infty$. Тому спектр (множина усіх неперервних комплекснозначних гомоморфізмів) M_{bs} алгебри $H_{bs}(L_\infty)$ є у взаємно однозначній відповідності із фактор-множиною $L_\infty/_\sim$, де відношення еквівалентності "$\sim$" на просторі L_∞ визначене наступним чином: $x \sim y \Leftrightarrow \delta_x = \delta_y$. Як наслідок, на M_{bs} можна задати фактор-топологію. З іншого боку, для M_{bs} існує природне подання у вигляді множини послідовностей, яка разом із заданими на ній операцією покоординатного додавання і фактор-топологією утворює абелеву топологічну групу. У статті доведено, що топологія на M_{bs} є метризовною і породжується метрикою $d(\xi,\eta) = \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n - \eta_n|}$, де $\xi = \{\xi_n\}_{n=1}^\infty, \eta = \{\eta_n\}_{n=1}^\infty \in M_{bs}$.

Ключові слова і фрази: симетрична функція, спектр алгебри.