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SOME FIXED POINT RESULTS IN COMPLETE GENERALIZED METRIC SPACES

The Banach contraction principle is the important result, that has many applications. Some authors were interested in this principle in various metric spaces. Branciari A. initiated the notion of the generalized metric space as a generalization of a metric space by replacing the triangle inequality by more general inequality, $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ for all pairwise distinct points x, y, u, v of X. As such, any metric space is a generalized metric space but the converse is not true. He proved the Banach fixed point theorem in such a space. Some authors proved different types of fixed point theorems by extending the Banach's result. Wardowski D. introduced a new contraction which generalizes the Banach contraction. Using a mapping $F : \mathbb{R}^+ \to \mathbb{R}$ he introduced a new type of contraction called F-contraction and proved a new fixed point theorem concerning F-contraction.

In this paper, we have dealt with F-contraction and F-weak contraction in complete generalized metric spaces. We prove some results for F-contraction and F-weak contraction and we establish the existence and uniqueness of fixed point for F-contraction and F-weak contraction in complete generalized metric spaces. Some examples are supplied in order to support the usability of our results. The obtained result is an extension and a generalization of many existing results in the literature.

Key words and phrases: F-contraction, F-weak contraction, generalized metric space.

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INTRODUCTION AND PRELIMINARIES

The Banach contraction principle is the simplest result in fixed point theory [4]. This principle has many applications and was extended by several authors (see [5–10, 12, 14–17, 19, 20]). Some authors gave the fundamental linear contractive conditions and the fundamental nonlinear contractive conditions by using the notion of F-contraction, and proved fixed point theorems which generalize Banach contraction principle.

Due to the nature of mathematics science, there have been many attempts to generalize the metric setting by modifying some of the axioms of metric spaces. Thus, several other types of spaces have been introduced and a lot of metric results have been extended to new settings. One of the interesting generalizations of the notion of metric space was introduced by Branciari A. Later, most of the authors dealing with such spaces made some additional requirements in order to deduce their results (see [1–3]).

In this paper, we prove fixed point theorems for F-contraction and F-weak contraction in complete generalized metric spaces. We also present uniqueness of the fixed point.

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Definition 1 ([13]). *Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:*

(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then X is called a generalized metric space.

The concepts of convergence, Cauchy sequence, completeness, and continuity on a generalized metric space are defined below.

Definition 2 ([1]). *Let (X, d) be a generalized metric space.*

(i) A sequence $\{x_n\}$ is called convergent to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we use the notation $x_n \to x$.

(ii) A sequence $\{x_n\}$ is called Cauchy if and only if for each $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.

(iii) A generalized metric space (X, d) is called complete if every Cauchy sequence is convergent in X.

(iv) A mapping $T : (X, d) \to (X, d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

Lemma 1 ([11]). *Let (X, d) be a generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then the sequence $\{x_n\}$ can converge to at most one point.*

Lemma 2 ([11]). *Let (X, d) be a generalized metric space and let $\{x_n\}$ be a sequence in X which is both Cauchy and convergent. Then the limit x of $\{x_n\}$ is unique. Moreover, if $z \in X$ is arbitrary, then $\lim_{n\to\infty} d(x_n, z) = d(x, z)$.*

Theorem 1 ([13]). *Let (X, d) be a complete generalized metric space and suppose the mapping $f : X \to X$ satisfies $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$ and fixed $k \in (0, 1)$. Then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for each $x \in X$.*

Definition 3 ([18]). *Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \longrightarrow \mathbb{R}$ such that:*

(F1) F is strictly increasing, that is, for all $\alpha, \beta \in (0, +\infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;

(F2) for each sequence $\{\alpha_n\}$ of positive numbers, the following holds: $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 4 ([18]). *Let (X, d) be a metric space. A map $T : X \to X$ is said to be an F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$*

$$\text{from } d(Tx, Ty) > 0 \text{ follows that } \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{1}$$

Theorem 2 ([18]). *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then*

(1) T has a unique fixed point x^*;

(2) for all $x \in X$ the sequence $\{T^n x\}$ is convergent to x^*.

Remark 1 ([18]). *Let T be an F-contraction. Then $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ such that $Tx \neq Ty$. Also, T is a continuous map.*

1 THE MAIN RESULTS

In this paper, we prove fixed point theorems for F-contraction and F-weak contraction in complete generalized metric spaces. We also present uniqueness of the fixed point.

Theorem 3. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be an F-contraction. If F is continuous, then*

(1) T has a unique fixed point $x^* \in X$;

(2) for all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.

Proof. Let $x_0 \in X$ be an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \text{ for all } n \in \mathbb{N}. \tag{2}$$

If there exists $n \in \mathbb{N}$, $x_n = x_{n+1}$, the proof is complete. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step 1. We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (1), we obtain

$$\tau + F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n)),$$

i.e., $F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n)) - \tau$. Repeating this process, we get

$$\begin{aligned} F(d(Tx_{n-1}, Tx_n)) &\leq F(d(x_{n-1}, x_n)) - \tau = F(d(Tx_{n-2}, Tx_{n-1})) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau = F(d(Tx_{n-3}, Tx_{n-2})) - 2\tau \\ &\leq F(d(x_{n-3}, x_{n-2})) - 3\tau \leq F(d(x_0, x_1)) - n\tau. \end{aligned} \tag{3}$$

From (3), we obtain $\lim_{n\to\infty} F(d(Tx_{n-1}, Tx_n)) = -\infty$, which together with (F2) and Definition 3 gives $\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0$, which implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4}$$

Step 2. We will prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. By (1), we have

$$\begin{aligned} F(d(Tx_{n-1}, Tx_{n+1})) &\leq F(d(x_{n-1}, x_{n+1})) - \tau = F(d(Tx_{n-2}, Tx_n)) - \tau \\ &\leq F(d(x_{n-2}, x_n)) - 2\tau = F(d(Tx_{n-3}, Tx_{n-1})) - 2\tau \\ &\leq F(d(x_{n-3}, x_{n-1})) - 3\tau \leq F(d(x_0, x_2)) - n\tau. \end{aligned} \tag{5}$$

From (5) we obtain $\lim_{n\to\infty} F(d(Tx_{n-1}, Tx_{n+1})) = -\infty$, which together with (F2) and Definition 3 gives $\lim_{n\to\infty} d(Tx_{n-1}, Tx_{n+1}) = 0$, which implies that,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{6}$$

Step 3. We will prove that $x_n \neq x_m$ for all $m \neq n$. We argue by contradiction. Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Since $d(x_p, x_{p+1}) > 0$, for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1$. Consider now

$$\begin{aligned}F(d(x_n, x_{n+1})) &= F(d(x_n, Tx_n)) = F(d(x_m, Tx_m)) = F(d(Tx_{m-1}, Tx_m)) \\ &\leq F(d(x_{m-1}, x_m)) - \tau \leq F(d(x_{n+1}, x_n)) - (m-n)\tau.\end{aligned}$$

It is a contradiction.

Step 4. We will show that in this case $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary. Then, there is an $\varepsilon > 0$ such that for an integer k, there exist natural numbers $m(k) > n(k) > k$ such that

$$d(x_{n(k)}, x_{m(k)}) > \varepsilon. \tag{7}$$

For every integer k let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (7), we get

$$d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon. \tag{8}$$

Now, using (7), (8) and the quadrilateral inequality, we find that

$$\begin{aligned}\varepsilon < d(x_{m(k)}, x_{n(k)}) &\leq d(x_{m(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + \varepsilon.\end{aligned}$$

Then, by (4) and (6), it follows that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{9}$$

Applying (1) with $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$, we have

$$F(d(x_{m(k)}, x_{n(k)})) = F(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \leq F(d(x_{m(k)-1}, x_{n(k)-1})) - \tau.$$

If $k \to \infty$ in the above inequality and using (9) we obtain$F(\varepsilon) \leq F(\varepsilon) - \tau$.

This contradiction shows that $\{x_n\}$ is a Cauchy sequence. (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{10}$$

Since T is continuous, we obtain from (10) that

$$\lim_{n\to\infty} d(x_{n+1}, Tx^*) = \lim_{n\to\infty} d(Tx_n, Tx^*) = 0.$$

That is $\lim\limits_{n\to\infty} x_{n+1} = Tx^*$. Taking into account Lemma 2 we conclude that $Tx^* = x^*$. That is x^* is a fixed point of T. Now, let us to show that T has at most one fixed point. Indeed if $x,y \in X$ be two distinct fixed points of T, that is, $Tx = x \neq y = Ty$. Therefore $d(Tx, Ty) = d(x, y) > 0$, then we get

$$F(d(x, y)) = F(d(Tx, Ty)) < \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

which is a contradiction. Therefore, the fixed point is unique. □

Definition 5. *Let (X, d) be a generalized metric space. A map $T : X \to X$ is said to be an F-weak contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$*

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(\max\{d(x, y), d(x, Tx), d(y, Ty)\}). \tag{11}$$

Remark 2. *Every F-contraction is an F-weak contraction on (X, d). But the converse is not true.*

Example 1. *Let $X = A \cup B$, where $A = \{1,2,3,4\}, B = [5,6]$. Define the generalized metric d on X as follows:*

$$\begin{aligned}
&d(x,y) = 0, \quad x = y \text{ and } x,y \in A,\\
&d(1,2) = d(3,4) = 2,\ d(1,3) = d(2,3) = 1,\ d(1,4) = d(2,4) = 5,\\
&d(x,y) = |x-y|\,,\ \text{for } x \in A, y \in B \text{ or } x \in B, y \in A \text{ or } x,y \in B.
\end{aligned}$$

It is easy to show that (X,d) is a complete generalized metric space, but (X,d) is not a metric space because d does not satisfy the triangle inequality for all $x,y,z \in X$. Indeed,

$$5 = d(1,4) > d(1,3) + d(3,4) = 1 + 2 = 3.$$

Let $T : X \to X$ be given by

$$Tx = \begin{cases} 3 & if \quad x \in A, \\ 1 & if \quad x \in B. \end{cases}$$

Since T is not continuous, T is not F-contraction by Remark 1. For $x \in A$ and $y \in B$, we have

$$d(Tx,Ty) = d(3,1) = 1 > 0$$

and $\max\{d(x,y), d(x,Tx), d(y,Ty)\} \geq 4$. *Therefore, by choosing $F\alpha = \ln\alpha$, $\alpha \in (0,+\infty)$ and $\tau = \ln 3$, we see that T is F -weak contraction.*

Theorem 4. *Let (X,d) be a complete generalized metric space and $T : X \to X$ be an F-weak contraction. If T or F is continuous, then*

(1) T has a unique fixed point $x^* \in X$;

(2) for all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.

Proof. Let $x_0 \in X$ be an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \ \text{ for all } n \in \mathbb{N}.$$

If there exists $n \in \mathbb{N}$, $x_n = x_{n+1}$, the proof is complete. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step 1. We will prove that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (11), we obtain

$$\begin{aligned}
F(d(x_{n+1},x_n)) &= F(d(Tx_n, Tx_{n-1}))\\
&\leq F(\max\{d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_{n-1})\}) - \tau\\
&= F(\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n)\}) - \tau\\
&= F(\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\}) - \tau.
\end{aligned} \tag{12}$$

If there exists $n \in \mathbb{N}$ such that $\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, from (12) becomes

$$F(d(x_{n+1},x_n)) \leq F(d(x_{n+1},x_n)) - \tau < F(d(x_{n+1},x_n)).$$

It is a contradiction. Therefore,

$$\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\} = d(x_n,x_{n-1}) \tag{13}$$

for all $n \in \mathbb{N}$. That is from (F1), (12) and (13), we get

$$d(x_n, x_{n+1}) < d(x_n, x_{n-1}). \tag{14}$$

Thus, from (12), we have $F(d(x_{n+1}, x_n)) \leq F(d(x_n, x_{n-1})) - \tau$ for all $n \in \mathbb{N}$. It implies that

$$F(d(x_{n+1}, x_n)) \leq F(d(x_1, x_0)) - n\tau \tag{15}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in (15), we get $\lim_{n\to\infty} F(d(x_{n+1}, x_n)) = -\infty$ that together with (F2) gives

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{16}$$

Step 2. We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{17}$$

By (11), we have

$$\begin{aligned} F(d(x_n, x_{n+2})) &= F(d(Tx_{n-1}, Tx_{n+1})) \\ &\leq F(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\}) - \tau \\ &= F(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}) - \tau. \end{aligned} \tag{18}$$

By (14) and from (F2), we have

$$\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\} = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}.$$

Take $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Thus, from (18)

$$\begin{aligned} F(a_n) &= F(d(x_n, x_{n+2})) = F(d(Tx_{n-1}, Tx_{n+1})) \\ &\leq F(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\}) - \tau \\ &= F(\max\{a_{n-1}, b_{n-1})\}) - \tau. \end{aligned} \tag{19}$$

Again, by (14) $b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}$. Therefore $\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}$, for all $n \in \mathbb{N}$. Then the sequence $\{\max\{a_n, b_n\}\}$ is monotone nonincreasing, so it converges to some $t \geq 0$. Assume that $t > 0$. Now, by (16)

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\} = t.$$

Taking $n \to \infty$ in (19), since F is continuous,

$$\begin{aligned} F(t) = \lim_{n\to\infty} \sup F(a_n) &\leq \lim_{n\to\infty} \sup(F(\max\{a_{n-1}, b_{n-1}\}) - \tau) \\ &\leq \lim_{n\to\infty} F(\max\{a_{n-1}, b_{n-1}\}) - \tau = F(t) - \tau, \end{aligned}$$

which is a contradiction, that is (17) is proved.

Step 3. We will prove that $x_n \neq x_m$ for all $m \neq n$.

We argue by contradiction. Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Since $d(x_p, x_{p+1}) > 0$, for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1$. Consider now

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(x_n, Tx_n)) = F(d(x_m, Tx_m)) = F(d(Tx_{m-1}, Tx_m)) \\ &\leq F(\max\{d(x_{m-1}, x_m), d(x_{m-1}, Tx_{m-1}), d(x_m, Tx_m)) - \tau \\ &= F(\max\{d(x_{m-1}, x_m), d(x_{m-1}, x_m), d(x_m, x_{m+1})\}) - \tau \\ &= F(\max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\}) - \tau. \end{aligned} \tag{20}$$

If $\max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\} = d(x_{m-1}, x_m)$, then from (20), we get

$$F(d(x_n, x_{n+1})) \leq F(d(x_{m-1}, x_m)) - \tau \leq F(d(x_n, x_{n+1})) - (m-n)\tau.$$

It is a contradiction. If $\max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\} = d(x_m, x_{m+1})$, then from (20), we get $F(d(x_n, x_{n+1})) \leq F(d(x_m, x_{m+1})) - \tau \leq F(d(x_n, x_{n+1})) - (m-n+1)\tau$. It is a contradiction.

Step 4. We will prove that $\{x_n\}$ is a Cauchy sequence, that is

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ for all } p \in \mathbb{N}.$$

From (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty}((d(x_{n+1}, x_n))^k F(d(x_{n+1}, x_n))) = 0. \tag{21}$$

By using (15) and from (21), we have

$$(d(x_{n+1}, x_n))^k (F(d(x_{n+1}, x_n)) - F(d(x_1, x_0))) \leq -(d(x_{n+1}, x_n))^k n\tau \leq 0 \tag{22}$$

for all $n \in \mathbb{N}$. By using (16), (21) and taking the limit as $n \to \infty$ in (22), we get

$$\lim_{n\to\infty}(n(d(x_{n+1}, x_n))^k) = 0. \tag{23}$$

Then there exists $n_1 \in \mathbb{N}$ such that $n(d(x_{n+1}, x_n))^k \leq 1$ for all $n \geq n_1$, that is

$$d(x_{n+1}, x_n) \leq \frac{1}{n^{\frac{1}{k}}}. \tag{24}$$

From (16) and (17) the cases $p = 1$ and $p = 2$ are proved. Now, take $p \geq 3$ arbitrary. It is sufficient to examine two cases.

Case 1. Suppose that $p = 2m + 1$ where $m \geq 1$. Then, by using step 3 and the quadrilateral inequality together with (24), we get

$$\begin{aligned} d(x_n, x_{n+p}) = d(x_n, x_{n+2m+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+2m}, x_{n+2m+1}) \\ &\leq \sum_{i=n}^{n+2m} d(x_{i+1}, x_i) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{25}$$

Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ is convergent, taking the limit as $n \to \infty$ in the above inequality, we obtain $\lim\limits_{n\to\infty} d(x_n, x_{n+p}) = 0$.

Case 2. Suppose that $p = 2m$ where $m \geq 2$. Then, by using step 3 and the quadrilateral inequality together with (24), we get

$$\begin{aligned} d(x_n, x_{n+p}) = d(x_n, x_{n+2m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+2m-1}, x_{n+2m}) \\ &\leq \sum_{i=n}^{n+2m-1} d(x_{i+1}, x_i) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{26}$$

Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ is convergent, taking the limit as $n \to \infty$ in the above inequality, we obtain $\lim\limits_{n\to\infty} d(x_n, x_{n+p}) = 0$.

This proves that $\{x_n\}$ is Cauchy sequence in X. Since X is complete, there exists x^*, that is a fixed point of T by two following cases.

Case 3. T is continuous. We have $d(x^*, Tx^*) = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. This proves that x^* is a fixed point of T.

Case 4. F is continuous. In this case, we consider two following subcases.

Subcase 1. For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_{n+1}} = Tx^*$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then we have

$$x^* = \lim_{n\to\infty} x_{i_{n+1}} = \lim_{n\to\infty} Tx^* = Tx^*.$$

This proves that x^* is a fixed point of T.

Subcase 2. There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$ for all $n \geq n_0$. That is $d(Tx_n, Tx^*) > 0$ for all $n \geq n_0$. It follows from (11) that

$$\begin{aligned} \tau + F(d(x_{n+1}, Tx^*)) &= \tau + F(d(Tx_n, Tx^*)) \leq F(\max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*)\}) \\ &= F(\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*)\}). \end{aligned} \tag{27}$$

If $d(x^*, Tx^*) > 0$ then by the fact

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(x^*, x_{n+1}) = 0,$$

there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have $\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*)\} = d(x^*, Tx^*)$. From (27), we get

$$\tau + F(d(x_{n+1}, Tx^*)) = F(d(x^*, Tx^*)), \tag{28}$$

for all $n \geq \max\{n_0, n_1\}$. Since F is continuous, taking the limit as $n \to \infty$ in (28), we obtain

$$\tau + F(d(x^*, Tx^*)) = F(d(x^*, Tx^*)).$$

It is contradiction. Therefore, $d(x^*, Tx^*) = 0$, that is, x^* is a fixed point of T. By two above cases, T has a fixed point x^*. Now, we prove that the fixed point of T is unique. Let x_1^*, x_2^* be two fixed points of T. Suppose to the contrary that $x_1^* \neq x_2^*$. Then $Tx_1^* \neq Tx_2^*$. It follows from (11) that

$$\begin{aligned} \tau + F(d(x_1^*, x_2^*)) &= \tau + F(d(Tx_1^*, Tx_2^*)) \leq F(\max\{d(x_1^*, x_2^*), d(x_1^*, Tx_1^*), d(x_2^*, Tx_2^*)\}) \\ &= F(\max\{d(x_1^*, x_2^*), d(x_1^*, x_1^*), d(x_2^*, x_2^*)\}) = F(d(x_1^*, x_2^*)). \end{aligned}$$

It is a contradiction. Then $d(x_1^*, x_2^*) = 0$, that is $x_1^* = x_2^*$. This proves that the fixed point of T is unique.

It follows from the proof of Theorem 4 that $\lim_{n\to\infty} T^n x = \lim_{n\to\infty} x_{n+1} = x^*$. □

Example 2. *Let F be given as in Example 1. Then T is an F-weak contraction. Therefore, Theorem 4 can be applicable to T and the unique fixed point of T is 3.*

Example 3. *Let $X = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$. Define the generalized metric d on X as follows:*

$$d(x,y) = 0, \quad x = y \text{ and } x, y \in X,$$
$$d\Big(\frac{1}{2}, \frac{2}{3}\Big) = d\Big(\frac{3}{4}, \frac{4}{5}\Big) = 0,2,\ d\Big(\frac{1}{2}, \frac{4}{5}\Big) = d\Big(\frac{2}{3}, \frac{3}{4}\Big) = 0,3,\ d\Big(\frac{1}{2}, \frac{3}{4}\Big) = d\Big(\frac{2}{3}, \frac{4}{5}\Big) = 0,6.$$

It is easy to show that (X,d) is a complete generalized metric space, but (X,d) is not a metric space because d does not satisfy the triangle inequality for all $x,y,z\in X$. Indeed,

$$0,6=d\Big(\frac{1}{2},\frac{3}{4}\Big)\geq d\Big(\frac{1}{2},\frac{2}{3}\Big)+d\Big(\frac{2}{3},\frac{3}{4}\Big)=0,2+0,3=0,5.$$

Let $T:X\to X$ be defined as follows:

$$Tx=\begin{cases}\frac{3}{4}, & x\in\{\frac{1}{2},\frac{2}{3},\frac{3}{4}\},\\ \frac{2}{3}, & x=\frac{4}{5}.\end{cases}$$

Let $F\alpha=\ln\alpha$, $\alpha\in(0,+\infty)$ and $\tau=\ln\frac{3}{2}$. Then, for $x\in\{\frac{1}{2},\frac{2}{3},\frac{3}{4}\}$ and $y=\frac{4}{5}$, we get

$$\begin{aligned}F(0,45)&=F\Big(d\Big(T\Big(\frac{1}{2}\Big),T\Big(\frac{4}{5}\Big)\Big)\Big)+\ln\frac{3}{2}\\ &\leq F\Big(\max\Big\{d\Big(\frac{1}{2},\frac{4}{5}\Big),d\Big(\frac{1}{2},T\Big(\frac{1}{2}\Big)\Big),d\Big(\frac{4}{5},T\Big(\frac{4}{5}\Big)\Big)\Big\}\Big)=F(0,6),\\ F(0,45)&=F\Big(d\Big(T\Big(\frac{2}{3}\Big),T\Big(\frac{4}{5}\Big)\Big)\Big)+\ln\frac{3}{2}\\ &\leq F\Big(\max\Big\{d\Big(\frac{2}{3},\frac{4}{5}\Big),d\Big(\frac{2}{3},T\Big(\frac{2}{3}\Big)\Big),d\Big(\frac{4}{5},T\Big(\frac{4}{5}\Big)\Big)\Big\}\Big)=F(0,6),\\ F(0,45)&=F\Big(d\Big(T\Big(\frac{3}{4}\Big),T\Big(\frac{4}{5}\Big)\Big)\Big)+\ln\frac{3}{2}\\ &\leq F\Big(\max\Big\{d\Big(\frac{3}{4},\frac{4}{5}\Big),d\Big(\frac{3}{4},T\Big(\frac{3}{4}\Big)\Big),d\Big(\frac{4}{5},T\Big(\frac{4}{5}\Big)\Big)\Big\}\Big)=F(0,6).\end{aligned}$$

Therefore, T is a F-weak contraction in generalized metric space. That is, Theorem 4 can be applicable to T and the unique fixed point of T is $\dfrac{3}{4}$.

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Санґурлу С.М., Тюркоґлу Д. *Деякі тереми про фіксовану точку в повних узагальнених метричних просторах* // Карпатські матем. публ. — 2017. — Т.9, №2. — C. 171–180.

Принцип стискуючих відображень є важливим результатом, що має багато застосувань. Деякі автори цікавились цим принципом в різних метричних просторах. Бранчіарі А. ввів поняття узагальненого метричного простору, замінивши нерівність трикутника більш загальною нерівністю $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ для всіх попарно різних точок x, y, u, v з X. Таким чином, будь-який метричний простір є узагальненим метричним простором, але не навпаки. Він довів теорему Банаха про фіксовану точку в таких просторах. Деякі автори довели різні типи теорем про фіксовану точку, розширюючи результат Банаха. Так Вардовський Д. представив новий вид стискуючих відображень, який узагальнює поняття стискуючого відображення Банаха. Використовуючи відображення $F : \mathbb{R}^+ \to \mathbb{R}$, він ввів новий тип стискуючих відображень, які називаються F-стиском. Також він довів теорему про фіксовану точку для F-стиску.

У даній роботі ми розглянули F-стиск та слабкий F-стиск у повних узагальнених метричних просторах. Доведено деякі результати для F-стисків і слабких F-стисків і встановлено існування та єдиність фіксованої точки для F-стискуючих і слабких F-стискуючих відображень у повних узагальних метричних просторах. Наведено деякі приклади для ілюстрації використання отриманих результатів. Дані результати є розширенням і узагальненням багатьох отриманих у літературі результатів.

Ключові слова і фрази: F-стиск, слабкий F-стиск, узагальнений метричний простір.