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THE STRUCTURE OF SOLUTIONS OF THE MATRIX LINEAR UNILATERAL POLYNOMIAL EQUATION WITH TWO VARIABLES

We investigate the structure of solutions of the matrix linear polynomial equation $A(\lambda)X(\lambda) + B(\lambda)Y(\lambda) = C(\lambda)$, in particular, possible degrees of the solutions. The solving of this equation is reduced to the solving of the equivalent matrix polynomial equation with matrix coefficients in triangular forms with invariant factors on the main diagonals, to which the matrices $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are reduced by means of semiscalar equivalent transformations. On the basis of it, we have pointed out the bounds of the degrees of the matrix polynomial equation solutions. Necessary and sufficient conditions for the uniqueness of a solution with a minimal degree are established. An effective method for constructing minimal degree solutions of the equations is suggested. In this article, unlike well-known results about the estimations of the degrees of the solutions of the matrix polynomial equations in which both matrix coefficients are regular or at least one of them is regular, we have considered the case when the matrix polynomial equation has arbitrary matrix coefficients $A(\lambda)$ and $B(\lambda)$.

Key words and phrases: matrix polynomial equation, solution of equation, semiscalar equivalence of polynomial matrices.

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INTRODUCTION

Let $\mathcal{F}$ be a field and $\mathcal{F}[\lambda]$ be a polynomial ring over $\mathcal{F}$. The matrix linear polynomial equations

$$A(\lambda)X(\lambda) + B(\lambda)Y(\lambda) = C(\lambda), \tag{1}$$

$$A(\lambda)X(\lambda) + Y(\lambda)B(\lambda) = C(\lambda), \tag{2}$$

where $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are known, $X(\lambda)$ and $Y(\lambda)$ are unknown $m \times m$ matrices over ring $\mathcal{F}[\lambda]$, find application in the dynamical systems theory, the optimal control theory and in other areas [6, 7, 12–14].

It is clear, that if equations (1) and (2) are solvable, then they have solutions of unlimited on top degrees. Therefore, when we describe the solutions of such equations, it is important to establish their minimal degrees. Some estimations of the degrees of the solutions of the matrix polynomial equation (2) are known in [1, 5, 9]. In [1], it has been established that if in the matrix polynomial equation (2) both matrices $A(\lambda), B(\lambda)$ are regular, then there exists a solution $X(\lambda)$, $Y(\lambda)$, such that

$$\deg X(\lambda) < \deg B(\lambda), \quad \deg Y(\lambda) < \deg A(\lambda) \tag{3}$$

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and it is unique if and only if

$$\deg C(\lambda) \leq \deg A(\lambda) + \deg B(\lambda) - 1 \quad \text{and} \quad (\det A(\lambda), \det B(\lambda)) = 1.$$

In [5], this result has been extended for the matrix equation (2) if at least one of the matrices $A(\lambda)$ or $B(\lambda)$ is regular. We don't know similar estimates of the degrees of the solutions of the matrix polynomial equation (1).

In [2, 8], the matrix linear unilateral and bilateral equations in the form (1) and (2) over other domains have been studied.

In [3], we have obtained some bounds of the degrees of the solutions of the matrix polynomial equation (1) with singular matrix coefficients. In this paper, we have continued studing the structure of solutions of this matrix polynomial equation. The triple of matrices $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ can be simultaneously reduced to triangular forms $T^A(\lambda)$, $T^B(\lambda)$ and $T^C(\lambda)$ with invariant factors on main diagonals by means of semiscalar equivalence transformations [10, 11]. Following this, the bounds of the degrees of the solutions of the matrix polynomial equation (1) have been pointed out. Necessary and sufficient conditions for the uniqueness of a solution with a minimal degree have been established. There is also suggested an effective method for constructing minimal degree solutions of such matrix polynomial equations.

1 Preliminary results

We denote the ring of $m \times m$ matrices over $\mathcal{F}[\lambda]$ by $M(m, \mathcal{F}[\lambda])$, groups of invertible matrices over $\mathcal{F}$ and $\mathcal{F}[\lambda]$ by $GL(m, \mathcal{F})$ and $GL(m, \mathcal{F}[\lambda])$, respectively.

It is well known, that every matrix $A(\lambda) \in M(m, \mathcal{F}[\lambda])$, $\operatorname{rank} A = r$, is equivalent to the Smith normal form $S^A(\lambda)$, that is,

$$S^A(\lambda) = U(\lambda)A(\lambda)V(\lambda) = \operatorname{diag}(\mu_1^A(\lambda), \ldots, \mu_r^A(\lambda), 0, \ldots, 0),$$

where $U(\lambda)$, $V(\lambda) \in GL(m, \mathcal{F}[\lambda])$, $\mu_i^A(\lambda) \mid \mu_{i+1}^A(\lambda)$, $i = 1, \ldots, r-1$. The polynomials $\mu_i^A(\lambda)$ are called the invariant factors of matrix $A(\lambda)$.

Definition 1 ([10, 11]). *Collection of polynomial matrices*

$$A_1(\lambda), \ldots, A_k(\lambda)$$

is called semiscalar equivalent to the collection of polynomial matrices

$$B_1(\lambda), \ldots, B_k(\lambda),$$

where $A_i(\lambda)$, $B_i(\lambda) \in M(m, \mathcal{F}[\lambda])$, *if there exist matrices* $Q \in GL(m, \mathcal{F})$ *and* $R_i(\lambda) \in GL(m, \mathcal{F}[\lambda])$ *such that* $B_i(\lambda) = QA_i(\lambda)R_i(\lambda)$, $i = 1, \ldots, k$.

Theorem 1 ([10, 11]). *Collection of nonsingular polynomial matrices*

$$A_1(\lambda), \ldots, A_k(\lambda), \ A_i(\lambda) \in M(m, \mathcal{F}[\lambda]),$$

$i = 1, \ldots, k$, *is semiscalar equivalent to the collection of triangular matrices*

$$T^{A_1}(\lambda), \ldots, T^{A_k}(\lambda),$$

that is, there exist an upper unitriangular matrix $Q \in GL(m,\mathcal{F})$ and invertible matrices $R^{A_i}(\lambda) \in GL(m,\mathcal{F}[\lambda])$ such that

$$T^{A_i}(\lambda) = QA_i(\lambda)R^{A_i}(\lambda) = \begin{Vmatrix} \mu_1^{A_i}(\lambda) & 0 & \cdots & 0 \\ t_{21}^{(i)}(\lambda)\mu_1^{A_i}(\lambda) & \mu_2^{A_i}(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ t_{m1}^{(i)}(\lambda)\mu_1^{A_i}(\lambda) & t_{m2}^{(i)}(\lambda)\mu_2^{A_i}(\lambda) & \cdots & \mu_m^{A_i}(\lambda) \end{Vmatrix}, \tag{4}$$

where $\deg t_{pq}^{(i)}(\lambda) < \deg \mu_p^{A_i}(\lambda) - \deg \mu_q^{A_i}(\lambda)$, *if* $\deg \mu_p^{A_i}(\lambda) > \deg \mu_q^{A_i}(\lambda)$ *and* $t_{pq}^{(i)}(\lambda) \equiv 0$, *if* $\mu_p^{A_i}(\lambda) = \mu_q^{A_i}(\lambda)$, *for all* $p,q = 1,\ldots,m$, $p > q$; $i = 1,\ldots,k$.

Triangular form $T^{A_i}(\lambda)$ is called **standard form** of polynomial matrix $A_i(\lambda)$ with respect to semiscalar equivalence. Note that the matrix $T^{A_i}(\lambda)$ may be written in the form $T^{A_i}(\lambda) = T_i(\lambda)S^{A_i}(\lambda)$, where $T_i(\lambda)$ is a lower unitriangular matrix, $S^{A_i}(\lambda)$ is the Smith normal form of matrix $A_i(\lambda)$.

It should be noted that this theorem holds if the field $\mathcal{F}$ is infinite or if it is finite but $\sum_{i=1}^{k} s_i < |\mathcal{F}|$, where $|\mathcal{F}|$ is the number of elements of finite field $\mathcal{F}$, $s_i = \deg\det A_i(\lambda)$, $i = 1, \ldots, k$.

2 SOLUTIONS OF MINIMAL DEGREE OF MATRIX POLYNOMIAL EQUATIONS

By Theorem 1, the triple of nonsingular polynomial matrices $A(\lambda)$, $B(\lambda)$, $C(\lambda) \in M(m,\mathcal{F}[\lambda])$ from equation (1) is semiscalar equivalent to the triple of triangular polynomial matrices $T^A(\lambda)$, $T^B(\lambda)$, $T^C(\lambda)$ in standard form, that is,

$$T^A(\lambda) = QA(\lambda)R^A(\lambda),\ T^B(\lambda) = QB(\lambda)R^B(\lambda),\ T^C(\lambda) = QC(\lambda)R^C(\lambda),$$

where $Q \in GL(m,\mathcal{F})$, $R^A(\lambda)$, $R^B(\lambda)$, $R^C(\lambda) \in GL(m,\mathcal{F}[\lambda])$.

Matrices $T^A(\lambda)$, $T^B(\lambda)$ and $T^C(\lambda)$ have the form (4), that is,

$$T^A(\lambda) = \begin{Vmatrix} \mu_1^A(\lambda) & 0 & \cdots & 0 \\ \widetilde{a}_{21}(\lambda)\mu_1^A(\lambda) & \mu_2^A(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{a}_{m1}(\lambda)\mu_1^A(\lambda) & \widetilde{a}_{m2}(\lambda)\mu_2^A(\lambda) & \cdots & \mu_m^A(\lambda) \end{Vmatrix},$$

$$T^B(\lambda) = \begin{Vmatrix} \mu_1^B(\lambda) & 0 & \cdots & 0 \\ \widetilde{b}_{21}(\lambda)\mu_1^B(\lambda) & \mu_2^B(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{b}_{m1}(\lambda)\mu_1^B(\lambda) & \widetilde{b}_{m2}(\lambda)\mu_2^B(\lambda) & \cdots & \mu_m^B(\lambda) \end{Vmatrix},$$

$$T^C(\lambda) = \begin{Vmatrix} \mu_1^C(\lambda) & 0 & \cdots & 0 \\ \widetilde{c}_{21}(\lambda)\mu_1^C(\lambda) & \mu_2^C(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{c}_{m1}(\lambda)\mu_1^C(\lambda) & \widetilde{c}_{m2}(\lambda)\mu_2^C(\lambda) & \cdots & \mu_m^C(\lambda) \end{Vmatrix}.$$

Then from equation (1) we obtain the matrix polynomial equation

$$T^A(\lambda)\widetilde{X}(\lambda) + T^B(\lambda)\widetilde{Y}(\lambda) = T^C(\lambda), \tag{5}$$

where $\widetilde{X}(\lambda) = (R^A(\lambda))^{-1}X(\lambda)R^C(\lambda)$, $\widetilde{Y}(\lambda) = (R^B(\lambda))^{-1}Y(\lambda)R^C(\lambda)$.

We will call the equation (5) **associate** to the equation (1).

Lemma 1. *The equation* (1) *is solvable if and only if the equation* (5) *is solvable. Each solution* $X(\lambda)$, $Y(\lambda)$ *of the equation* (1) *corresponds to a solution* $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ *of the equation* (5) *and the converse each solution* $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ *of the equation* (5) *corresponds to a solution* $X(\lambda)$, $Y(\lambda)$ *of the equation* (1).

Proof. It is well known [6, 13], that the matrix equation (1) is solvable if and only if the left greatest common divisor $D(\lambda)$ of matrices $A(\lambda)$ and $B(\lambda)$ is the left divisor of the matrix $C(\lambda)$. Then the greatest common divisor of triangular forms $T^A(\lambda)$ and $T^B(\lambda)$ is $D_1(\lambda) = QD(\lambda)$. Is it easy to see that if the matrix $D(\lambda)$ is the left divisor of the matrix $C(\lambda)$, then $D_1(\lambda)$ is the divisor of the matrix $T^C(\lambda)$ and conversely.

Furthermore, each solution $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ of the equation (5) corresponds to the solution

$$X(\lambda) = R^A(\lambda)\widetilde{X}(\lambda)(R^C(\lambda))^{-1}, \quad Y(\lambda) = R^B(\lambda)\widetilde{Y}(\lambda)(R^C(\lambda))^{-1}$$

of the equation (1) and conversely. □

Thus, the description of solutions of the matrix equation (1) is reduced to the description of solutions of the associated equation (5).

Solutions $X(\lambda)$, $Y(\lambda)$ and $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ of the matrix equations (1) and (5) are associate.

We denote the i-th row of matrix A by $row_i(A)$.

Theorem 2. *Let the matrix equation* (5) *be solvable. Then, it has the solution*

$$\widetilde{X}_1(\lambda) = \|\widetilde{x}_{ij}^{(1)}(\lambda)\|_1^m, \quad \widetilde{Y}_1(\lambda) = \|\widetilde{y}_{ij}^{(1)}(\lambda)\|_1^m$$

such that

$$row_i(\widetilde{X}_1(\lambda)) = \mathbf{0} \quad \textit{if} \quad \deg\mu_i^B(\lambda) = 0 \quad (\mu_i^B(\lambda) = 1),\ i = 1, \ldots, k, \tag{6}$$

$$\deg row_i(\widetilde{X}_1(\lambda)) < \deg\mu_i^B(\lambda) \quad \textit{if} \quad \deg\mu_i^B(\lambda) \geq 1,\ i = k+1, \ldots, m, \tag{7}$$

and the solution $\widetilde{X}_2(\lambda) = \|\widetilde{x}_{ij}^{(2)}(\lambda)\|_1^m$, $\widetilde{Y}_2(\lambda) = \|\widetilde{y}_{ij}^{(2)}(\lambda)\|_1^m$ *such that*

$$row_i(\widetilde{Y}_2(\lambda)) = \mathbf{0} \quad \textit{if} \quad \deg\mu_i^A(\lambda) = 0 \quad (\mu_i^A(\lambda) = 1),\ i = 1, \ldots, l, \tag{8}$$

$$\deg row_i(\widetilde{Y}_2(\lambda)) < \deg\mu_i^A(\lambda) \quad \textit{if} \quad \deg\mu_i^A(\lambda) \geq 1,\ i = l+1, \ldots, m. \tag{9}$$

Proof. From the matrix equation (5), we obtain the system of linear polynomial equations

$$\sum_{k=1}^{i} \left(\mu_k^A(\lambda)\widetilde{a}_{ik}(\lambda)\widetilde{x}_{kj}(\lambda) + \mu_k^B(\lambda)\widetilde{b}_{ik}(\lambda)\widetilde{y}_{kj}(\lambda)\right) = \mu_j^C(\lambda)\widetilde{c}_{ij}(\lambda), \tag{10}$$

$i,\ j = 1, \ldots, m$, where $\widetilde{a}_{ii(\lambda)} = \widetilde{b}_{ii}(\lambda) = \widetilde{c}_{ii}(\lambda) = 1$.

The description of solutions of this system is reduced to the description of solutions of linear polynomial equations in the following form

$$\mu_i^A(\lambda)\widetilde{x}_{ij}(\lambda) + \mu_i^B(\lambda)\widetilde{y}_{ij}(\lambda) = \widehat{c}_{ij}(\lambda), \quad i, j = 1, \ldots, m. \tag{11}$$

If the equation (11) is solvable, then it has the solution $\widetilde{x}_{ij}(\lambda) = \widetilde{x}_{ij}^{(1)}(\lambda)$, $\widetilde{y}_{ij}(\lambda) = \widetilde{y}_{ij}^{(1)}(\lambda)$ such that $\deg\widetilde{x}_{ij}^{(1)}(\lambda) < \deg\mu_i^B(\lambda)$ and the solution $\widetilde{x}_{ij}(\lambda) = \widetilde{x}_{ij}^{(2)}(\lambda)$, $\widetilde{y}_{ij}(\lambda) = \widetilde{y}_{ij}^{(2)}(\lambda)$ such that $\deg\widetilde{y}_{ij}^{(2)}(\lambda) < \deg\mu_i^A(\lambda)$ [4,7]. If $\deg\mu_i^B(\lambda) \geq 1$, $i = k+1, \ldots, m$, then for each element in the row $row_i(\widetilde{X}_1(\lambda))$ the condition (7) of the theorem is true. Similarly, if $\deg\mu_i^A(\lambda) \geq 1$, $i = l+1, \ldots, m$, the condition (9) is true.

Among equations of the system (10) there are such polynomial equations

$$\mu_i^A(\lambda)\widetilde{x}_{ii}(\lambda) + \mu_i^B(\lambda)\widetilde{y}_{ii}(\lambda) = \mu_i^C(\lambda). \tag{12}$$

If $\mu_i^A(\lambda) = 1$ and $\mu_i^B(\lambda) = 1$, then this equation has solutions $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = \mu_i^C(\lambda)$ and $\widetilde{x}_{ii}(\lambda) = \mu_i^C(\lambda)$, $\widetilde{y}_{ii}(\lambda) = 0$. If only one of $\mu_i^A(\lambda) = 1$ or $\mu_i^B(\lambda) = 1$, then this equation has solutions $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = \frac{\mu_i^C(\lambda)}{\mu_i^B(\lambda)}$ and $\widetilde{x}_{ii}(\lambda) = \frac{\mu_i^C(\lambda)}{\mu_i^A(\lambda)}$, $\widetilde{y}_{ii}(\lambda) = 0$, respectively.

The system (10) also has polynomial equations in the following form

$$\mu_i^A(\lambda)\widetilde{x}_{ij}(\lambda) + \mu_i^B(\lambda)\widetilde{y}_{ij}(\lambda) = 0, \quad i < j,\ i = 1, \ldots, m-1,\ j = 2, \ldots, m. \tag{13}$$

These equations always have a zero solution, that is, $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = 0$. Thus, the conditions (6) and (8) of the theorem are true. This completes the proof. □

From the proof of this theorem, we get a method for constructing solutions of the matrix equation (5). Since, the following inequalities $\deg\mu_i^A(\lambda) \leq \deg\mu_m^A(\lambda)$, $i = 1, \ldots, m-1$, are true for the invariant factors of matrix $A(\lambda)$, then $\deg S^A(\lambda) = \deg\mu_m^A(\lambda)$. Therefore, from Theorem 2 we get the following corollary.

Corollary 1. *Let the matrix equation* (5) *be solvable. Then it has the solution*

$$\widetilde{X}_1(\lambda), \quad \widetilde{Y}_1(\lambda)$$

such that

$$\widetilde{X}_1(\lambda) = \mathbf{0} \quad \textit{if} \quad \deg S^B(\lambda) = 0\ (B(\lambda)\ \textit{is an invertible matrix}),$$
$$\deg\widetilde{X}_1(\lambda) < \deg S^B(\lambda) \quad \textit{if} \quad \deg S^B(\lambda) \geq 1,$$

and the solution

$$\widetilde{X}_2(\lambda), \quad \widetilde{Y}_2(\lambda)$$

such that

$$\widetilde{Y}_2(\lambda) = \mathbf{0} \quad \textit{if} \quad \deg S^A(\lambda) = 0\ (A(\lambda)\ \textit{is an invertible matrix}),$$
$$\deg\widetilde{Y}_2(\lambda) < \deg S^A(\lambda) \quad \textit{if} \quad \deg S^A(\lambda) \geq 1.$$

Theorem 3. *Let*

$$S^A(\lambda) = \operatorname{diag}(\underbrace{1, \ldots, 1}_{k}, \mu_{k+1}^A(\lambda), \ldots, \mu_m^A(\lambda)),\ k \geq 0, \tag{14}$$

and

$$S^B(\lambda) = \operatorname{diag}(\underbrace{1, \ldots, 1}_{l}, \mu_{l+1}^B(\lambda), \ldots, \mu_m^B(\lambda)),\ l \geq 0, \tag{15}$$

be the Smith normal forms of the matrices $A(\lambda)$ *and* $B(\lambda)$, *respectively, and let the matrix equation* (5) *be solvable. Without loss of generality, let* $k \geq l$.

(i) If $\deg \mu_i^C(\lambda) \geq \deg \mu_i^A(\lambda) + \deg \mu_i^B(\lambda)$, $\mu_i^A(\lambda) \neq 1$, $\mu_i^B(\lambda) \neq 1$, $i = 1, \ldots, m$, *then the matrix equation* (5) *has the solution*

$$\widetilde{X}(\lambda) = \|\widetilde{x}_{ij}(\lambda)\|_1^m, \quad \widetilde{Y}(\lambda) = \|\widetilde{y}_{ij}(\lambda)\|_1^m$$

such that

$$\deg row_i(\widetilde{X}(\lambda)) < \deg \mu_i^B(\lambda), \deg row_i(\widetilde{Y}(\lambda)) = \deg \mu_i^C(\lambda) - \deg \mu_i^B(\lambda), \quad (16)$$

(ii) if $\deg \mu_i^C(\lambda) = \deg \mu_i^A(\lambda) + \deg \mu_i^B(\lambda)$, $\mu_i^A(\lambda) = 1$ *or* $\mu_i^B(\lambda) = 1$, $i = 1, \ldots, k$, *then the matrix equation* (5) *has solutions* $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ *such that*

$$row_i(\widetilde{X}(\lambda)) = \mathbf{0}, \deg row_i(\widetilde{Y}(\lambda)) \leq \deg \mu_i^C(\lambda) - \deg \mu_i^B(\lambda), \quad (17)$$

and

$$\deg row_i(\widetilde{X}(\lambda)) \leq \deg \mu_i^C(\lambda) - \deg \mu_i^A(\lambda), \ row_i(\widetilde{Y}(\lambda)) = \mathbf{0}, \quad (18)$$

(iii) if $\deg \mu_i^C(\lambda) < \deg \mu_i^A(\lambda) + \deg \mu_i^B(\lambda)$, $i = k+1, \ldots, m$, *then the matrix equation* (5) *has the solution* $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ *such that*

$$\deg row_i(\widetilde{X}(\lambda)) < \deg \mu_i^B(\lambda), \ \deg row_i(\widetilde{Y}(\lambda)) < \deg \mu_i^A(\lambda). \quad (19)$$

Proof. Case (i). In the proof of Theorem 2, it has been shown that the solving of the matrix equation (5) is reduced to the solving of the system of linear polynomial equations (10). This system has equations (12). Then, there exists a solution with the condition $\deg \widetilde{x}_{ii}(\lambda) < \deg \mu_i^B(\lambda)$ of the i-th equation (12) [4,7]. So, $\deg \widetilde{y}_{ij}(\lambda) = \deg \mu_i^C(\lambda) - \deg \mu_i^B(\lambda)$ for a fixed value of i and all values of $j = 1, \ldots, m$. Thus, the matrix equation (5) has the solution $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ with the condition (16).

Case (ii). In this case the condition has the form $\deg \mu_i^C(\lambda) = \deg \mu_i^A(\lambda)$ or $\deg \mu_i^C(\lambda) = \deg \mu_i^B(\lambda)$ if $\mu_i^B(\lambda) = 1$ or $\mu_i^A(\lambda) = 1$ for a fixed value of i. If $\mu_i^B(\lambda) = 1$ and $\mu_i^A(\lambda) = 1$ for a fixed value of i, then the condition has the form $\deg \mu_i^C(\lambda) = 0$. In the proof of Theorem 2, it has been shown that the system of linear polynomial equations (11) has equations (12) and (13). In this case, these equations have zero solutions. Thus, the matrix equation (5) has solutions $\widetilde{X}(\lambda)$, $\widetilde{Y}(\lambda)$ with the conditions (17) and (18).

Case (iii). There exists a solution of the equation (11) with the condition $\deg \widetilde{x}_{ij}(\lambda) < \deg \mu_j^B(\lambda)$, $\deg \widetilde{y}_{ij}(\lambda) < \deg \mu_i^A(\lambda)$ if the condition $\deg \mu_i^C(\lambda) < \deg \mu_i^A(\lambda) + \deg \mu_i^B(\lambda)$ is true for a fixed value of i and all values of $j = 1, \ldots, m$ [4,7]. This completes the proof. □

Remark 1. *We should note that in cases (ii) and (iii), opposite propositions hold, that is, their conditions are necessary for the existence of solutions with the conditions* (17)—(19).

Theorem 4. *Let the equation* (5) *be solvable. Then it has solutions*

$$\widetilde{X}(\lambda) = \|\widetilde{x}_{ij}(\lambda)\|_1^m, \ \widetilde{Y}(\lambda) = \|\widetilde{y}_{ij}(\lambda)\|_1^m$$

of lower triangular forms such that

(i) $\deg \widetilde{x}_{ii}(\lambda) < \deg \mu_i^B(\lambda)$, $\deg \widetilde{y}_{ii}(\lambda) < \deg \mu_i^A(\lambda)$

$$\text{if} \quad \deg \mu_i^C(\lambda) < \deg \mu_i^A(\lambda) + \deg \mu_i^B(\lambda), \ i = 1, \ldots, m;$$

(ii) $\deg\widetilde{x}_{ii}(\lambda) < \deg\mu_i^B(\lambda)$, $\deg\widetilde{y}_{ii}(\lambda) = \deg\mu_i^C(\lambda) - \deg\mu_i^B(\lambda)$

$$\textit{if} \quad \deg\mu_i^C(\lambda) \geq \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda), \ i = 1, \ldots, m.$$

Proof. We prove this theorem in a similar way to Theorem 2 and Theorem 3. □

We get solutions of the matrix equation (1) from solutions of the matrix equation (5):

$$X(\lambda) = R^A(\lambda)\widetilde{X}(\lambda)(R^C(\lambda))^{-1}, \ Y(\lambda) = R^B(\lambda)\widetilde{Y}(\lambda)(R^C(\lambda))^{-1}.$$

3 THE UNIQUENESS OF SOLUTIONS OF MINIMAL DEGREES OF MATRIX POLYNOMIAL EQUATIONS

We will establish the conditions for the uniqueness of solutions of minimal degrees of the matrix equation (5).

Theorem 5. *The matrix equation* (5) *has a unique solution*

$$\widetilde{X}_0^{(1)}(\lambda) = \|\widetilde{x}_{ij}^{(1)}(\lambda)\|_1^m, \ \widetilde{Y}_0^{(1)}(\lambda) = \|\widetilde{y}_{ij}^{(1)}(\lambda)\|_1^m$$

and

$$\widetilde{X}_0^{(2)}(\lambda) = \|\widetilde{x}_{ij}^{(2)}(\lambda)\|_1^m, \ \widetilde{Y}_0^{(2)}(\lambda) = \|\widetilde{y}_{ij}^{(2)}(\lambda)\|_1^m$$

such that

$$row_i(\widetilde{X}_0^{(1)}(\lambda)) = \mathbf{0} \quad \textit{if} \quad \deg\mu_i^B(\lambda) = 0, \ i = 1, \ldots, k, \tag{20}$$

$$\deg row_i(\widetilde{X}_0^{(1)}(\lambda)) < \deg\mu_i^B(\lambda) \quad \textit{if} \quad \deg\mu_i^B(\lambda) \geq 1, \ i = k+1, \ldots, m, \tag{21}$$

and

$$row_i(\widetilde{Y}_0^{(2)}(\lambda)) = \mathbf{0} \quad \textit{if} \quad \deg\mu_i^A(\lambda) = 0, \ i = 1, \ldots, k, \tag{22}$$

$$\deg row_i(\widetilde{Y}_0^{(2)}(\lambda)) < \deg\mu_i^A(\lambda) \quad \textit{if} \quad \deg\mu_i^A(\lambda) \geq 1, \ i = k+1, \ldots, m, \tag{23}$$

if and only if

$$(\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1.$$

Proof. It is clear that the matrix equation (5) has a unique solution $\widetilde{X}_0^{(1)}(\lambda)$, $\widetilde{Y}_0^{(1)}(\lambda)$ with the condition (21) if and only if each equation (11) has a unique solution $\widetilde{x}_{ij}^{(1)}(\lambda)$, $\widetilde{y}_{ij}^{(1)}(\lambda)$ such that $\deg\widetilde{x}_{ij}^{(1)} < \deg\mu_i^B(\lambda)$. This solution of the equation (11) is unique if and only if $(\mu_i^A(\lambda), \mu_j^B(\lambda)) = 1$ for all $i, j = 1, \ldots, m$ [4, 7]. The last condition holds if and only if $(\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1$.

As it has been shown in the proof of Theorem 2, the system (10) has equations (12) and (13). By the condition of the theorem, these equations have a zero solution, which is unique. Thus, the solution $\widetilde{X}_0^{(1)}(\lambda)$, $\widetilde{Y}_0^{(1)}(\lambda)$ with the condition (20) is unique.

Similarly we prove the existence of a unique solution $\widetilde{X}_0^{(2)}(\lambda)$, $\widetilde{Y}_0^{(2)}(\lambda)$ with the conditions (22) and (23). This completes the proof. □

Theorem 6. *Let the matrix equation* (5) *be solvable and let* $S^A(\lambda)$, *and* $S^B(\lambda)$ *be the Smith normal forms* (14) *and* (15) *of the matrices* $A(\lambda)$ *and* $B(\lambda)$, *respectively. Then, there exists a unique solution*

$$\widetilde{X}(\lambda) = \|\widetilde{x}_{ij}(\lambda)\|_1^m,\ \widetilde{Y}(\lambda) = \|\widetilde{y}_{ij}(\lambda)\|_1^m$$

of the matrix equation (5) *with the conditions* (17) *and* (18) *if and only if*

$$\deg\mu_i^C(\lambda) = \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda), \quad i = 1, \ldots, k, \quad \text{and} \quad (\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1,$$

and with the condition (19) *if and only if*

$$\deg\mu_i^C(\lambda) < \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda), \quad i = k+1, \ldots, m, \quad \text{and} \quad (\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1.$$

Proof. It is clear that a unique solution of the matrix equation (5) exists if and only if a unique solution of the system of linear polynomial equations (10) exists, that is, a unique solution of each linear polynomial equation (11) exists. This system has equations (12). If $\mu_i^A(\lambda) = 1$ and $\mu_i^B(\lambda) = 1$, then by the conditions of the theorem, this equation has solutions $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = \mu_i^C(\lambda)$ and $\widetilde{x}_{ii}(\lambda) = \mu_i^C(\lambda)$, $\widetilde{y}_{ii}(\lambda) = 0$. If only one of $\mu_i^A(\lambda) = 1$ or $\mu_i^B(\lambda) = 1$, then this equation has solutions

$$\widetilde{x}_{ii}(\lambda) = 0,\ \widetilde{y}_{ii}(\lambda) = \frac{\mu_i^C(\lambda)}{\mu_i^B(\lambda)} \quad \text{and} \quad \widetilde{x}_{ii}(\lambda) = \frac{\mu_i^C(\lambda)}{\mu_i^A(\lambda)},\ \widetilde{y}_{ii}(\lambda) = 0,$$

respectively. The equations (13) always have a zero solution, that is, $\widetilde{x}_{ii}(\lambda) = 0$, $\widetilde{y}_{ii}(\lambda) = 0$. This solution is unique. So, there exists a unique solution with the conditions (17) and (18) of the matrix equation (5).

If $\mu_i^A(\lambda) \neq 1$ and $\mu_i^B(\lambda) \neq 1$, then by the results [4, 7] the solution with the condition (19) of the matrix equation (5) is unique if and only if the solution $\widetilde{x}_{ij}(\lambda)$, $\widetilde{y}_{ij}(\lambda)$ such that $\deg\widetilde{x}_{ij}(\lambda) < \deg\mu_i^B(\lambda)$ and $\deg\widetilde{y}_{ij}(\lambda) < \deg\mu_i^A(\lambda)$ of the equation (11) is unique. There exist such solutions and they are unique if and only if $\deg\mu_i^C(\lambda) < \deg\mu_i^A(\lambda) + \deg\mu_i^B(\lambda)$ and

$$(\mu_i^A(\lambda), \mu_j^B(\lambda)) = 1 \ \ i, j = 1, \ldots, m.$$

The last conditions are true if and only if $(\mu_m^A(\lambda), \mu_m^B(\lambda)) = 1$. This completes the proof. □

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Джалюк Н.С., Петричкович В.М. *Структура розв'язків матричного лінійного однобічного поліноміального рівняння від двох змінних* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 48–56.

Досліджується структура розв'язків матричного лінійного поліноміального рівняння $A(\lambda)X(\lambda) + B(\lambda)Y(\lambda) = C(\lambda)$, зокрема можливі степені цих розв'язків. Розв'язування цього матричного поліноміального рівняння зводиться до розв'язування еквівалетного матричного поліноміального рівняння з матрицями-коефіцієнтами у трикутних формах з інваріантними множниками на головних діагоналях, до яких зводяться поліноміальні матриці $A(\lambda)$, $B(\lambda)$ і $C(\lambda)$ напівскалярними еквівалентними перетвореннями. На основі цього вказано межі для степенів розв'язків матричних поліноміальних рівнянь. Встановлено необхідні і достатні умови єдиності розв'язку мінімального степеня. Запропоновано ефективний метод побудови розв'язків мінімальних степенів цих рівнянь. На відміну від відомих результатів про оцінки степенів розв'язків матричних поліноміальних рівнянь, в яких обидва або принаймні один із коефіцієнтів є регулярною матрицею, у цій статті розглянуто випадок матричного поліноміального рівняння з довільними коефіцієнтами $A(\lambda)$ і $B(\lambda)$.

Ключові слова і фрази: матричне поліноміальне рівняння, розв'язок рівняння, напівскалярна еквівалентність поліноміальних матриць.