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ON ESTIMATES FOR THE JACOBI TRANSFORM IN THE SPACE $L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$

For the Jacobi transform in the space $L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$ we prove the estimates in some classes of functions, characterized by a generalized modulus of continuity.

Key words and phrases: Jacobi operator, Jacobi transform, Jacobi generalized translation, generalized modulus of continuity.

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1 INTRODUCTION AND PRELIMINARIES

The main aim of this paper is to generalize the Theorem 1 in [3].

Let $\alpha > \frac{-1}{2}$, $\alpha \geq \beta \geq \frac{-1}{2}$ and $J^{\alpha,\beta}(x) := (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$ for $x \in \mathbb{R}^+$. We define $L^p_{(\alpha,\beta)}(\mathbb{R}^+) := L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$, $1 < p \leq 2$, as the Banach space of measurable functions $f(x)$ on $\mathbb{R}^+$ with the finite norm

$$\|f\|_{p,(\alpha,\beta)} = \left(\int_0^{+\infty} |f(x)|^p J^{\alpha,\beta}(x)dx\right)^{\frac{1}{p}}.$$

Let

$$D_{\alpha,\beta} := \frac{d^2}{dx^2} + ((2\alpha+1)\cos x + (2\beta+1)\operatorname{tg} x)\frac{d}{dx}$$

be the Jacobi differential operator and denote by $\varphi_\lambda^{(\alpha,\beta)}(x)$, $\lambda \in \mathbb{C}, x \in \mathbb{R}^+$, the Jacobi function of order (α, β). The function $\varphi_\lambda^{(\alpha,\beta)}(x)$ satisfies the differential equation

$$(D_{\alpha,\beta} + \lambda^2 + \rho^2)\varphi_\lambda^{(\alpha,\beta)}(x) = 0,$$

where $\rho = \alpha + \beta + 1$.

Lemma 1.1. *Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, $\rho = \alpha + \beta + 1$, and let $x_0 > 0$. Then for $|\eta| \leq \rho$ there exists a positive constant $C_1 = C_1(\alpha, \beta, x_0)$ such that*

$$|1 - \varphi_{\mu+i\eta}^{(\alpha,\beta)}(x)| \geq C_1|1 - j_\alpha(\mu x)|,$$

for all $0 \leq x \leq x_0$ and $\mu \in \mathbb{R}$, where $j_\alpha(x)$ is a normalized Bessel function of the first kind.

Proof. (See [2], Lemma 9). □

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In $L^p_{(\alpha,\beta)}(\mathbb{R}^+)$ consider the Jacobi generalized translation T_h

$$T_h f(x) = \int_0^{+\infty} f(z)\mathcal{K}_{\alpha,\beta}(x,h,z)J^{\alpha,\beta}(z)dz,$$

where the kernel $\mathcal{K}_{\alpha,\beta}$ is explicitly known (see [5]).

The Jacobi transform is defined by formula

$$\widehat{f}(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda^{(\alpha,\beta)}(x)J^{\alpha,\beta}(x)dx.$$

The inversion formula is

$$f(x) = \frac{1}{2\pi}\int_0^{+\infty} \widehat{f}(\lambda)\varphi_\lambda^{(\alpha,\beta)}(x)d\mu(\lambda),$$

where $d\mu(\lambda) :=\mid C(\lambda)\mid^{-2} d\lambda$ and the C-function $C(\lambda)$ is defined by

$$C(\lambda) = \frac{2^\rho\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)}.$$

We have the Young inequality

$$\|\widehat{f}\|_{q,(\alpha,\beta)} \leq K\|f\|_{p,(\alpha,\beta)}, \tag{1}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and K is positive constant.

We note the important property of the Jacobi transform: if $f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+)$, then

$$\widehat{D_{\alpha,\beta}f}(\lambda) = -(\lambda^2+\rho^2)\widehat{f}(\lambda). \tag{2}$$

The following relation connects the Jacobi generalized translation and the Jacobi transform:

$$\widehat{T_h f}(\lambda) = \varphi_\lambda^{(\alpha,\beta)}(h)\widehat{f}(\lambda). \tag{3}$$

The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = T_h f(x) - f(x) = (T_h - I)f(x),$$

where I is the identity operator in $L^p_{(\alpha,\beta)}(\mathbb{R}^+)$ and

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1}f(x)) = (T_h-1)^k f(x) = \sum_{i=0}^{k}(-1)^{k-1}\tbinom{k}{i}T_h^i f(x), \tag{4}$$

where $T_h^0 f(x) = f(x)$, $T_h^i f(x) = T_h(T_h^{i-1}f(x))$, $i=1,2,\ldots,k$ and $k=1,2,\ldots$.

The k-th order generalized modulus of continuity of a function $f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+)$ is defined by

$$\Omega_k(f,\delta) = \sup_{0<h\leq\delta}\|\Delta_h^k f\|_{p,(\alpha,\beta)}, \qquad \delta>0.$$

Let $W^{r,k}_{p,\varphi}(D_{\alpha,\beta})$ denote the class of functions $f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+)$ that have generalized derivatives in the sense of Levi (see [4]) satisfying the estimate

$$\Omega_k(D^r_{\alpha,\beta}f,\delta) = O(\varphi(\delta^k)), \qquad \delta\to 0;$$

i.e.,

$$W^{r,k}_{p,\varphi}(D_{\alpha,\beta}) := \{f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+) : D^r_{\alpha,\beta}f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+) \text{ and } \Omega_k(D^r_{\alpha,\beta}f,\delta) = O(\varphi(\delta^k)),\ \delta\to 0\},$$

where $\varphi(x)$ is any nonnegative function given on $[0,\infty)$, and $D^0_{\alpha,\beta}f = f$, $D^r_{\alpha,\beta}f = D_{\alpha,\beta}(D^{r-1}_{\alpha,\beta}f)$; $r=1,2,\ldots$

2 MAIN RESULTS

In this section we estimate the integral

$$\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)$$

in certain classes of functions in $L^p_{(\alpha,\beta)}(\mathbb{R}^+)$.

Lemma 2.1. *Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}, \rho = \alpha+\beta+1$, and let $f \in L^p_{(\alpha,\beta)}(\mathbb{R}^+)$. Then*

$$\left(\int_0^\infty (\lambda^2+\rho^2)^{qr}|1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{qk}|\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{q}} \leq K\|\Delta_h^k D^r_{\alpha,\beta} f(x)\|_{p,(\alpha,\beta)},$$

where $1 < p \leq 2$ and q such that $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From formula (2) we obtain

$$\widehat{D^r_{\alpha,\beta} f}(\lambda) = (-1)^r(\lambda^2+\rho^2)^r \widehat{f}(\lambda); \quad r = 0,1,\ldots \tag{5}$$

We use the formulas (3) and (5) and conclude

$$\widehat{T_h^i D^r_{\alpha,\beta}} f(\lambda) = (-1)^r(\varphi_\lambda^{(\alpha,\beta)}(h))^i(\lambda^2+\rho^2)^r\widehat{f}(\lambda), \qquad 1 \leq i \leq k. \tag{6}$$

From the definition of finite difference (4) and formula (6) the image $\Delta_h^k D^r_{\alpha,\beta} f(x)$ under the Jacobi transform has the form

$$\Delta_h^k \widehat{D^r_{\alpha,\beta} f}(\lambda) = (-1)^r(\varphi_\lambda^{(\alpha,\beta)}(h)-1)^k(\lambda^2+\rho^2)^r\widehat{f}(\lambda).$$

Now by the inequality (1) we have the result. □

Theorem 1. *Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}, \rho = \alpha+\beta+1$ and let $f \in W^{r,k}_{p,\varphi}(D_{\alpha,\beta})$. Then*

$$\sup_{W^{r,k}_{p,\varphi}(D_{\alpha,\beta})} \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{q}} = O\Big(N^{-2r}\varphi\Big(\big(\frac{c}{N}\big)^k\Big)\Big) \qquad as \quad N \to \infty,$$

where $r = 0,1,2,\ldots; k = 1,2,\ldots, c > 0$ is a fixed constant, and $\varphi(t)$ is any nonnegative function defined on the interval $[0,\infty)$.

Proof. In the terms of $j_\alpha(x)$, for the normalized Bessel function of the first kind we have (see [1])

$$1-j_\alpha(x) = O(1), \quad x \geq 1, \tag{7}$$
$$1-j_\alpha(x) = O(x^2), \quad 0 \leq x \leq 1, \tag{8}$$
$$\sqrt{hx}J_\alpha(hx) = O(1), \quad hx \geq 0, \tag{9}$$

where $J_\alpha(x)$ is Bessel function of the first kind, and

$$j_\alpha(x) = \frac{2^\alpha\Gamma(\alpha+1)}{x^\alpha}J_\alpha(x). \tag{10}$$

Let $f \in W^{r,k}_{p,\varphi}(D_{\alpha,\beta})$. By the Hölder inequality and Lemma 1.1, we have

$$\begin{aligned}
&\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) - \int_N^\infty j_\alpha(\lambda h)|\widehat{f}(\lambda)|^q d\mu(\lambda) = \int_N^\infty (1 - j_\alpha(\lambda h))|\widehat{f}(\lambda)|^q d\mu(\lambda)\\
&\quad = \int_N^\infty (1 - j_\alpha(\lambda h)) \left(|\widehat{f}(\lambda)| \mid C(\lambda) \mid^{\frac{-2}{q}}\right)^q d\lambda\\
&\quad = \int_N^\infty (1 - j_\alpha(\lambda h)) \left(|\widehat{f}(\lambda)| \mid C(\lambda) \mid^{\frac{-2}{q}}\right)^{q-\frac{1}{k}} \left(|\widehat{f}(\lambda)| \mid C(\lambda) \mid^{\frac{-2}{q}}\right)^{\frac{1}{k}} d\lambda\\
&\quad \leq \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \left(\int_N^\infty |1 - j_\alpha(\lambda h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{qk}}\\
&\quad \leq \frac{1}{C_1} \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \left(\int_N^\infty |1 - \varphi_\lambda^{(\alpha,\beta)}(h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{qk}}\\
&\quad \leq \frac{1}{C_1} \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \left(\int_N^\infty (\lambda^2+\rho^2)^{-rq+rq} |1 - \varphi_\lambda^{(\alpha,\beta)}(h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{qk}}\\
&\quad \leq \frac{(N^2+\rho^2)^{\frac{-r}{k}}}{C_1} \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \left(\int_N^\infty (\lambda^2+\rho^2)^{qr} |1 - \varphi_\lambda^{(\alpha,\beta)}(h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{qk}}.
\end{aligned}$$

In view of Lemma 2.1, we conclude that

$$\int_N^\infty (\lambda^2+\rho^2)^{qr} |1 - \varphi_\lambda^{(\alpha,\beta)}(h)|^{qk} |\widehat{f}(\lambda)|^q d\mu(\lambda) \leq K^q \|\Delta_h^k D^r_{\alpha,\beta} f(x)\|^q_{p,(\alpha,\beta)}.$$

Therefore

$$\begin{aligned}
\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) \ &\leq \ \int_N^\infty j_\alpha(\lambda h)|\widehat{f}(\lambda)|^q d\mu(\lambda)\\
&+ \ K^{\frac{1}{k}} \frac{(N^2+\rho^2)^{\frac{-r}{k}}}{C_1} \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \|\Delta_h^k D^r_{\alpha,\beta} f(x)\|^{\frac{1}{k}}_{p,(\alpha,\beta)}.
\end{aligned}$$

From formulas (9) and (10), we have $j_\alpha(\lambda h) = O((\lambda h)^{-\alpha-\frac{1}{2}})$. Then

$$\begin{aligned}
&\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\\
&\quad = O\left(\int_N^\infty (\lambda h)^{-\alpha-\frac{1}{2}} |\widehat{f}(\lambda)|^q d\mu(\lambda) + N^{-\frac{2r}{k}} \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \|\Delta_h^k D^r_{\alpha,\beta} f(x)\|^{\frac{1}{k}}_{p,(\alpha,\beta)}\right)\\
&\quad = O\left((Nh)^{-\alpha-\frac{1}{2}} \int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) + N^{-\frac{2r}{k}} \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \|\Delta_h^k D^r_{\alpha,\beta} f(x)\|^{\frac{1}{k}}_{p,(\alpha,\beta)}\right),
\end{aligned}$$

or

$$(1 - (Nh)^{-\alpha-\frac{1}{2}}) \int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) = O(N^{-\frac{2r}{k}}) \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \|\Delta_h^k D^r_{\alpha,\beta} f(x)\|^{\frac{1}{k}}_{p,(\alpha,\beta)}.$$

Choose a constant c such that the number $1 - c^{-\alpha-\frac{1}{2}}$ is positive. Setting $h = c/N$ in the last inequality, we obtain

$$\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda) = O(N^{-\frac{2r}{k}}) \left(\int_N^\infty |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{qk-1}{qk}} \varphi^{\frac{1}{k}}\Big((\frac{c}{N})^k\Big).$$

Then
$$\int_N^{\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda) = O\Big(N^{-2rq}\varphi^q\Big(\big(\frac{c}{N}\big)^k\Big)\Big),$$
which completes the proof. □

Corollary 2.1. *Let* $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}, \rho = \alpha+\beta+1, \varphi(t) = t^{\nu}, \nu > 0$, *and let* $f \in W^{r,k}_{p,t^{\alpha}}(D_{\alpha,\beta})$. *Then*
$$\left(\int_N^{\infty} |\widehat{f}(\lambda)|^q d\mu(\lambda)\right)^{\frac{1}{q}} = O(N^{-2r-k\nu}) \quad as \quad N \to \infty,$$
where $1 < p \leq 2$ *and* q *such that* $\frac{1}{p} + \frac{1}{q} = 1$.

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Для перетворення Якобі в просторі $L^p(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$ доведено оцінки в деяких класах функцій, що характеризуються узагальненим модулем неперервності.

Ключові слова і фрази: оператор Якобі, перетворення Якобі, узагальнений зсув Якобі, узагальнений модуль неперервності.