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THE BARGMANN TYPE REDUCTION FOR SOME LAX INTEGRABLE TWO-DIMENSIONAL GENERALIZATION OF THE RELATIVISTIC TODA LATTICE

The possibility of applying the method of reducing upon finite-dimensional invariant subspaces, generated by the eigenvalues of the associated spectral problem, to some two-dimensional generalization of the relativistic Toda lattice with the triple matrix Lax type linearization is investigated. The Hamiltonian property and Lax-Liouville integrability of the vector fields, given by this system, on the invariant subspace related with the Bargmann type reduction are found out.

Key words and phrases: relativistic Toda lattice, triple Lax type linearization, invariant reduction, symplectic structure, Liouville integrability.

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INTRODUCTION

By use of the different Lie-algebraic approaches the Lax integrable $(2 + 1)$-dimensional nonlinear differential-difference systems given on functional manifolds of one discrete and one continuous independent variables have been obtained in [4], [10], [16], [26], [27]. The systems represented in the papers [10], [16], [26], [27] possess the triple Lax type linearizations and infinite sequences of local conservation laws. The $(2 + 1)$-dimensional nonlinear dynamical systems with such type properties on functional manifolds of two continuous independent variables have been investigated by means of the invariant reduction technique in the paper [14]. In this connection it is interesting to know whether the invariant reduction technique can be applied to the Lax integrable $(2 + 1)$-dimensional differential-difference systems obtained in [10], [16], [26], [27]. The reductions of the $(1 + 1)$-dimensional nonlinear differential-difference systems with the matrix Lax representations upon the finite-dimensional invariant subspaces generated by the critical points of the related local conservation laws and the associated spectral problem eigenvalues, have been considered in [13].

The aim of the present paper is to investigate the applicability of the invariant reduction technique to the $(2 + 1)$-dimensional differential-difference systems with the triple matrix Lax type linearizations, which can be obtained by means of two so called eigenfunction symmetries related with the same eigenvalue of the corresponding spectral problem (see [10]). This research is based on the approach to the study of the finite-dimensional invariant reductions for the $(1 + 1)$-dimensional nonlinear dynamical systems, possessing the matrix Lax type representations [6], [11], [23], and their superanalogs with the same properties, which has been devised in the papers [2], [8], [9], [11], [22], [21]. In the framework of such approach the exact

symplectic structure on the invariant subspace can be found by means of the Gelfand-Dikii type relationship [5], [19] for the differential of the related Lagrangian function on a suitably extended phase space. The discrete analog of the Gelfand-Dikii relationship has been considered in [18], [19], [20].

In the present article the approach mentioned above is used to study the Bargmann type reduction [14] of the Lax integrable two-dimensional generalization of the relativistic Toda lattice [25], which has been constructed in [10].

The paper is organized in the following way. Section 1 contains the triple matrix Lax type linearization for this $(2+1)$-dimensional differential-difference system that will be used in further investigations. In section 2 we establish the existence of an exact symplectic structure on the Bargmann type invariant subspace by means of the discrete analog of the Gelfand-Dikii relationship as well as the Hamiltonian representations for the reduced commuting vector fields given by the system. In section 3, basing on the differential-geometric properties of the trace gradient for the monodromy matrix of the associated periodic matrix linear spectral problem, we obtain the corresponding Lax representations for the reduced vector fields. The complete set of the functionally independent conservation laws which are involutive with respect to the corresponding Poisson bracket and as a consequence ensure the Liouville integrability [1], [17] of the reduced vector fields is also found.

1   THE TRIPLE MATRIX LAX TYPE LINEARIZATION FOR THE TWO-DIMENSIONAL GENERALIZATION OF THE RELATIVISTIC TODA LATTICE

In the paper [10] we have constructed the set of the hierarchies of the eigenfunction symmetries

$$dl/d\tau_{s,m} = -[M_m^s, l], \ df_j/d\tau_{s,m} = (-M_m^s + \delta_s^j l^s)f_j, \ df_j^*/d\tau_{s,m} = (M_m^s - \delta_s^j l^s)^* f_j^*, \quad (1)$$

which are additional homogeneous symmetries of the Lax type hierarchy on the extended dual space to the Lie algebra [3] of Laurent series by the usual shift operator $\mathcal{E}$

$$dl/dt_s = [l_+^s, l], \ df_j/dt_s = l_+^s f_j, \ df_j^*/dt_s = -(l_+^s)^* f_j^*, \quad (2)$$

where $l := \mathcal{E} + \sum_{j=1}^R f_j \mathcal{E}(\mathcal{E}-1)^{-1} f_j^*$, $\mathbf{f} = (f_1, f_2, \ldots, f_R, f_1^*, f_2^*, \ldots, f_R^*)^\top \in \mathcal{M}^{2R}$,

$$\mathcal{M}^{2R} := \{\mathbf{g}: \ \mathbf{g}(n) \in \mathbb{C}^{2R}, \ \mathbf{g}(n+q) = \mathbf{g}(n), \ n \in \mathbb{Z}\}, \quad q \in \mathbb{N},$$

$$M_m^s := \sum_{\beta=0}^{s-1} (l^\beta f_m)(\mathcal{E}-1)^{-1}(l^{*(s-1-\beta)} f_m^*),$$

δ_m^j is the Kronecker symbol, $j, m = \overline{1, R}$, and the lower index "+" denotes a projection of the corresponding operator on the Lie subalgebra of power series, $t_m, \tau_{s,m} \in \mathbb{R}, s \in \mathbb{N}$. Here any operator $\bar{\mathcal{A}}^*$ is assumed to be adjoint to the super-integro-differential one $\mathcal{A}$ with respect to the scalar product

$$(x, y) = \sum_{n \in \mathbb{Z}} y(n)\bar{z}(n),$$

where $y, z \in \ell_2(\mathbb{Z}; \mathbb{C}), n \in \mathbb{Z}$. In the paper the line over any variable denotes the complex conjugation of this variable.

In the case of $R = 1$ and $s = 2$ the evolutions of the functions describe the relativistic Toda lattice.

The vector fields (2) have been considered as the Hamiltonian flows generated by the Casimir functionals

$$\gamma_s = \frac{1}{s+1} \sum_{n=0}^{q-1} res \, l^{s+1}[\mathbf{f}(n)], \quad s \in \mathbb{Z}_+, \tag{3}$$

where the symbol "res" denotes the coefficient at $\mathcal{E}^0$ in the expansion of the corresponding operator, and Poisson structure found in [10]. In that paper the hierarchies (3) have been established to be Hamiltonian with respect to the natural powers of some different eigenvalues of the associated spectral problem and Poisson structure mentioned above. It has been shown also that for each $j = \overline{1, N}$ the first eigenfunction symmetry and any other which belong both to the hierarchy related with the same eigenvalue can be applied to construct (2+1)-dimensional differential-difference systems with the triple matrix Lax linearizations. These systems have been obtained by introducing some new functions which denote the expressions with inverse operator to the difference one into the equations of the eigenfunction symmetries.

In the present paper we consider two additional homogeneous symmetries for the Lax type hierarchy (2) such that

$$df_j/d\tau = (-M_1^1 + \delta_1^j l)f_j, \; df_j^*/d\tau = (M_1^1 - \delta_1^j l)^* f_j^* \tag{4}$$

and

$$df_j/dT = (l_+^2 - M_1^2 + \delta_1^j l^2)f_j, \; df_j^*/dT = (-l_+^2 + M_1^2 - \delta_1^j l^2)^* f_j^*, \tag{5}$$

where $\tau := \tau_{1,1}$ and $d/dT := d/dt_2 + \tau_{1,2}$, in the case of $R = 2$. The vector fields $d/d\tau$ and d/dT are commuting because of the relation

$$dl_+^2/d\tau = [l_+^2, \, M_1^1]_+ \tag{6}$$

where $l_+^2 := \mathcal{E}^2 + w_1 \mathcal{E} + w_0$, $w_1 := (\mathcal{E}P) + P$, $w_0 := P^2 + \sum_{j=1}^2 ((\mathcal{E}f_j)f_j^* + f_j(\mathcal{E}^{-1}f_j^*))$ and $P = \sum_{j=1}^2 f_j f_j^*$.

The dynamical systems (4), (5) and commutability condition (6) are written as

$$\begin{aligned} f_{1,\tau} &= (\mathcal{E}f_1) + Pf_1 + uf_2, f_{1,\tau}^* = -(\mathcal{E}^{-1}f_1^*) - Pf_1^* + (\mathcal{E}\tilde{u})f_2^*), \\ f_{2,\tau} &= -\tilde{u}f_1, \quad f_{2,\tau}^* = -(\mathcal{E}u)f_1^*, \end{aligned} \tag{7}$$

$$\begin{aligned} f_{1,T} &= f_{1,\tau\tau} + (\mathcal{E}^2 f_1) + w_1(\mathcal{E}f_1) + w_0 f_1 + 2(f_1(\mathcal{E}^{-1}f_1^*) + u\tilde{u})f_1, \\ f_{1,T}^* &= -f_{1,\tau\tau}^* - (\mathcal{E}^{-2}f_1^*) - (\mathcal{E}^{-1}w_1)(\mathcal{E}^{-1}f_1^*) - w_0 f_1^* - 2(f_1(\mathcal{E}^{-1}f_1^*) + u\tilde{u})f_1^*, \\ f_{2,T} &= (\mathcal{E}^2 f_2) + w_1(\mathcal{E}f_2) + w_0 f_2 - \tilde{u}f_{1,\tau} + \tilde{u}_\tau f_1, \\ f_{2,T}^* &= -(\mathcal{E}^{-2}f_2^*) - (\mathcal{E}^{-1}w_1)(\mathcal{E}^{-1}f_2^*) - w_0 f_2^* + uf_{1,\tau}^* - u_\tau f_1^*, \\ (\mathcal{E} - 1)u &= f_1 f_2^*, \quad (\mathcal{E} - 1)\tilde{u} = f_1^* f_2, \end{aligned} \tag{8}$$

$$\begin{aligned} w_{0,\tau} &= (\mathcal{E}^2 f_1)f_1^* - f_1(\mathcal{E}^{-2}f_1^*) + w_1(\mathcal{E}f_1)f_1^* - f_1(\mathcal{E}^{-1}w_1)(\mathcal{E}^{-1}f_1^*), \\ w_{1,\tau} &= (\mathcal{E}^2 f_1)(\mathcal{E}f_1^*) - f_1(\mathcal{E}^{-1}f_1^*), \end{aligned} \tag{9}$$

where $u, \tilde{u}$ are some q-periodical complex-valued functions. The dynamical system (8) and relationships (9) describe the Lax integrable (2+1)-dimensional differential-difference system [10], which can be considered as some two-dimensional generalization of the relativistic Toda lattice.

Its triple Lax type linearization [10] is formed by the spectral relationship

$$ly = \lambda y, \tag{10}$$

where $y \in \ell_2(\mathbb{Z};\mathbb{C})$, $\lambda \in \mathbb{C}$ is a spectral parameter, and evolution equations

$$dy/d\tau = -M_1^1 y, \tag{11}$$
$$dy/dT = (l_+^2 - M_1^2)y. \tag{12}$$

The corresponding adjoint spectral relationship and adjoint evolutions take following forms:

$$l^* z = \lambda z, \tag{13}$$
$$dz/d\tau = M_1^{1*} z, \tag{14}$$
$$dz/dT = -(l_+^2 - M_1^2)^* z, \tag{15}$$

where $l^* = \mathcal{E}^{-1} - \sum_{j=1}^{2}(f_j^*(\mathcal{E} - 1)^{-1}f_j)$. The spectral relationships (10) and (13) have the equivalent matrix forms

$$\mathcal{E}Y = AY, \tag{16}$$
$$\mathcal{E}^{-1}Z = (\mathcal{E}^{-1}A^\top)Z, \tag{17}$$

where Y, $Z \in \ell_2(\mathbb{Z};\mathbb{C}^3)$, $Y = (y_1, y_2, y_3)^\top$, $y_3 := y$, $Z = (z_1, z_2, z_3)^\top$, $z_3 := (\mathcal{E}^{-1}z)$, $A := A[\mathbf{f}; \lambda]$ and

$$A = \begin{pmatrix} 1 & 0 & f_1^* \\ 0 & 1 & f_2^* \\ -f_1 & -f_2 & \lambda - P \end{pmatrix}.$$

The corresponding evolutions are written as

$$dY/d\tau = B^{(\tau)}Y, \quad dZ/d\tau = -(B^{(\tau)})^\top Z, \tag{18}$$
$$dY/dT = B^{(T)}Y, \quad dZ/dT = -(B^{(T)})^\top Z, \tag{19}$$

where $B^{(\tau)} := B^{(\tau)}[\mathbf{f}; \lambda]$, $B^{(T)} := B^{(T)}[\mathbf{f}; \lambda]$, and

$$B^{(\tau)} = \begin{pmatrix} -\lambda & \tilde{u} & (\mathcal{E}^{-1}f_1^*) \\ -u & 0 & 0 \\ -f_1 & 0 & 0 \end{pmatrix},$$

$$B^{(T)} = \begin{pmatrix} -\lambda^2 - u\tilde{u} - & \lambda\tilde{u} - \tilde{u}_\tau - & 2\lambda(\mathcal{E}^{-1}f_1^*) - \tilde{u}(\mathcal{E}^{-1}f_2^*) + \\ -2f_1(\mathcal{E}^{-1}f_1^*) & -f_2(\mathcal{E}^{-1}f_1^*) & +2(\mathcal{E}^{-1}P)(\mathcal{E}^{-1}f_1^*) + \\ & & +2(\mathcal{E}^{-2}f_1^*) \\ & & \\ -\lambda u - u_\tau - & u\tilde{u} - & \lambda(\mathcal{E}^{-1}f_2^*) + u(\mathcal{E}^{-1}f_1^*) + \\ -f_1(\mathcal{E}^{-1}f_2^*) & -f_2(\mathcal{E}^{-1}f_2^*) & +(\mathcal{E}^{-2}f_2^*) + \\ & & +(\mathcal{E}^{-1}P)(\mathcal{E}^{-1}f_2^*) \\ & & \\ -2\lambda f_1 - uf_2 & -\lambda f_2 - \tilde{u}f_1 - & \lambda^2 + 2f_1(\mathcal{E}^{-1}f_1^*) + \\ -2(\mathcal{E}f_1) - 2Pf_1 & -(\mathcal{E}f_2) - Pf_2 & +f_2(\mathcal{E}^{-1}f_2^*) \end{pmatrix}.$$

The matrices $B^{(\tau)}$ and $B^{(T)}$ satisfy the compatibility conditions

$$dA/d\tau = (\mathcal{E}B^{(\tau)})A - AB^{(\tau)}, \tag{20}$$

$$dA/dT = (\mathcal{E}B^{(T)})A - AB^{(T)}. \tag{21}$$

The system (8)-(9) possesses the infinite sequence of the local conservation laws (3).

2   THE SYMPLECTIC STRUCTURE ON SOME INVARIANT SUBSPACE

We will study below the differential-geometric properties of the commuting vector fields $d/d\tau$ and d/dT on their common invariant finite-dimensional subspace $\mathcal{M}_N^4 \subset \mathcal{M}^4$ such as

$$\mathcal{M}_N^4 = \left\{ \mathbf{f} \in \mathcal{M}^4 : \operatorname{grad} \mathcal{L}_N[\mathbf{f}(n)] = 0 \right\}, \quad \mathcal{L}_N := \sum_{n=0}^{q-1} \mathcal{L}_N[\mathbf{f}(n)] = -\gamma_0 + \sum_{i=1}^{N} c_i \lambda_i,$$

where $\gamma_0 = \sum_{n=0}^{q-1} \sum_{j=1}^{2} f_j(n) f_j^*(n)$, $\lambda_i \in \mathbb{C}$, $i = \overline{1, N}$, are different eigenvalues of the periodic spectral problem (16) with the corresponding eigenvectors $Y_i = (y_{1i}, y_{2i}, y_{3i})^\top \in \mathcal{W}$ and adjoint eigenvectors $Z_i = (z_{1i}, z_{2i}, z_{3i})^\top \in \mathcal{W}$, $\mathcal{W} := \{ \mathbf{a} = (a_1, a_2, a_3)^\top : \mathbf{a}(n) \in \mathbb{C}^3, \mathbf{a}(n+q) = \mathbf{a}(n+q), n \in \mathbb{Z} \} \subset \ell_2(\mathbb{Z}; \mathbb{C})$, and $c_i \in \mathbb{C} \setminus \{0\}$, $i = \overline{1, N}$, are some fixed constants, which will be chosen later. Here the eigenvalues $\lambda_i \in \mathbb{C}$, $i = \overline{1, N}$, are considered as smooth by Frechet functionals on $\mathcal{M}^4$.

We will first analyze the differential-geometric structure of the invariant subspace $\mathcal{M}_N^4 \subset \mathcal{M}^4$. To describe this subspace explicitly we will find the gradients of the eigenvalues $\lambda_i \in \mathcal{D}(\mathcal{M}^4)$, $i = \overline{1, N}$.

Because of the relations

$$\sum_{n=0}^{q-1} (\mathcal{E}Y_i(n))^\top (\mathcal{E}Z_i(n)) = \sum_{n=0}^{q-1} (A[\mathbf{f}(n); \lambda_i] Y_i(n))^\top (\mathcal{E}Z_i(n)), \quad i = \overline{1, N}, \tag{22}$$

that follow from the spectral problem (16), we can derive the explicit form of the gradient of the eigenvalue λ_i for any $i = \overline{1, N}$ only on the level surface $\{ (\mathbf{f}, \mathcal{Y}, \mathcal{Z})^\top \in \hat{\mathcal{M}}^4 : \mu_i = a_i, a_i \in \mathbb{C} \setminus \{0\} \}$ of the functional $\mu_i := -\sum_{n=0}^{q-1} y_{3i}(n)(\mathcal{E}z_{3i}(n))$, which is invariant with respect to the vector fields $d/d\tau$ and d/dT. Thus, for any $i = \overline{1, N}$ the gradient of the eigenvalue λ_i on this level surface is written as

$$\operatorname{grad} \lambda_i = \begin{pmatrix} \delta\lambda_i/\delta f_1 \\ \delta\lambda_i/\delta f_2 \\ \delta\lambda_i/\delta f_1^* \\ \delta\lambda_i/\delta f_2^* \end{pmatrix} = -\frac{1}{a_i} \begin{pmatrix} \bar{f}_1^* \bar{y}_{3i}(\mathcal{E}\bar{z}_{3i}) + \bar{y}_{1i}(\mathcal{E}\bar{z}_{3i}) \\ \bar{f}_2^* \bar{y}_{3i}(\mathcal{E}\bar{z}_{3i}) + \bar{y}_{2i}(\mathcal{E}\bar{z}_{3i}) \\ \bar{f}_1 \bar{y}_{3i}(\mathcal{E}\bar{z}_{3i}) - \bar{y}_{3i}(\mathcal{E}\bar{z}_{1i}) \\ \bar{f}_2 \bar{y}_{3i}(\mathcal{E}\bar{z}_{3i}) - \bar{y}_{3i}(\mathcal{E}\bar{z}_{2i}) \end{pmatrix},$$

where $\bar{Y}_i = (\bar{y}_{1i}, \bar{y}_{2i}, \bar{y}_{3i})^\top$, $\bar{Z}_i = (\bar{z}_{1i}, \bar{z}_{2i}, \bar{z}_{3i})^\top$, $i = \overline{1, N}$.

Let us choose $a_i = -c_i$, $i = \overline{1, N}$, and investigate the vector fields $d/d\tau$ and d/dT on the invariant finite-dimensional subspace $\mathcal{M}_N^4 \bigcap H_c \subset \mathcal{M}^4$ given by the following Bargmann type constraints

$$\mathcal{M}_N^4 \bigcap H_c = \left\{ \mathbf{f} \in \mathcal{M}^4 : \rho f_1 = -\sum_{i=1}^{N} y_{3i} \bar{z}_{1i}, \ \rho f_2 = -\sum_{i=1}^{N} y_{3i} \bar{z}_{2i}, \right.$$

$$\left. \rho f_1^* = \sum_{i=1}^{N} y_{1i} \bar{z}_{3i}, \ \rho f_2^* = \sum_{i=1}^{N} y_{2i} \bar{z}_{3i} \right\},$$

where $H_c := \{(\mathbf{f}, \mathcal{Y}, \mathcal{Z})^\top \in \hat{\mathcal{M}}^4 : \mu_i = -c_i, \, c_i \in \mathbb{C} \setminus \{0\}, \, i = \overline{1, N}\}$ is a common level surface of the invariant functionals μ_i, $i = \overline{1, N}$, in the extended phase space $\hat{\mathcal{M}}^4 := \mathcal{M}^4 \times W^{2N}$ of the coupled dynamical systems (8), (9), (18) and (19) with the parameter $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, and $\mathcal{Y} := (Y_1, Y_2, \ldots, Y_N)^\top$, $\mathcal{Z} := (Z_1, Z_2, \ldots, Z_N)^\top$, $\tilde{Z}_i := \mathcal{E} Z_i = (\tilde{z}_{1i}, \tilde{z}_{2i}, \tilde{z}_{3i})^\top$, $i = \overline{1, N}$, $\rho = 1 - \sum_{i=1}^N y_{3i} \tilde{z}_{3i}$. This invariant subspace can be described by means of the equivalent relationships

$$
\mathcal{M}_N^4 \bigcap H_c = \left\{ \mathbf{f} \in \mathcal{M}^4 : f_1 = - \sum_{i=1}^N y_{3i} z_{1i}, \quad f_2 = - \sum_{i=1}^N y_{3i} z_{2i}, \right.
$$
$$
\left. \mathcal{E}^{-1} f_1^* = \sum_{i=1}^N y_{1i} z_{3i}, \quad \mathcal{E}^{-1} f_2^* = \sum_{i=1}^N y_{2i} z_{3i} \right\},
$$

(23)

From (23) it follows that the functions f_1, f_2, $\mathcal{E}^{-1} f_1^*$, $\mathcal{E}^{-1} f_2^*$ are expressed via the coordinates of the eigenvectors Y_i and Z_i, $i = \overline{1, N}$, on the invariant subspace $\mathcal{M}_N^4 \bigcap H_c$. The relation

$$
\sum_{i=1}^N y_{1i} z_{2i} = 0, \quad \sum_{i=1}^N y_{2i} z_{1i} = 0, \quad (\mathcal{E} - 1) \sum_{i=1}^N y_{3i} z_{3i} = 0,
$$

$$
u \sum_{i=1}^N (y_{2i} z_{2i} - y_{1i} z_{1i}) = - \sum_{i=1}^N \lambda_i y_{2i} z_{1i} - f_1(\mathcal{E}^{-1} f_2^*),
$$

$$
\tilde{u} \sum_{i=1}^N (y_{2i} z_{2i} - y_{1i} z_{1i}) = \sum_{i=1}^N \lambda_i y_{1i} z_{2i} + f_2(\mathcal{E}^{-1} f_1^*),
$$

$$
u_\tau \sum_{i=1}^N (y_{1i} z_{1i} - y_{2i} z_{2i}) = \sum_{i=1}^N \lambda_i^2 y_{2i} z_{1i} + u \sum_{i=1}^N (\lambda_i y_{2i} z_{2i} - \lambda_i y_{1i} z_{1i})
$$
$$
+ u(-f_1(\mathcal{E}^{-1} f_1^*) + f_2(\mathcal{E}^{-1} f_2^*)) + (\mathcal{E}^{-1} f_2^*) \sum_{i=1}^N \lambda_i y_{3i} z_{1i} + 2 f_1 \sum_{i=1}^N \lambda_i y_{2i} z_{3i}
$$
$$
+ f_1(\mathcal{E}^{-1} f_2^*) \sum_{i=1}^N (y_{1i} z_{1i} - y_{2i} z_{2i}) - (\mathcal{E}^{-1} P)(\mathcal{E}^{-1} f_2^*) f_1,
$$

$$
\tilde{u}_\tau \sum_{i=1}^N (y_{2i} z_{2i} - y_{1i} z_{1i}) = \sum_{i=1}^N \lambda_i^2 y_{1i} z_{2i} + \tilde{u} \sum_{i=1}^N (\lambda_i y_{1i} z_{1i} - \lambda_i y_{2i} z_{2i})
$$
(24)
$$
- \tilde{u}(-f_1(\mathcal{E}^{-1} f_1^*) + f_2(\mathcal{E}^{-1} f_2^*)) + (\mathcal{E}^{-1} f_1^*) \sum_{i=1}^N \lambda_i y_{3i} z_{2i} - f_2 \sum_{i=1}^N \lambda_i y_{1i} z_{3i}
$$
$$
- f_2(\mathcal{E}^{-1} f_1^*) \sum_{i=1}^N (y_{1i} z_{1i} + y_{2i} z_{2i} - 2 y_{3i} z_{3i}) - 2(\mathcal{E}^{-1} P)(\mathcal{E}^{-1} f_1^*) f_2,
$$

$$
\mathcal{E} f_1 = - \sum_{i=1}^N \lambda_i y_{3i} z_{1i} + f_1 \sum_{i=1}^N (y_{1i} z_{1i} - y_{3i} z_{3i}) - P f_1,
$$

$$
\mathcal{E} f_2 = - \sum_{i=1}^N \lambda_i y_{3i} z_{2i} + f_2 \sum_{i=1}^N (y_{2i} z_{2i} - y_{3i} z_{3i}) - P f_2,
$$

$$
\mathcal{E}^{-2} f_1^* = \sum_{i=1}^N \lambda_i y_{1i} z_{3i} + (\mathcal{E}^{-1} f_1^*) \left(\sum_{i=1}^N (y_{1i} z_{1i} - y_{3i} z_{3i}) - (\mathcal{E}^{-1} P) \right),
$$

$$
\mathcal{E}^{-2} f_2^* = \sum_{i=1}^N \lambda_i y_{2i} z_{3i} + (\mathcal{E}^{-1} f_2^*) \left(\sum_{i=1}^N (y_{2i} z_{2i} - y_{3i} z_{3i}) - (\mathcal{E}^{-1} P) \right),
$$

obtained with taking into account the equations (8), (9), spectral problems (16), (17) and evolutions (23), allow to express the entries of the matrices $B^\tau[\mathbf{f}; \lambda]$ and $B^T[\mathbf{f}; \lambda]$, reduced upon $\mathcal{M}_N^4 \cap H_c$, via the coordinates of the eigenvectors Y_i and Z_i, $i = \overline{1, N}$. In addition, from the spectral problems (16), (17) and evolution equations (23), when $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ we have

$$\frac{d}{d\tau} \sum_{\chi=1}^{3} y_{\chi i} z_{\chi i} = 0, \quad \frac{d}{dT} \sum_{\chi=1}^{3} y_{\chi i} z_{\chi i} = 0, \quad \frac{d}{d\tau} \sum_{i=1}^{N} y_{3i} \tilde{z}_{3i} = 0, \quad \frac{d}{dT} \sum_{i=1}^{N} y_{3i} \tilde{z}_{3i} = 0.$$

Therefore, we are in a position to formulate the following theorem.

Theorem 1. *The commuting vector fields $d/d\tau$ and d/dT, given by the system (8)-(9), allow the invariant reductions upon the finite-dimensional subspaces $\mathcal{M}_N^4 \cap H_c \subset \mathcal{M}^4$, $N \in \mathbb{N}$. These subspaces are diffeomorphic to the finite-dimensional space $\mathcal{M}_\mathcal{F}$, which is smoothly embedded into the space $\mathbb{R}^{6N}$ and endowed with the Poisson bracket $\{.,.\}_{\omega_\mathcal{F}^{(2)}}$, being the Dirac reduction of the Poisson bracket $\{.,.\}_{\hat{\omega}^{(2)}}$ related with the symplectic structure*

$$\hat{\omega}^{(2)} = \sum_{i=1}^{N} \sum_{s=1}^{3} d(\mathcal{E}^{-1} \tilde{z}_{si}) \wedge dy_{si} = \sum_{i=1}^{N} \sum_{s=1}^{3} dz_{si} \wedge dy_{si}, \tag{25}$$

where "$\wedge$" is a symbol of the exterior product on the Grassmann algebra of differential forms on $\mathbb{C}^{6N}$. The reduced vector fields $d/d\tau$ and d/dT, given by the equations (18) and (19) when $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, are Hamiltonian with respect to the Poisson bracket $\{.,.\}_{\omega_\mathcal{F}^{(2)}}$. The corresponding Hamiltonians $h^{(\tau)}$, $h^{(T)} \in C^\infty(\mathbb{R}^{6N}; \mathbb{R})$ are reductions of the functions $\hat{h}^{(\tau)}$, $\hat{h}^{(T)} \in D(\hat{\mathcal{M}}^4)$, satisfying the equalities

$$\left\langle \left(d\mathbf{f}/d\tau, d\mathcal{Y}/d\tau, d\tilde{\mathcal{Z}}/d\tau\right)^\top, \operatorname{grad} \hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \tilde{\mathcal{Z}}] \right\rangle = -(\mathcal{E} - 1)\hat{h}^{(\tau)}, \tag{26}$$

$$\left\langle \left(d\mathbf{f}/dT, d\mathcal{Y}/dT, d\tilde{\mathcal{Z}}/dT\right)^\top, \operatorname{grad} \hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \tilde{\mathcal{Z}}] \right\rangle = -(\mathcal{E} - 1)\hat{h}^{(T)}, \tag{27}$$

where the brackets $\langle , \rangle$ denote the standard scalar product on $\mathbb{C}^{6N+4}$, and involutive with respect to the Poisson bracket $\{.,.\}_{\omega^{(2)}}$. The relationships (23) describe all periodic and quasi-periodic solutions of the system (8), (9) on the subspaces $\mathcal{M}_N^4 \cap H_c$, $N \in \mathbb{N}$.

Proof. The exact symplectic structure on the invariant subspace $\mathcal{M}_N^4 \subset \mathcal{M}^4$ can be found by means of the discrete analog [18], [19], [20] of the Gelfand-Dikii relationship on the functional manifold $\mathcal{M}^4$ in the same manner as has been done in the paper [19] for the subspaces of critical points of local conservation laws.

To make use this relationship we need the explicit forms of the smooth by Frechet functionals λ_i, $i = \overline{1, N}$, on H_c. From the equalities (22) we have

$$\lambda_i' = \sum_{n=0}^{q-1} \left(\sum_{s=1}^{3} (\mathcal{E} y_{si}(n)) \tilde{z}_{si}(n) - y_{1i}(n) \tilde{z}_{1i}(n) - y_{2i}(n) \tilde{z}_{2i}(n) - f_1^*(n) y_{3i}(n) \tilde{z}_{1i}(n) \right.$$

$$\left. - f_2^*(n) y_{3i}(n) \tilde{z}_{2i}(n) + f_1(n) y_{1i}(n) \tilde{z}_{3i}(n) + f_2(n) y_{2i}(n) \tilde{z}_{3i}(n) + P(n) y_{3i}(n) \tilde{z}_{3i}(n) \right),$$

where $\lambda_i' := \lambda_i|_{H_c}$, $i = \overline{1, N}$, on the level surface H_c in the extended phase space $\hat{\mathcal{M}}^4$. Since the functionals $\lambda_i' \in \mathcal{D}(\hat{\mathcal{M}}^4)$, $i = \overline{1, N}$, depend on the functions $(f, \mathcal{Y}, \mathcal{Z})^\top \in \hat{\mathcal{M}}^4$, it is expedient

to apply the discrete analog of the Gelfand-Dikii relationship to the Lagrangian functional $\hat{\mathcal{L}}_N := \sum_{n=0}^{q-1} \hat{\mathcal{L}}_N[\mathbf{f}(n), \mathcal{Y}(n), \tilde{\mathcal{Z}}(n)] \in \mathcal{D}(\hat{\mathcal{M}}^4)$ of the form

$$\tilde{\mathcal{L}}_N = -\gamma_0 + \sum_{i=1}^{N} \lambda_i' + \sum_{i=1}^{N} \varsigma_i \mu_i,$$

where $\varsigma_i \in \mathbb{C}$ are Lagrangian multipliers, $\mu_i = -\sum_{n=0}^{q-1} y_{3i}(n)\tilde{z}_{3i}(n)$, $i = \overline{1, N}$.

Because of the Lax theorem [11], [12] the condition $grad\, \hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \tilde{\mathcal{Z}}] = 0$ determines the invariant subspace $\tilde{\mathcal{M}}_N^4 \subset \hat{\mathcal{M}}^4$,

$$\tilde{\mathcal{M}}_N^4 = \left\{ (\mathbf{f}, \mathcal{Y}, \tilde{\mathcal{Z}})^\top \in \mathcal{M}^4 : f_1 = -\sum_{i=1}^{N} y_{3i} z_{1i}, \ f_2 = -\sum_{i=1}^{N} y_{3i} z_{2i}, \ \mathcal{E}^{-1} f_1^* = \sum_{i=1}^{N} y_{1i} z_{3i}, \right.$$

$$\left. \mathcal{E}^{-1} f_2^* = \sum_{i=1}^{N} y_{2i} z_{3i}, \mathcal{E} Y_i = A[\mathbf{f}; \varsigma_i] Y_i, \ \mathcal{E}^{-1} \tilde{Z}_i = A^\top[\mathbf{f}; \varsigma_i] \tilde{Z}_i, \ i = \overline{1, N} \right\},$$

of the coupled dynamical systems (8), (9), (18) and (19) with the parameter $\lambda \in \{\varsigma_1, \ldots, \varsigma_N\}$. Thus, for every $N \in \mathbb{N}$ the invariant subspace $\mathcal{M}_N^4 \bigcap H_c \subset \mathcal{M}^4$ is diffeomorphic to the subspace $\hat{\mathcal{M}}_N^4 \subset \hat{\mathcal{M}}^4$ when $\varsigma_i = \lambda_i, i = \overline{1, N}$.

By means of the discrete analog of the Gelfand-Dikii differential relationship [18], [19], [20] for $\hat{\mathcal{L}}_N \in \mathcal{D}(\hat{\mathcal{M}}^4)$ such as

$$d\hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \tilde{\mathcal{Z}}] = \langle (d\mathbf{f}, d\mathcal{Y}, d\tilde{\mathcal{Z}})^\top, grad\, \hat{\mathcal{L}}_N[\mathbf{f}, \mathcal{Y}, \tilde{\mathcal{Z}}] \rangle + ((\mathcal{E} - 1)\alpha^{(1)}), \tag{28}$$

where $(\mathcal{Y}, \tilde{\mathcal{Z}})^\top$ are coordinates on the suitably truncated manifold $\hat{\mathcal{M}}_N^4$ and the brackets $\langle , \rangle$ denote the standard scalar product on $\mathbb{C}^{6N+4}$, we can find the exact two-form (25)

$$\hat{\omega}^{(2)} := d\alpha.^{(1)}$$

The reduced two-form $\omega^{(2)} := \hat{\omega}^{(2)} \big|_{\hat{\mathcal{M}}_N^4}$ defines the symplectic structure on the invariant subspace $\mathcal{M}_N^4 \bigcap H_c \simeq \tilde{\mathcal{M}}_N^4 \subset \hat{\mathcal{M}}_N^4$, which is smoothly embedded into $\hat{\mathcal{M}}_N^4$ due to the relationships (23).

The formula (28) ensures the invariance of the reduced two-form $\omega^{(2)}$ with respect to the operator $(\mathcal{E} - 1)$, that is

$$\sum_{i=1}^{N} \sum_{s=1}^{3} d(\mathcal{E} z_{si}) \wedge d(\mathcal{E} y_{si}) = \sum_{i=1}^{N} \sum_{s=1}^{3} dz_{si} \wedge dy_{si}.$$

Taking into account that the subspace $\mathcal{M}_N^4 \bigcap H_c \subset \mathcal{M}^4$ is diffeomorphic to the finite-dimensional submanifold $\mathcal{M}_{\mathcal{F}} \subset \mathbb{R}^{6N}$ determined by the constraints

$$F_1 := \sum_{i=1}^{N} y_{1i} z_{2i} = 0, \quad F_2 := \sum_{i=1}^{N} y_{2i} z_{1i} = 0,$$

in the space $\mathbb{R}^{6N}$, we can obtain the symplectic structure on $\mathcal{M}_N^4 \bigcap H_c$ as a natural Dirac type reduction of the two-form $\hat{\omega}^{(2)}$ on $\mathcal{M}_{\mathcal{F}}$.

The two-form $\hat{\omega}^{(2)}$ generates the standard Poisson bracket $\{., .\}_{\hat{\omega}^{(2)}}$ on $\mathbb{R}^{6N}$. As the matrix of constraints $\{F_{\kappa_1}, F_{\kappa_2}\}_{\hat{\omega}^{(2)}}, \kappa_1, \kappa_2 = 1, 2$, is nondegenerate when $Q := \sum_{i=1}^{N} (y_{1i} z_{1i} - y_{2i} z_{2i}) \neq 0$,

the standard Dirac type reduction procedure [7,11] entails the Poisson bracket related with the symplectic structure $\omega_{\mathcal{F}}^{(2)} := \omega^{(2)}$ such that

$$\{F,G\}_{\omega_{\mathcal{F}}^{(2)}} = \{F,G\}_{\hat{\omega}^{(2)}} + \frac{1}{Q}\{F,F_1\}_{\hat{\omega}^{(2)}}\{F_2,G\}_{\hat{\omega}^{(2)}} - \frac{1}{Q}\{F,F_2\}_{\hat{\omega}^{(2)}}\{F_1,G\}_{\hat{\omega}^{(2)}}$$

$$= \{F,G\}_{\hat{\omega}^{(2)}} + \frac{1}{Q}\sum_{i_1=1}^{N}\left(z_{2i_1}\frac{\partial F}{\partial z_{1i_1}} - y_{1i_1}\frac{\partial F}{\partial y_{2i_1}}\right)\sum_{i_2=1}^{N}\left(y_{2i_2}\frac{\partial G}{\partial y_{1i_2}} - z_{1i_2}\frac{\partial G}{\partial z_{2i_2}}\right)$$

$$- \frac{1}{Q}\sum_{i_1=1}^{N}\left(-y_{2i_1}\frac{\partial F}{\partial y_{1i_1}} + z_{1i_1}\frac{\partial F}{\partial z_{2i_1}}\right)\sum_{i_2=1}^{N}\left(-z_{2i_2}\frac{\partial G}{\partial z_{1i_2}} + y_{1i_2}\frac{\partial_l G}{\partial y_{2i_2}}\right),$$

where F, $G \in C^\infty(\mathbb{R}^{6N};\mathbb{R})$ are arbitrary smooth functions. Since

$$d\hat{\mathcal{L}}_N/d\tau = 0, \quad d\hat{\mathcal{L}}_N/dT = 0,$$

with taking into account the results obtained in the papers [18], [19] we can state the existence of the smooth by Frechet functions $\hat{h}^{(\tau)}$, $\hat{h}^{(T)} \in \mathcal{D}(\mathcal{M}^4)$, which satisfy the relations (26) and (27) correspondingly. Then for the functions $h^{(\tau)} := \hat{h}^{(\tau)}\big|_{\mathcal{M}_N^4}$ and $h^{(T)} := \hat{h}^{(T)}\big|_{\mathcal{M}_N^4}$ we have

$$i_{d/d\tau}\omega^{(2)} = -dh^{(\tau)}, \quad i_{d/dT}\omega^{(2)} = -dh^{(T)},$$

where $i_{d/d\tau}$ and $i_{d/dT}$ are inner differentiations with respect to the vector fields $d/d\tau : \mathcal{M}_N^4 \to T(\mathcal{M}_N^4)$ and $d/dT : \mathcal{M}_N^4 \to T(\mathcal{M}_N^4)$ in the Grassmann algebra of differential forms on $\mathbb{R}^{6N}$.

Therefore, the functions $h^{(\tau)}$ and $h^{(T)}$ are Hamiltonians of the reduced upon $\mathcal{M}_N^4 \bigcap H_c \subset \mathcal{M}^4$ vector fields $d/d\tau$ and d/dT when $\varsigma_i = \lambda_i$, $i = \overline{1,N}$. They take the following forms

$$h^{(\tau)} = -\sum_{i=1}^{N}\lambda_i y_{1i}z_{1i} - f_1(\mathcal{E}^{-1}f_1^*),$$

$$h^{(T)} = \sum_{i=1}^{N}(\lambda_i^2 y_{3i}z_{3i} - \lambda_i^2 y_{1i}z_{1i})$$

$$+ \frac{\left(\sum_{i=1}^{N}\lambda_i y_{1i}z_{2i} + f_2(\mathcal{E}^{-1}f_1^*)\right)\left(\sum_{i=1}^{N}\lambda_i y_{2i}z_{1i} - f_1(\mathcal{E}^{-1}f_2^*)\right)}{\sum_{i=1}^{N}(y_{2i}z_{2i} - y_{1i}z_{1i})}$$

$$+ 2(\mathcal{E}^{-1}f_1^*)\sum_{i=1}^{N}\lambda_i y_{3i}z_{1i} + (\mathcal{E}^{-1}f_2^*)\sum_{i=1}^{N}\lambda_i y_{3i}z_{2i} - 2f_1\sum_{i=1}^{N}\lambda_i y_{1i}z_{3i} - f_2\sum_{i=1}^{N}\lambda_i y_{2i}z_{3i}$$

$$+ 2f_1(\mathcal{E}^{-1}f_1^*)\sum_{i=1}^{N}(y_{3i}z_{3i} - y_{1i}z_{1i}) + f_2(\mathcal{E}^{-1}f_2^*)\sum_{i=1}^{N}(y_{3i}z_{3i} - y_{1i}z_{1i}),$$

where the functions f_1, f_2, $\mathcal{E}^{-1}f_1^*$, $\mathcal{E}^{-1}f_2^*$ have the forms (23).

By means of the direct calculations it is easily to verify that

$$\{h^{(\tau)},h^{(T)}\}_{\omega^{(2)}} = -\frac{d}{d\tau}h^{(T)} = 0.$$

$\square$

Let us consider the vector field d/dt_1, commuting with the vector fields $d/d\tau$ and d/dT, on the functional manifold $\mathcal{M}^4$ and investigate its reduction upon the invariant subspace $\mathcal{M}^4_N \cap H_c \subset \mathcal{M}^4$, $N \in \mathbb{N}$. In the same manner as in the proof of Theorem 1 we can find the Hamiltonian representation for the reduced vector field d/dt_1. The corresponding Hamiltonian $h^{(t_1)}$ takes the form

$$h^{(t_1)} = \sum_{i=1}^{N} \lambda_i y_{3i} z_{3i} - f_1(\mathcal{E}^{-1} f_1^*) - f_2(\mathcal{E}^{-1} f_2^*).$$

Since

$$\{h^{(t_1)}, h^{(\tau)}\}_{\omega^{(2)}} = -\frac{d}{dt_1} h^{(\tau)} = 0, \quad \{h^{(t_1)}, h^{(T)}\}_{\omega^{(2)}} = -\frac{d}{dt_1} h^{(T)} = 0,$$

the reduced vector fields d/dt_1, $d/d\tau$ and d/dT are integrable in the case of $N = 1$ due to the Liouville theorem [1], [17].

3   THE LAX-LIOUVILLE INTEGRABILITY OF REDUCED VECTOR FIELDS

To state the Liouville integrability of the Hamiltonian vector fields $d/d\tau$ and d/dT on $\mathcal{M}^4_N \cap H_c \subset \mathcal{M}^4$ for all $N \in \mathbb{N}$ we need to construct the related matrix Lax representations, which depend on the spectral parameter $\lambda \in \mathbb{C}$, making use the reduction procedure for the monodromy matrix of the periodic spectral problem (16). Thus, the following theorem holds.

Theorem 2. *For every* $N \in \mathbb{N}$ *on the intersections of the finite-dimensional subspace* $\mathcal{M}^4_N \cap H_c \simeq \mathcal{M}_\mathcal{F}$ *with the level surfaces* $h_C := \{(\mathcal{Y}, \mathcal{Z})^\top \in \mathbb{R}^{6N} : \sum_{i=1}^{N} y_{3i} \check{z}_{31} = C, C \in \mathbb{C}\}$ *of the invariant function* $1 - \rho = \sum_{i=1}^{N} y_{3i} z_{3i}$ *the matrix Lax representations for the Hamiltonian vector fields* $d/d\tau$ *and* d/dT *have the following forms*

$$d\check{S}_N/d\tau = [B_N^{(\tau)}, \check{S}_N], \tag{29}$$

$$d\check{S}_N/dT = [B_N^{(T)}, \check{S}_N], \tag{30}$$

where $B_N^{(\tau)} := B_N^{(\tau)}(\mathcal{Y}, \mathcal{Z}; \lambda) = B^{(\tau)}[\mathbf{f}; \lambda]\big|_{\mathcal{M}_\mathcal{F} \cap h_C}$, $B_N^{(T)} := B_N^{(T)}(\mathcal{Y}, \mathcal{Z}; \lambda) = B_N^{(T)}[\mathbf{f}; \lambda]\big|_{\mathcal{M}_\mathcal{F} \cap h_C}$ *are projections of the corresponding matrices on* $\mathcal{M}_\mathcal{F} \cap h_C$ *and*

$$\check{S}_N = \sum_{i=1}^{N} \frac{S_i}{\lambda - \lambda_i} + S_0$$

$$= \sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i} \begin{pmatrix} y_{1i} z_{1i} & y_{1i} z_{2i} & y_{1i} z_{3i} \\ y_{2i} z_{1i} & y_{2i} z_{2i} & y_{2i} z_{3i} \\ y_{3i} z_{1i} & y_{3i} z_{2i} & y_{3i} z_{3i} \end{pmatrix} + \begin{pmatrix} -C & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & 1 - C \end{pmatrix}. \tag{31}$$

Proof. Making use the spectral problem (16), we can express the gradient $\varphi(n; \bar{\lambda}) := \operatorname{grad} \operatorname{tr} S$ of the trace of the corresponding monodromy matrix

$$S := S(n; \lambda) = A[\mathbf{f}(n + q - 1); \lambda] A[\mathbf{f}(n + q - 2); \lambda] \times \ldots \times A[\mathbf{f}(n); \lambda]$$

via the entries of the matrix $V = SA^{-1}$ by such a way

$$\varphi(n; \bar{\lambda}) = \begin{pmatrix} \operatorname{tr}(\bar{V} \bar{A}_{f_1}) \\ \operatorname{tr}(\bar{V} \bar{A}_{f_2}) \\ \operatorname{tr}(\bar{V} \bar{A}_{f_1^*}) \\ \operatorname{tr}(\bar{V} \bar{A}_{f_2^*}) \end{pmatrix} = \begin{pmatrix} -\bar{V}_{13} - \bar{f}_1^* \bar{V}_{33} \\ -\bar{V}_{23} - \bar{f}_2^* \bar{V}_{33} \\ \bar{V}_{31} - \bar{f}_1 \bar{V}_{33} \\ \bar{V}_{32} - \bar{f}_2 \bar{V}_{33} \end{pmatrix},$$

where $\varphi(n; \bar{\lambda}) \simeq \sum_{r \in \mathbb{Z}_+} \varphi_r(n) \bar{\lambda}^{-(r+1)}$, $\varphi_r = grad\,\gamma_r[\mathbf{f}]$, when $|\lambda| \to \infty$, $\bar{V}$ is a matrix with the entries, being complex conjugate to the corresponding ones of the matrix

$$V := \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix},$$

and $\bar{A}_{f_1}$, $\bar{A}_{f_2}$, $\bar{A}_{f_1^*}$, $\bar{A}_{f_2^*}$ are matrices with the enteries, being complex conjugate to the corresponding ones of A_{f_1}, A_{f_2}, $A_{f_1^*}$, $A_{f_2^*}$ respectively.

From the equation for the matrix V

$$\mathcal{E}(VA) = AV, \tag{32}$$

we can obtain the Magri type relationships [15]

$$\vartheta\,\varphi(n; \bar{\lambda}) = \bar{\lambda}\,\eta\,\varphi(n; \bar{\lambda}) - \eta\,\varphi_0, \tag{33}$$

where ϑ, $\eta : T^*(\mathcal{M}^4) \to T(\mathcal{M}^4)$ are a pair of linear Poisson operators of the forms

$$\eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\vartheta = \begin{pmatrix} -\bar{f}_1 \Pi \bar{f}_1 & \begin{matrix} -\bar{f}_2 \Delta^{-1} \mathcal{E} \bar{f}_1 - \\ -\bar{f}_1 \Delta^{-1} \bar{f}_2 \end{matrix} & \begin{matrix} \mathcal{E} + \bar{f}_1 \Pi \bar{f}_1^* + \\ +\bar{f}_2 \Delta^{-1} \bar{f}_2^* + \\ +\bar{P} \end{matrix} & \bar{f}_1 \Delta^{-1} \mathcal{E} \bar{f}_2^* \\[2em] \begin{matrix} -\bar{f}_1 \Delta^{-1} \bar{f}_2 - \\ -\bar{f}_2 \Delta^{-1} \mathcal{E} \bar{f}_1 \end{matrix} & -\bar{f}_2 \Pi \bar{f}_2 & \bar{f}_2 \Delta^{-1} \mathcal{E} \bar{f}_1^* & \begin{matrix} \mathcal{E} + \bar{f}_2 \Pi \bar{f}_2^* + \\ +\bar{f}_1 \Delta^{-1} \bar{f}_1^* + \\ +\bar{P} \end{matrix} \\[2em] \begin{matrix} -\mathcal{E}^{-1} + \bar{f}_1^* \Pi \bar{f}_1 + \\ +\bar{f}_2^* \Delta^{-1} \mathcal{E} \bar{f}_2 - \bar{P} \end{matrix} & \bar{f}_1^* \Delta^{-1} \bar{f}_2 & -\bar{f}_1^* \Pi \bar{f}_1^* & \begin{matrix} -\bar{f}_1^* \Delta^{-1} \mathcal{E} \bar{f}_2^* - \\ -\bar{f}_2^* \Delta^{-1} \bar{f}_1^* \end{matrix} \\[2em] \bar{f}_2^* \Delta^{-1} \bar{f}_1 & \begin{matrix} -\mathcal{E}^{-1} + \bar{f}_2^* \Pi \bar{f}_2 + \\ +\bar{f}_1^* \Delta^{-1} \mathcal{E} \bar{f}_1 - \bar{P} \end{matrix} & \begin{matrix} -\bar{f}_2^* \Delta^{-1} \bar{f}_1^* - \\ -\bar{f}_1^* \Delta^{-1} \mathcal{E} \bar{f}_2^* \end{matrix} & -\bar{f}_2^* \Pi \bar{f}_2^* \end{pmatrix}.$$

Here $\Delta = (\mathcal{E} - 1)$, $\Pi = \Delta^{-1}(\mathcal{E} + 1)$. Taking into account the equality

$$\varphi(n; \bar{\lambda}_i) = \left(\overline{\left. \frac{d}{d\lambda} \operatorname{tr} S(n; \lambda) \right|_{\lambda = \lambda_i}} \right) grad\,\lambda_i,$$

we find for every $i = \overline{1, N}$ that

$$\Lambda\,grad\,\lambda_i = \bar{\lambda}_i\,grad\,\lambda_i + \left(\overline{\left. \frac{d}{d\lambda} \operatorname{tr} S(n; \lambda) \right|_{\lambda = \lambda_i}} \right)^{-1} \varphi_0, \quad \Lambda = \eta^{-1} \vartheta,$$

where $\sigma_i := \left(\dfrac{d}{d\lambda} \operatorname{tr} S(n;\lambda) \Big|_{\lambda=\lambda_i} \right)^{-1} = \sum_{\chi=1}^{3} y_{\chi i} z_{\chi i}$ is invariant with respect to the vector fields $d/d\tau$ and d/dT.

Then on the invariant subspace $\mathcal{M}_N^4 \cap H_c \subset \mathcal{M}^4$ the gradients of the conservation laws $\gamma_m \in \mathcal{D}(\mathcal{M}^4)$, $m \in \mathbb{Z}_+$, take the forms

$$\varphi_0 = \sum_{i=1}^{N} grad\, \lambda_i, \quad \varphi_1 = \Lambda \varphi_0 = \sum_{i=1}^{N} \bar{\lambda}_i\, grad\, \lambda_i + \bar{J}_1 \sum_{i=1}^{N} grad\, \lambda_i, \quad \dots,$$

$$\varphi_r = \Lambda \varphi_{r-1} = \sum_{i=1}^{N} \bar{\lambda}_i^r\, grad\, \lambda_i + \sum_{p=1}^{r} \bar{J}_p \sum_{i=1}^{N} \bar{\lambda}_i^{r-p}\, grad\, \lambda_i, \quad \text{etc.,} \tag{34}$$

where

$$J_1 = \sum_{i=1}^{N} \sigma_i, \quad J_2 = \sum_{i=1}^{N} \lambda_i \sigma_i + J_1 \sum_{i=1}^{N} \sigma_i, \quad \dots, J_r = \sum_{i=1}^{N} \lambda_i^r \sigma_i + \sum_{p=1}^{r} J_p \sum_{i=1}^{N} \lambda_i^{r-p} \sigma_i, \quad \text{etc.}$$

From the relationships (33) and (34) we obtain directly the explicit forms of the entries V_{13}, V_{23}, V_{31}, V_{31}, V_{32}, V_{33} on $\mathcal{M}_N^4 \cap H_c$ such as

$$V_{13} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r} \right) \sum_{i=1}^{N} \frac{y_{1i}\tilde{z}_{3i}}{\lambda - \lambda_i}, \quad V_{23} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r} \right) \sum_{i=1}^{N} \frac{y_{2i}\tilde{z}_{3i}}{\lambda - \lambda_i},$$

$$V_{31} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r} \right) \sum_{i=1}^{N} \frac{y_{3i}\tilde{z}_{1i}}{\lambda - \lambda_i}, \quad V_{32} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r} \right) \sum_{i=1}^{N} \frac{y_{3i}\tilde{z}_{2i}}{\lambda - \lambda_i},$$

$$V_{33} = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r} \right) \sum_{i=1}^{N} \frac{y_{3i}\tilde{z}_{3i}}{\lambda - \lambda_i},$$

where $1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r} = \left(1 - \sum_{i=1}^{N} \dfrac{\sigma_i}{\lambda - \lambda_i} \right)^{-1}$.

The remaining entries of the reduced matrix $V_N := V|_{\mathcal{M}_N^4 \cap H_c}$ can be derived from the equation (32), considered on the level surfaces h_C, $C \in \mathbb{C}$, of the invariant function $1 - \rho$. On these surfaces the functions f_1^*, f_2^* satisfy the following equalities

$$f_1^* \left(1 - C + \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{1i} z_{1i} \right) + f_2^* \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{1i} z_{2i} = \sum_{i_1=1}^{N} \frac{1}{\lambda_i} y_{1i} z_{3i},$$

$$f_1^* \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{2i} z_{1i} + f_2^* \left(1 - C + \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{2i} z_{2i} \right) = \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{2i} z_{3i},$$

$$- f_1^* \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{3i} z_{1i} - f_2^* \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{3i} z_{2i} + \sum_{i=1}^{N} \frac{1}{\lambda_i} y_{3i} z_{3i} = C.$$

Thus, the reduced matrix V_N on $\mathcal{M}_N^4 \cap H_c \cap h_C$, $C \in \mathbb{C}$, is written as

$$V_N = \left(1 + \sum_{r\in\mathbb{N}} J_r \lambda^{-r} \right) \check{V}_N,$$

where

$$
\check{V}_N = \sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i} \begin{pmatrix} y_{1i}\check{z}_{1i} & y_{1i}\check{z}_{2i} & y_{1i}\check{z}_{3i} \\ y_{2i}\check{z}_{1i} & y_{2i}\check{z}_{2i} & y_{2i}\check{z}_{3i} \\ y_{3i}\check{z}_{1i} & y_{3i}\check{z}_{2i} & y_{3i}\check{z}_{3i} \end{pmatrix} + \begin{pmatrix} -C & 0 & 0 \\ 0 & -C & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

The explicit form (31) of the monodromy matrix S_N on $\mathcal{M}_N^4 \cap H_c \cap h_C$, $C \in \mathbb{C}$, follows from the relationship

$$
S_N = V_N A_N,
$$

where the matrix $A_N := A_N(\mathcal{Y}, \mathcal{Z}; \lambda) = A[\mathbf{f}; \lambda]|_{\mathcal{M}_{\mathcal{F}} \cap h_C}$ is a projection of the matrix A on $\mathcal{M}_N^4 \cap H_c \cap h_C$. Thus,

$$
S_N = \left(1 + \sum_{r \in \mathbb{N}} J_r \lambda^{-r} \right) \check{S}_N,
$$

where the matrix $\check{S}_N$ has the form (31) when $|\lambda| > \max_{i=\overline{1,N}} |\lambda_i|$ and $\left(\sum_{i=1}^{N} \frac{\sigma_i}{\lambda - \lambda_i} \right) \neq 1$.

The relations (29) and (30) are derived from the monodromy matrix equation [6]

$$
(\mathcal{E}S)A = AS
$$

and compatibility conditions (20)-(21).   □

Due to the equations (29) and (30) the functionals $\frac{1}{\alpha} \operatorname{tr} \check{S}_N^\alpha$, $\alpha \in \mathbb{N}$, are invariant with respect to the vector fields $d/d\tau$ and d/dT. Then the coefficients in the expansions of these functionals by poles appear to be conservation laws of the reduced upon $\mathcal{M}_{\mathcal{F}} \cap h_C$, $C \in \mathbb{C}$, vector fields given by the system (8), (9). The coefficients $\sigma_i, \hat{\sigma}_i, \tilde{\sigma}_i \in C^\infty(\mathbb{R}^{6N}; \mathbb{R})$, $i = \overline{1, N}$, in the expansions of the invariant functionals $\operatorname{tr} \check{S}_N$, $\frac{1}{2} \operatorname{tr} \check{S}_N^2$ and $\frac{1}{3} \operatorname{tr} \check{S}_N^3$ such that

$$
\operatorname{tr} \check{S}_N = \sum_{i=1}^{N} \frac{\sigma_i}{\lambda - \lambda_i} - 3C + 1,
$$

$$
\frac{1}{2} \operatorname{tr} \check{S}_N^2 = \frac{1}{2} \sum_{i=1}^{N} \frac{\sigma_i^2}{(\lambda - \lambda_i)^2} + \sum_{i=1}^{N} \frac{\hat{\sigma}_i}{\lambda - \lambda_i} + \frac{1}{2}(3C^2 - 2C + 1),
$$

$$
\hat{\sigma}_i = \sum_{k=1, k \neq i}^{N} \frac{\operatorname{tr}(S_i S_k)}{\lambda_i - \lambda_k} + \operatorname{tr}(S_0 S_i) = \sum_{k=1, k \neq i}^{N} \frac{\left(\sum_{\chi_1=1}^{3} y_{\chi_1 i} z_{\chi_1 k} \right) \left(\sum_{\chi_2=1}^{3} y_{\chi_2 k} z_{\chi_2 i} \right)}{\lambda_i - \lambda_k}
$$
$$
- 3C(y_{1i} z_{1i} + y_{2i} z_{2i} + y_{3i} z_{3i}) + y_{3i} z_{3i},
$$

and

$$\frac{1}{3}\operatorname{tr}\check{S}_N^3 = \frac{1}{3}\sum_{i=1}^{N}\frac{\sigma_i^3}{(\lambda-\lambda_i)^3} + \sum_{i=1}^{N}\frac{\sigma_i\hat{\sigma}_i}{(\lambda-\lambda_i)^2} + \sum_{i=1}^{N}\frac{\tilde{\sigma}_i}{\lambda-\lambda_i} - \frac{1}{3}(3C^3 - 3C^2 + 3C - 1),$$

$$\tilde{\sigma}_i = \sum_{\substack{k,\ell=1,\\ k,\ell\neq i,\, k\neq \ell}}^{N}\frac{\operatorname{tr}(S_iS_kS_\ell)}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_\ell)} + \sum_{k=1,\,k\neq i}^{N}\frac{\operatorname{tr}(S_iS_k)(\sigma_k-\sigma_i)}{(\lambda_i-\lambda_k)^2}$$

$$+ \sum_{k=1,\,k\neq i}^{N}\frac{\operatorname{tr}(S_0(S_iS_k+S_kS_i))}{\lambda_i-\lambda_k} + \operatorname{tr}(S_0^2 S_i)$$

$$= \sum_{\substack{k,\ell=1,\\ k,\ell\neq i,\, k\neq \ell}}^{N}\frac{\left(\sum_{\chi_1=1}^{3}y_{\chi_1 i}z_{\chi_1 \ell}\right)\left(\sum_{\chi_2=1}^{3}y_{\chi_2 \ell}z_{\chi_2 k}\right)\left(\sum_{\chi_3=1}^{3}y_{\chi_3 k}z_{\chi_3 i}\right)}{(\lambda_i-\lambda_k)(\lambda_i-\lambda_\ell)}$$

$$+ \sum_{k=1,\,k\neq i}^{N}\frac{\left(\sum_{\chi_1=1}^{3}y_{\chi_1 i}z_{\chi_1 k}\right)\left(\sum_{\chi_2=1}^{3}y_{\chi_2 k}z_{\chi_2 i}\right)\sum_{\chi_3=1}^{3}(y_{\chi_3 i}z_{\chi_3 i} - y_{\chi_3 k}z_{\chi_3 k})}{(\lambda_i-\lambda_k)^2}$$

$$+ \sum_{k=1,\,k\neq i}^{N}\frac{(-C(y_{1i}z_{1k} + y_{2i}z_{2k} + y_{3i}z_{3k}) + y_{3i}z_{3k})\left(\sum_{\chi=1}^{3}y_{\chi k}z_{\chi i}\right)}{\lambda_i-\lambda_k}$$

$$+ \sum_{k=1,\,k\neq i}^{N}\frac{(-C(y_{1k}z_{1i} + y_{2k}z_{2i} + y_{3k}z_{3i}) + y_{3k}z_{3i})\left(\sum_{\chi=1}^{3}y_{\chi i}z_{\chi k}\right)}{\lambda_i-\lambda_k}$$

$$+ (C^2 y_{1i}z_{1i} + C^2 y_{2i}z_{2i} + (1-C)^2 y_{3i}z_{3i}),$$

are functionally independent on $\mathcal{M}_{\mathcal{F}}\bigcap h_C$, $C \in \mathbb{C}$. Being involutive with respect to the Poisson bracket $\{.,.\}_{\omega^{(2)}}$, the coefficients σ_i, $\hat{\sigma}_i$, $\tilde{\sigma}_i \in C^\infty(\mathbb{R}^{6N};\mathbb{R})$, $i = \overline{1,N}$, ensure the Liouville integrability of the vector fields $d/d\tau$ and d/dT on the finite-dimensional subspaces $\mathcal{M}_{\mathcal{F}}\bigcap h_C$, $C \in \mathbb{C}$ (see [1], [17]). The surfaces h_C, $C \in \mathbb{C}$, mentioned in Theorem 2, are determined by the conditions

$$\left(\sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{1i_1}z_{3i_1}\left(1 - C + \sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{2i_2}z_{2i_2}\right) - \sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{2i_1}z_{3i_1}\left(\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{1i_2}z_{2i_2}\right)\right)\sum_{i=1}^{N}\frac{1}{\lambda_{i_3}}y_{3i_3}z_{1i_3}$$

$$- \left(\left(1 - C + \sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{1i_1}z_{1i_1}\right)\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{2i_2}z_{3i_2} - \sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{1i_1}z_{2i_1}\left(\sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{2i_2}z_{1i_2}\right)\right)\sum_{i=1}^{N}\frac{1}{\lambda_{i_3}}y_{3i_3}z_{2i_3}$$

$$+ \left(C - \sum_{i_1=1}^{N}\frac{1}{\lambda_{i_1}}y_{3i_1}z_{3i_3}\right)\left(\left(1 - C + \sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{1i_2}z_{1i_2}\right)\right.$$

$$\times \left.\left(1 - C + \sum_{i_3=1}^{N}\frac{1}{\lambda_{i_3}}y_{2i_3}z_{2i_3}\right) - \sum_{i_2=1}^{N}\frac{1}{\lambda_{i_2}}y_{1i_2}z_{2i_2}\left(\sum_{i_3=1}^{N}\frac{1}{\lambda_{i_3}}y_{2i_3}z_{1i_3}\right)\right) = 0,$$

when

$$\left(1 - C + \sum_{i_1=1}^{N} \frac{1}{\lambda_{i_1}} y_{1i_1} z_{1i_1}\right) \left(1 - C + \sum_{i_2=1}^{N} \frac{1}{\lambda_{i_2}} y_{2i_2} z_{2i_2}\right)$$
$$- \left(\sum_{i_1=1}^{N} \frac{1}{\lambda_{i_1}} y_{1i_1} z_{2i_1}\right) \left(\sum_{i_2=1}^{N} \frac{1}{\lambda_{i_2}} y_{2i_2} z_{1i_2}\right) \neq 0.$$

4   CONCLUSION

In the present paper by use of the method [2], [8], [9], [11], [22], [21] of reducing upon the special finite-dimensional invariant subspaces we have investigated the Bargmann type reduction of the Lax integrable two-dimensional generalization of the relativistic Toda lattice [10]. We have shown that the symplectic structure on the corresponding finite-dimensional invariant subspace can be found by means of the discrete analog of the Gelfand-Dikii relationship for the related Lagrangian function on a suitably extended phase space. This invariant subspace has been established to be diffeomorphic to the symplectic manifold smoothly embedded into space R^{6N}, $N \in \mathbb{N}$, with the canonical symplectic structure. The Lax-Liouville integrability of the reduced vector fields given by the system has been proven.

If $R = 2$, for every $s \in \mathbb{N}$, $s \geq 2$, the evolutions of the vector-function $(f_1,\ f_2,\ f_1^*,\ f_2^*)^\top \in \mathcal{M}^4$, which are generated by the vector fields $d/dT_s := d/dt_s + d/d\tau_{s,1}$ and $d/dT_2 := d/dT$ and written out with taking into account the equalities

$$l^s f_1 = (d/d\tau + M_1^1)^s f_1, \quad l^s f_1^* = (-d/d\tau + M_1^{1*})^s f_1^*,$$

together with the relationship

$$dl_+^s/d\tau_{s,1} = [l_+^s,\ M_1^1]_+,$$

determine $(2 + 1)$-dimensional nonlinear dynamical system with the triple Lax type linearization. The symplectic finite-dimensional manifold described in the paper is a common invariant subspace of the vector fields $d/dT_s := d/dt_s + d/d\tau_{s,1}$, $s \in \mathbb{N}$, on which they are Hamiltonian and integrable by Liouville. Thus, it is interesting to investigate the possibility of applying the integration procedure, developed for the Liouville integrable finite-dimensional systems in [24], to the vector fields reduced upon this invariant subspace. The integration procedure [24] is based on the specially constructed Picard-Fuchs type differential-functional equations which generate the Hamiltonian-Jacobi transformations.

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Гентош О.Є. *Редукція Баргмана для деякого інтегровного за Лаксом двовимірного узагальнення релятивістського ланцюжка Тоди* // Карпатські матем. публ. — 2015. — Т.7, №2. — С. 155–171.

Досліджується можливість застосування методу редукування на скінченновимірні інваріантні підпростори, породжені власними значеннями асоційованої спектральної задачі, для деякого двовимірного узагальнення релятивістського ланцюжка Тоди з потрійною матричною лінеаризацією типу Лакса. Встановлено гамільтоновість та інтегровність за Лаксом-Ліувіллем заданих цією системою векторних полів на інваріантному підпростірі, пов'язаному з редукцією типу Баргмана.

Ключові слова і фрази: релятивістський ланцюжок Тоди, потрійна лінеаризація типу Лакса, інваріантна редукція, симплектична структура, інтегровність за Ліувіллем.