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ON THE STRUCTURE OF SOME MINIMAX-ANTIFINITARY MODULES

Let R be a ring and G be a group. An R-module A is said to be *minimax* if A includes a noetherian submodule B such that A/B is artinian. It is studied a $\mathbb{Z}_{p^\infty}G$-module A such that $A/C_A(H)$ is minimax as a $\mathbb{Z}_{p^\infty}$-module for every proper not finitely generated subgroup H.

Key words and phrases: minimax module, cocentralizer, module over group ring, minimax-antifinitary RG-module, generalized radical group.

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INTRODUCTION

The modules over group rings RG are classical objects of study with well established links to various areas of algebra. The case when G is a finite group has been studying in sufficient details for a long time. For the case when G is an infinite group, the situation is different. The investigation of modules over polycyclic-by-finite groups was initiated in the classical works of P. Hall [3, 4]. Nowadays, the theory of modules over polycyclic-by-finite groups is highly developed and rich on interesting results. This was largely due to the fact that a group ring RG of a polycyclic-by-finite group G over a noetherian ring R is also noetherian. This allowed developing an advanced theory of such rings and obtain deep results about their structure. For group rings over some other groups (even over well-studied groups, as for instance, the Chernikov groups) the situation is not so good since these rings have quite a sophisticated structure. In particular, they are neither Noetherian nor Artinian. In such cases, it is not always possible to conduct the study of modules based only on the ring properties. So naturally there is a need for other approaches. Application of the finiteness conditions, particularly the use of the minimal and maximal conditions, proved to be very effective in the classical theory of rings and modules. Noetherian and artinian modules over group rings are also very well investigated. Many aspects of the theory of artinian modules over group rings are well reflected in the book [9]. Lately the so-called finitary approach is under intensive development. This is mainly due to the progress which its applications have found in the theory of infinite dimensional linear groups.

Let R be a ring, G a group and A an RG-module. For a subgroup H of G we consider the R- submodule $C_A(H)$. Then H acts on $A/C_A(H)$. The R-factor-module $A/C_A(H)$ is called the *cocentralizer of H in A*. The factor-group $H/C_H(A/C_A(H))$ is isomorphic to a subgroup of automorphisms group of an R-module $A/C_A(H)$. If x is an element of $C_H(A/C_A(H))$, then x acts trivially on factors of the series $\langle 0\rangle \le C_A(H) \le A$. It follows that $C_H(A/C_A(H))$

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is abelian. This shows that the structure of H to a greater extent is defined by the structure of $C_H(A/C_A(H))$, and hence by the structure of the automorphisms group of the R-module $A/C_A(H)$.

Let $\mathfrak{M}$ be a class of R-modules. We say that A is an *$\mathfrak{M}$-finitary module over RG* if $A/C_A(x) \in \mathfrak{M}$ for each element $x \in G$. If R is a field, $C_G(A) = \langle 1 \rangle$, and $\mathfrak{M}$ is the class of all finite dimensional vector spaces over R, then we come to the finitary linear groups. The theory of finitary linear groups is quite well developed (see, for example, the survey [11]). B.A.F. Wehrfritz began considering the cases when $\mathfrak{M}$ is the class of finite R-modules [13, 15, 16, 18], when $\mathfrak{M}$ is the class of noetherian R-modules [14], when $\mathfrak{M}$ is the class of artinian R- modules [16–20]. The artinian-finitary modules have been considered also in the paper [10]. The artinian and noetherian modules can be united into the following type of modules. An R-module A is said to be *minimax* if A has a finite series of submodules, whose factors are either noetherian or artinian. It is not hard to show that if R is an integral domain, then every minimax R-module A includes a noetherian submodule B such that A/B is artinian. The first natural case here is the case when $R = \mathbb{Z}$ is the ring of all integers. B.A.F. Wehrfritz has began the study of noetherian-finitary and artinian-finitary modules with separate consideration of this case. This case is of particular importance in applications, for instance, it is very important in the theory of generalized soluble groups.

Let G be a group, A an RG-module, and $\mathfrak{M}$ a class of R-modules. Put

$$\mathcal{C}_{\mathfrak{M}}(G) = \{H \mid H \text{ is a subgroup of } G \text{ such that } A/C_A(H) \in \mathfrak{M}\}.$$

If A is an $\mathfrak{M}$-finitary module, then $\mathcal{C}_{\mathfrak{M}}(G)$ contains every cyclic subgroup (moreover, every finitely generated subgroup whenever $\mathfrak{M}$ satisfies some natural restrictions). It is clear that the structure of G depends significantly on which important subfamilies of the family $\Lambda(G)$ of all proper subgroups of G include $\mathcal{C}_{\mathfrak{M}}(G)$. Therefore it is interesting to consider the cases when the family $\mathcal{C}_{\mathfrak{M}}(G)$ is large. In almost all groups (with exception of noetherian groups), the family of subgroups which is not finitely generated is much larger than the family of finitely generated subgroups. It is therefore interesting to consider the case, which is dual to the case of an $\mathfrak{M}$-finitary module.

Let R be a ring, G be a group and A be an RG-module. We say that A is *minimax-antifinitary RG-module* if the factor-module $A/C_A(H)$ is minimax as an R-module for each not finitely generated proper subgroup H and the R-module $A/C_A(G)$ is not minimax.

This current work is devoted to the study of the minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-modules. Here $\mathbb{Z}_{p^\infty}$ denotes a ring of p-adic number. The ring $\mathbb{Z}_{p^\infty}$ play a very specific role in the theory of modules over group rings. It is principal ideal domain and, in the other hand, it is a valuation ring. The study breaks down naturally into the following cases. Put

$$\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx(G)} = \{x \mid A/C_A(x) \text{ is a minimax } \mathbb{Z}_{p^\infty} - \text{module}\}.$$

The first case is the case when $G = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx(G)}$. In this case, every proper subgroup of G has a minimax cocentralizer. This case was considered separately in another paper. The second case is the case when $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ and the group G is not finitely generated. The third case is the case when $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ and the group G is finitely generated. The current article is dedicated to the second case. Here we consider the modules over groups, which belong to the following very large class of groups.

A group G is called *generalized radical*, if G has an ascending series whose factors are locally nilpotent or locally finite. Hence a generalized radical group G either has an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the first case, the locally nilpotent radical $\mathbf{Lnr}(G)$ of G is non–identity. In the second case, it is not hard to see that G includes a non–identity normal locally finite subgroup. Clearly in every group G the subgroup $\mathbf{Lfr}(G)$ generated by all normal locally finite subgroups is the largest normal locally finite subgroup (the *locally finite radical*). Thus every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors. A group G is called *locally generalized radical group*, if every finitely generated subgroup is generalized radical. The class of locally radical group is very large, in particular, it includes all locally finite groups and all locally soluble groups.

The main result is a following.

Theorem 1. *Let G be a locally generalized radical group, A a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module, and $D = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. Suppose that G is not finitely generated, $G \neq D$ and $C_G(A) = \langle 1 \rangle$. Then G is a group of one of the following types.*

1. *G is a quasicyclic q-group for some prime q.*
2. *$G = Q \times \langle g \rangle$ where Q is a quasicyclic p-subgroup, g is a d-element and $g^d \in D$, p, d are prime (not necessary different).*
3. *G includes a normal divisible Chernikov p-subgroup Q, such that $G = Q\langle g \rangle$ where g is a d-element, p, d are prime (not necessary different). Moreover, G satisfies the following conditions:*
   - (a) *$g^d \in D$;*
   - (b) *Q is G-quasifinite;*
   - (c) *if $p = d$, then Q has special rank $d^{m-1}(d-1)$ where $d^m = |\langle g \rangle / C_{\langle g \rangle}(Q)|$;*
   - (d) *if $p \neq d$, then Q has special rank $\mathbf{o}(p, d^m)$ where again $d^m = |\langle g \rangle / C_{\langle g \rangle}(Q)|$ and $\mathbf{o}(p, d^m)$ is the order of p modulo d^m.*

Furthermore, for the types 2, 3 $A(\omega\mathbb{Z}_{p^\infty}D)$ is a Chernikov p-subgroup.

Here ωRG be the *augmentation ideal* of the group ring RG, the two-sided ideal of RG generated by all elements $g - 1$, $g \in G$.

Recall also that an abelian normal subgroup A of a group G is called *G-quasifinite* if every proper G-invariant subgroup of A is finite. Clearly that in this case either A is a union of its finite G-invariant subgroups or A includes a finite G-invariant subgroup B such that the factor A/B is a G-chief. At the end of the article, we provide the examples showing that all the situations that arise in the theorem can be realized.

1 Some preliminary results

Let R be a ring and $\mathfrak{M}$ a class of R-modules. Then $\mathfrak{M}$ is said to be a *formation* if it satisfies the following conditions:

F1. if $A \in \mathfrak{M}$ and B is an R-submodule of A, then $A/B \in \mathfrak{M}$;

F2. if A is an R-module and $B_1, ..., B_k$ are R-submodules of A such that $A/B_j \in \mathfrak{M}$, $1 \leq j \leq k$, then $A/(B_1 \cap ... \cap B_k) \in \mathfrak{M}$.

Lemma 1. *Let R be a ring, $\mathfrak{M}$ a formation of R-modules, G a group and A an RG-module.*

(i) If L, H are subgroups of G such that $L \leq H$ and $A/C_A(H) \in \mathfrak{M}$, then $A/C_A(L) \in \mathfrak{M}$.

(ii) If L, H are subgroups of G whose cocentralizers belong to $\mathfrak{M}$, then $A/C_A(\langle H, L\rangle) \in \mathfrak{M}$.

Proof. The inclusion $L \leq H$ implies that $C_A(L) \geq C_A(H)$. Since $A/C_A(H) \in \mathfrak{M}$ and $\mathfrak{M}$ is a formation, $A/C_A(L) \in \mathfrak{M}$. Clearly $C_A(\langle H, L\rangle) \leq C_A(H) \cap C_A(L)$. Since $\mathfrak{M}$ is a formation, $A/(C_A(H) \cap C_A(L)) \in \mathfrak{M}$. Then we have $A/C_A(\langle H, L\rangle) \in \mathfrak{M}$. □

Lemma 2. *Let R be a ring, $\mathfrak{M}$ a formation of R-modules, G a group and A an RG-module. Then*

$$\mathbf{Coc}_{\mathfrak{M}}(G) = \{x \in G \mid A/C_A(x) \in \mathfrak{M}\}$$

is a normal subgroup of G.

Proof. By Lemma 1 $\mathbf{Coc}_{\mathfrak{M}}(G)$ is a subgroup of G. Now let $x \in \mathbf{Coc}_{\mathfrak{M}}(G)$, $g \in G$. Then $C_A(x^g) = C_A(x)g$. Since the mapping $a \mapsto ag$, $a \in A$, is R-linear,

$$A/C_A(x) \cong_R Ag/C_A(x)g = A/C_A(x)g = A/C_A(x^g),$$

which shows that $A/C_A(x^g) \in \mathfrak{M}$, and hence $x^g \in \mathbf{Coc}_{\mathfrak{M}}(G)$. □

Clearly the class of minimax modules over an integral domain R is a formation and so we obtain the following result.

Corollary 1. *Let R be a ring, G a group and A an RG-module.*

(i) L, H are subgroups of G such that $L \leq H$ and a factor-module $A/C_A(H)$ is minimax, then $A/C_A(L)$ is also minimax.

(ii) If L, H are subgroups of G whose cocentralizers are minimax, then $A/C_A(\langle H, L\rangle)$ is also minimax.

Corollary 2. *Let R be a ring, G a group and A an RG-module. Then*

$$\mathbf{Coc}_{R-mmx}(G) = \{x \in G \mid A/C_A(x) \textit{is minimax}\}$$

is a normal subgroup of G.

A group G is said to be *$\mathfrak{F}$-perfect* if G does not include proper subgroups of finite index.

Lemma 3. *Let G be a locally generalized radical group and A be a $\mathbb{Z}_{p^\infty}G$-module. Suppose that A includes a $\mathbb{Z}_{p^\infty}$–minimax $\mathbb{Z}_{p^\infty}G$-submodule B, which is minimax. Then the following assertions hold:*

(i) $G/C_G(B)$ is soluble-by-finite;

(ii) if $G/C_G(B)$ is periodic, then it is nilpotent-by-finite;

(iii) if $G/C_G(B)$ is $\mathfrak{F}$-perfect and periodic, then it is abelian; moreover $[[B, G], G] = \langle 0\rangle$.

Proof. Without loss of generality we can suppose that $C_G(B) = \langle 1 \rangle$. Since B is minimax, it has a series of G-invariant subgroups $\langle 0 \rangle \leq D \leq K \leq B$ where D is divisible Chernikov subgroup, K/D is finite, and B/K is torsion-free and has finite $\mathbb{Z}_{p^\infty}$-rank. In particular, the $\Pi(D) = \{p\}$. Clearly D is G-invariant. The factor-group $G/C_G(D)$ is isomorphic to a subgroup of $\mathbf{GL}_m(\mathbb{Q}_{p^\infty})$ where $\mathbb{Q}_{p^\infty}$ is the field of fractions of $\mathbb{Z}_{p^\infty}$ and m satisfies $q^m = |\mathbf{\Omega}_1(D)|$. Let $\mathbb{Q}_{p^\infty}$ be a field of fractions of $\mathbb{Z}_{p^\infty}$, then $G/C_G(D)$ is isomorphic to a subgroup of $\mathbf{GL}_m(\mathbb{Q}_{p^\infty})$. Note that $\mathbf{char}(\mathbb{Q}_{p^\infty}) = 0$. Being locally generalized radical, $G/C_G(D)$ does not include the non-cyclic free subgroup; thus an application of Tits Theorem (see, for example, [12, Corollary 10.17]) shows that $G/C_G(D)$ is soluble-by-finite. If G is periodic, then $G/C_G(D)$ is finite (see, for example, [12, Theorem 9.33]). Since K/D is finite, $G/C_G(K/D)$ is finite. Finally, $G/C_G(B/K)$ is isomorphic to a subgroup of $\mathbf{GL}_r(\mathbb{Q}_{p^\infty})$, where $\mathbf{r} = \mathbf{r}_{\mathbb{Z}_{p^\infty}}(B/K)$. Using again the fact that $G/C_G(A/K)$ does not include the non-cyclic free subgroup and Tits Theorem or Theorem 9.33 of the book [12] (for periodic G), we obtain that $G/C_G(B/K)$ is soluble-by-finite (respectively finite whenever G is periodic). Put

$$Z = C_G(D) \cap C_G(K/D) \cap C_G(B/K).$$

Then G/Z is embedded in $G/C_G(D) \cap G/C_G(K/D) \cap G/C_G(B/K)$, in particular, G/Z is soluble-by- finite (respectively finite).

If $x \in Z$, then x acts trivially in every factors of the series $\langle 0 \rangle \leq D \leq K \leq A$. By Kaloujnin's theorem [7] Z is nilpotent. It follows that G is soluble-by-finite (respectively nilpotent-by-finite).

Suppose now that G is an $\mathfrak{F}$-perfect group. Again consider the series of G-invariant subgroups $\langle 0 \rangle \leq K \leq B$. Being abelian and Chernikov, K is a union of the ascending series

$$\langle 0 \rangle = K_0 \leq K_1 \leq \dots \leq K_n \leq K_{n+1} \leq \dots$$

of G-invariant finite subgroups K_n, $n \in \mathbb{N}$. Then the factor-group $G/C_G(K_n)$ is finite for every $n \in \mathbb{N}$. Since G is $\mathfrak{F}$-perfect, $G = C_G(K_n)$ for each $n \in \mathbb{N}$. The equality $K = \bigcup_{n\in\mathbb{N}} K_n$ implies that $G = C_G(K)$. As proved above, since $G/C_G(B/K)$ is soluble–by–finite and $\mathfrak{F}$–perfect, it is soluble. Then $G/C_G(B/K)$ includes normal subgroups U, V such that $C_G(B/K) \leq U \leq V$, $U/C_G(B/K)$ is isomorphic to a subgroup of $\mathbf{UT_r}(\mathbb{Q}_{p^\infty})$, V/U includes a free abelian subgroup of finite index [1, Theorem 2]. Since $G/C_G(B/K)$ is $\mathfrak{F}$-perfect, it follows that $G/C_G(B/K)$ is torsion-free. Then $G/C_G(B/K)$ must be identity, because it is periodic. In other words, $G = C_G(B/K)$. Hence G acts trivially in every factors of a series $\langle 0 \rangle \leq K \leq B$, so that $[[B, G], G] = \langle 0 \rangle$, and using again Kaloujnin's theorem [7], we obtain that G is abelian. □

Lemma 4. *Let G be a Chernikov q-group and A a $\mathbb{Z}_{p^\infty}G$-module. If $A/C_A(G)$ is minimax (as a $\mathbb{Z}_{p^\infty}$-module), then the additive group of $A(\omega\mathbb{Z}_{p^\infty}G)$ is a Chernikov p-subgroup and $q = p$.*

Proof. For each element x of G consider the mapping $\delta_x : A \to A$, defined by the rule $\delta_x(a) = a(x-1)$, $a \in A$. Clearly this mapping is a $\mathbb{Z}_{p^\infty}$-endomorphism of A, $\mathbf{Ker}(\delta_x) = C_A(x)$ and $\mathbf{Im}(\delta_x) = A(\omega\mathbb{Z}_{p^\infty}\langle x \rangle) = A(x-1)$. Hence

$$A(x-1) = \mathbf{Im}(\delta_x) \cong_{\mathbb{Z}_{p^\infty}} A/\mathbf{Ker}(\delta_x) = A/C_A(x).$$

Since $A/C_A(G)$ is minimax, it has finite special rank r for some positive integer r. An inclusion $C_A(G) \leq C_A(x)$ follows that $A/C_A(x)$ has a special rank at most r. Then $\mathbf{r}(A(x-1)) \leq r$.

Let k be a positive integer such that $|\Omega_1(G)| = q^k$. Then G has an ascending series of finite subgroups

$$L_1 = \Omega_1(G) \leq L_2 \leq \ldots \leq L_n \leq L_{n+1} \leq \ldots$$

such that $L_n = \mathbf{Dr}_{1 \leq j \leq k}\langle x_{n_j} \rangle$, where $|x_{n_j}| \leq q^n$ for each j, and $G = \bigcup_{n \in \mathbb{N}} L_n$. The equality

$$A(\omega \mathbb{Z}_{p^\infty} L_n) = A(\omega \mathbb{Z}_{p^\infty} \langle x_{n_1} \rangle) + \ldots + A(\omega \mathbb{Z}_{p^\infty} \langle x_{n_k} \rangle) = A(x_{n_1} - 1) + \ldots + A(x_{n_k} - 1)$$

together with $\mathbf{r}(A(x_{n_j} - 1)) \leq r$, $1 \leq j \leq k$, shows that $\mathbf{r}(A(\omega \mathbb{Z} L_n)) \leq rk$ for every $n \in \mathbb{N}$. Since $G = \bigcup_{n \in \mathbb{N}} L_n$ we have $A(\omega \mathbb{Z}_{p^\infty} G) = \bigcup_{n \in \mathbb{N}} A(\omega \mathbb{Z}_{p^\infty} L_n)$, moreover $L_n \leq L_{n+1}$ implies that $A(\omega \mathbb{Z}_{p^\infty} L_n) \leq A(\omega \mathbb{Z}_{p^\infty} L_{n+1})$ for every $n \in \mathbb{N}$. Let B be an arbitrary finitely generated subgroup of $A(\omega \mathbb{Z}_{p^\infty} G)$. Then there exists a positive integer m such that $B \leq A(\omega \mathbb{Z}_{p^\infty} L_m)$. By proved above B has at most rk generators. It follows that $A(\omega \mathbb{Z}_{p^\infty} G)$ has a finite special rank at most rk.

Let Q be the divisible part of G. Since $A/C_A(Q)$ is minimax, A has a series of $\mathbb{Z}_{p^\infty}G$-submodules $C_A(Q) = C \leq T \leq A$ where $T/C = \mathbf{Tor}(A/C)$ is a Chernikov group and A/T is torsion-free and has finite $\mathbb{Z}_{p^\infty}$-rank. Repeating the final part of the proof of Lemma 3, we obtain that $Q = C_Q(T)$ and $Q = C_Q(A/T)$.

Let a be an arbitrary element of T. Consider the mapping $\gamma_a : Q \to A(\omega \mathbb{Z}_{p^\infty} Q)$, defined by the rule $\gamma_a(x) = a(x-1)$. By $(x-1)(y-1) = (xy-1) - (x-1) - (y-1)$. We have $a(xy-1) = a(x-1) + a(y-1) + a(x-1)(y-1) = a(x-1) + a(y-1)$. An equality $Q = C_Q(T)$ implies that $a(x-1)(y-1) = 0$. In other words, $\gamma_a(xy) = \gamma_a(x) + \gamma_a(y)$, thus γ_a is a homomorphism. Furthermore, $\mathbf{Ker}(\gamma_a) = C_Q(a)$ and $\mathbf{Im}(\gamma_a) = \langle a \rangle(\omega \mathbb{Z}_{p^\infty} Q) = [a, Q]$, so that $[a, Q] \cong Q/C_Q(a)$. It follows that if $[a, Q] \neq \langle 0 \rangle$, then it is a divisible Chernikov subgroup and $\Pi([a, Q]) \subseteq \Pi(Q) = \{q\}$. Since it is valid for every $a \in T$, $T(\omega \mathbb{Z}_{p^\infty} Q)$ is a divisible subgroup (if it is non-trivial) and $\Pi(T(\omega \mathbb{Z}_{p^\infty} Q)) \subseteq \Pi(Q) = \{q\}$. By proved above, $T(\omega \mathbb{Z}_{p^\infty} Q)$ has finite special rank, and therefore $T(\omega \mathbb{Z}_{p^\infty} Q)$ is a Chernikov subgroup.

Consider now the factor-module A/V where $V = T(\omega \mathbb{Z}_{p^\infty} Q)$. Then the inclusion $T/V \leq C_{A/V}(Q)$ implies that $(A/V)(\omega \mathbb{Z}_{p^\infty} Q) \leq T/V$. Using the above arguments, we obtain that $(A/V)(\omega \mathbb{Z}_{p^\infty} Q)$ is a Chernikov divisible group such that $\Pi((A/V)(\omega \mathbb{Z}_{p^\infty} Q)) \subseteq \Pi(Q)$. We have

$$(A/V)(\omega \mathbb{Z}_{p^\infty} Q) = (A(\omega \mathbb{Z}_{p^\infty} Q) + V)/V = (A(\omega \mathbb{Z}_{p^\infty} Q) + T(\omega \mathbb{Z}_{p^\infty} Q))/(T(\omega \mathbb{Z}_{p^\infty} Q),$$

which follows that $A(\omega \mathbb{Z}_{p^\infty} Q)$ is a Chernikov divisible subgroup such that $\Pi(A(\omega \mathbb{Z}_{p^\infty} Q)) \subseteq \Pi(Q)$.

Let $M = A(\omega \mathbb{Z}_{p^\infty} Q)$, then $Q \leq C_G(A/M)$, in particular, $G/C_G(A/M)$ is finite. By proved above $(A/M)(\omega \mathbb{Z}_{p^\infty} G)$ has finite special rank. Using the above arguments, we obtain that $\langle a + M \rangle(\omega \mathbb{Z}_{p^\infty} G)$ is a finite group and $\Pi(\langle a + M \rangle(\omega \mathbb{Z}_{p^\infty} G)) \subseteq \Pi(G) = \{q\}$ for every element $a \in A$. The finiteness of $\Pi(G)$ implies that $(A/M)(\omega \mathbb{Z}_{p^\infty} G)$ is a Chernikov subgroup of A/M and $\Pi((A/M)(\omega \mathbb{Z}_{p^\infty} G)) \subseteq \Pi(G) = \{q\}$. Hence $A(\omega \mathbb{Z}_{p^\infty} G)$ is Chernikov and $\Pi(A(\omega \mathbb{Z}_{p^\infty} G)) \subseteq \Pi(G) = \{q\}$ but $\Pi(A(\omega \mathbb{Z}_{p^\infty} G)) \subseteq \{p\}$ so we have $q = p$. □

Corollary 3. *Let G be a group and A a $\mathbb{Z}_{p^\infty}G$-module. If $A/C_A(G)$ is minimax as $\mathbb{Z}_{p^\infty}$-module, then every locally generalized radical subgroup of $G/C_G(A)$ is soluble-by- finite, and every periodic subgroup of $G/C_G(A)$ is nilpotent-by-finite.*

Proof. Indeed, Lemma 3 shows that $G/C_G(A/C_A(G))$ is soluble-by-finite. Every element $x \in C_G(A/C_A(G))$ acts trivially in the factors of the series $\langle 0\rangle \leq C_A(G) \leq A$. It follows that $C_G(A/C_A(G))$ is abelian. Suppose now that $H/C_G(A)$ is a periodic subgroup of $G/C_G(A)$. Since $A/C_A(G)$ is minimax, A has a series of H-invariant subgroups

$$\langle 0\rangle \leq C_A(G) \leq D \leq K \leq A,$$

where $D/C_A(G)$ is a divisible Chernikov subgroup, K/D is finite and A/K is torsion-free and has finite $\mathbb{Z}_{p^\infty}$-rank. In Lemma 3 we have already proved that $G/C_G(D/C_A(G))$, $G/C_G(K/D)$ and $G/C_G(A/K)$ are finite. Let $Z = C_G(D/C_A(G)) \cap C_G(K/D) \cap C_G(A/K)$. Then G/Z is finite. If $x \in Z$, then x acts trivially in every factors of a series $\langle 0\rangle \leq C_A(G) \leq D \leq K \leq A$. By Kaloujnin's theorem [7] Z is nilpotent. □

Let G be a generalized radical group and let R_1 be a normal subgroup of G, satisfying the following conditions: R_1 is radical, G/R_1 does not include the non-trivial locally nilpotent normal subgroups. Then G/R_1 must include a non-trivial normal locally finite subgroup. It follows that the locally finite radical R_2/R_1 is non-trivial. If we suppose that G/R_2 includes a non-trivial normal locally finite subgroup L/R_2, then L/R_1 is also locally finite, which contradicts the choice of R_2. This contradiction shows that G/R_2 does not include a non-identity normal locally finite subgroup, and therefore it must include a non-identity normal locally nilpotent subgroup. Let R_3/R_2 be a normal subgroup of G/R_2 satisfying the following conditions: R_3/R_2 is radical, G/R_3 does not include non-identity locally nilpotent normal subgroups. Using similar arguments, we construct the ascending series of normal subgroups

$$\langle 1\rangle = R_0 \leq R_1 \leq \dots R_\alpha \leq R_{\alpha+1} \leq \dots R_\gamma = G,$$

whose factors are radical or locally finite, and if $R_{\alpha+1}/R_\alpha$ is radical (respectively locally finite), then $R_{\alpha+2}/R_{\alpha+1}$ is locally finite (respectively radical).

This series is called a *standard series* of a generalized radical group G.

Lemma 5. *Let G be a group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. Then every proper generalized radical subgroup of $G/C_G(A)$ is soluble-by-finite.*

Proof. Again we will suppose that $C_G(A) = \langle 1\rangle$. Let L be an arbitrary proper generalized radical subgroup of G. Let

$$\langle 1\rangle = R_0 \leq R_1 \leq \dots R_\alpha \leq R_{\alpha+1} \leq \dots R_\gamma = L,$$

be a standard series of L. Suppose that $\gamma \geq \omega$ (ω is the first infinite ordinal) and consider the subgroup R_ω. Assume that R_ω is finitely generated, that is $R_\omega = \langle u_1, \dots, u_t\rangle$ for some elements $u_1, \dots, u_t$. The equality $R_\omega = \bigcup_{n\in\mathbb{N}} R_n$ shows that there exists a positive integer m such that $u_1, \dots, u_t \in R_m$. But in this case, $R_\omega = R_m$ and we obtain a contradiction. This contradiction shows that R_ω is not finitely generated. It follows that $A/C_A(R_\omega)$ is minimax. Corollary 3 shows that R_ω is soluble-by-finite. In this case $R_\omega = R_2$ and we again obtain a contradiction. This contradiction shows that $\gamma = k$ for some positive integer.

Now we will use induction by k for a proof of our assertion. Consider the subgroup R_1. Then either R_1 is radical or locally finite. If R_1 is not finitely generated, then $A/C_A(R_\omega)$ is minimax. Corollary 3 shows that R_1 is soluble-by-finite. Suppose that R_1 is finitely generated. If R_1 is locally finite, then it is finite. Therefore assume that R_1 is radical. Let

$$\langle 1 \rangle = V_0 \leq V_1 \leq \dots V_\alpha \leq V_{\alpha+1} \leq \dots V_\eta = R_1,$$

be an ascending series of R_1 where $V_{\alpha+1}/V_\alpha$ is the locally nilpotent radical of R_1/V_α, $\alpha < \eta$. Using the above arguments we obtain that $\eta = d$ for some positive integer d. Let m be a number such that all factors $V_{m+1}/V_m, V_{m+2}/V_{m+1}, \dots, V_d/V_{d-1}$ are finitely generated. Since they are locally nilpotent, they must be polycyclic. It follows that V_d/V_m is polycyclic. In particular if every subgroup V_j is finitely generated, $1 \leq j \leq d$, then R_1 is polycyclic. Therefore assume that there is a positive integer s such that V_s is not finitely generated, but a subgroup V_j is finitely generated for all $j > s$. Then $A/C_A(V_s)$ is minimax and Corollary 3 yields that V_s is soluble. In this case R_1/V_s is polycyclic, so that R_1 is soluble.

Suppose that we have already proved that all subgroups $R_1, R_2, \dots, R_{k-1}$ are soluble-by-finite. Repeating the above arguments, we obtain that and R_k is soluble-by-finite, and the result is proved. □

Lemma 6. *Let G be a group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. If H is a proper subgroup of G and $\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ does not include H, then H is finitely generated.*

Proof. Indeed if we suppose that H is not finitely generated, then $A/C_A(H)$ is minimax. Corollary 1 shows that $A/C_A(h)$ is minimax for each element $h \in H$. It follows that $H \leq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, and we obtain a contradiction with the choice of H. □

2 PROOFS OF THE MAIN RESULTS

Proposition 1. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. If $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is not finitely generated, then $G/C_G(A)$ is a quasicyclic q-group for some prime q.*

Proof. Again suppose that $C_G(A) = \langle 1 \rangle$. Let $M = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. Let H be a proper subgroup of G. If M does not include H, then Lemma 6 shows that H is finitely generated. In particular if $M \leq H$, then H/M is finitely generated. In other words, every proper subgroup of G/M is finitely generated. By [8, Proposition 2.7] G/M is a quasicyclic q-group for some prime q.

Let L/M be a proper subgroup of G/M, then L/M is a finite cyclic subgroup. An application of Lemma 6 shows that the subgroup L is finitely generated. The finiteness of index $|L : M|$ implies that M is finitely generated (see, for example, [5, Corollary 7.2.1]). Using Lemma 5 we obtain that M is soluble-by-finite. Then M includes a maximal normal soluble subgroup S such that M/S is finite. It is not hard to see, that S is G-invariant. Let $D = S'$, then M/D is abelian-by-finite and finitely generated, therefore it is noetherian. Let V/D be a proper subgroup of G/D. If M/D does not include V/D, then M does not include V, and as above, V is finitely generated. Then V/D is also finitely generated. If $V/D \leq M/D$, then again V/D is finitely generated. Thus every proper subgroup of G/D is finitely generated, and application of [8, Proposition 2.7] shows that G/D is a quasicyclic group. Since M/D is a proper subgroup of G/D, M/D is a finite cyclic subgroup. Suppose that $D \neq \langle 1 \rangle$, then $K = D' \neq D$. Repeating the above arguments, we obtain that G/K is a quasicyclic group. In particular, it is abelian.

Then S/K is abelian, which follows that $K \geq S' = D$. This contradiction shows that $D = \langle 1 \rangle$, so that G is a quasicyclic group. $\square$

Lemma 7. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. Suppose that $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, G is not finitely generated, and $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is finitely generated. Then G is soluble and $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is a group of a prime order q.*

Proof. Again suppose that $C_G(A) = \langle 1 \rangle$. Let $D = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. Since G/D is finitely generated, $G = \langle M, D \rangle$ for some finite subset M of G. We may suppose that M is minimal finite set with this property, that is $G \neq \langle S, D \rangle$ for each proper subset S of M. Now suppose that $|M| \geq 2$. Then M includes two proper subsets X, Y such that $M = X \cup Y$. By the choice of M, the subgroups $\langle X, D \rangle$ and $\langle Y, D \rangle$ are proper and also $\langle X, D \rangle \neq D$, $\langle Y, D \rangle \neq D$. By Lemma 6 both subgroups $\langle X, D \rangle$ and $\langle Y, D \rangle$ are finitely generated. An equality $X \cup Y = M$ implies that $G = \langle X, Y, D \rangle$ is finitely generated. This contradiction shows that $|M| = 1$. In other words, G/D is cyclic. Suppose that $|G/D|$ is not a prime. Then G includes a proper subgroup B such that $D \leq B$, $B \neq D$, and G/B has a prime order. Using Lemma 6 we obtain that B is finitely generated. The finiteness of G/B gives that G is finitely generated. This final contradiction proves that G/D has a prime order. Choose an element g such that $G = \langle g, D \rangle$.

Since G is not finitely generated, D cannot be finitely generated. Using Lemma 5, we obtain that D is soluble-by-finite. Let S be a maximal normal soluble subgroup of D having finite index. Suppose that $D \neq S$. Clearly S is G-invariant. Since D/S is finite and non-soluble, $S\langle g^q \rangle \neq D$. It follows that $S\langle g \rangle$ is a proper subgroup of G. Since D does not include $S\langle g \rangle$, $S\langle g \rangle$ is finitely generated by Lemma 6. Then $S\langle g^q \rangle$ is finitely generated (see, for example, [5, Corollary 7.2.1]). Since the index $|D : S|$ is finite, D is finitely generated. This contradiction shows that D is soluble. Hence G is soluble. $\square$

Let K be a finite group. We have $|K| = p_1^{t_1} \dots p_s^{t_s}$ where $p_1, \dots, p_s$ are primes and $p_m \neq p_j$ whenever $m \neq j$. Put $\Pi(K) = \{p_1, \dots, p_s\}$.

Corollary 4. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module and $D = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. Suppose that $G \neq D$, G is not finitely generated and G/D is finitely generated. Let g be an element of G with a property $G = \langle g \rangle D$. If H is a normal subgroup of G, having finite index, then $H\langle g \rangle = G$. Moreover, G/H is a q-group.*

Proof. If we assume that $H\langle g \rangle$ is a proper subgroup of G, then the choice of g yields that D does not include $H\langle g \rangle$. By Lemma 6, $H\langle g \rangle$ is finitely generated. Since $H\langle g \rangle$ has finite index, G must be finitely generated. This contradiction shows that $H\langle g \rangle = G$.

Suppose that $\Pi(G/H) \neq \{q\}$. Let P/H be a Sylow q-subgroup of G/H. Then P/H is a proper subgroup of G/H. Since P has finite index in G, P is not finitely generated. Then $A/C_A(P)$ is minimax. It follows that $P \leq D$. On the other hand, G/D is a non-trivial q-group and therefore D cannot include P. This contradiction proves that G/H is a q-group. $\square$

Let G be a group, denote by $\mathbf{Tor}(G)$ the maximal normal periodic subgroup of G (periodic part of G).

Proposition 2. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. Suppose that $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, G is not finitely generated and*

$G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is finitely generated. If G/G' is infinite, then $G = Q \times \langle g\rangle$ where Q is a quasicyclic p-subgroup, g is a d-element and $g^d \in \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, where p, d are primes (not necessary different).

Proof. As usual we suppose that $C_G(A) = \langle 1\rangle$. Let $D = \mathbf{Coc}_{\mathbb{Z}-mmx}(G)$. By Lemma 7, G is soluble and G/D is a group of a prime order q. Choose an element g such that $G = \langle g, D\rangle$.

Put $K = G'$, then $K \leq D$. Suppose that $K\langle g\rangle = G$. Since G/K is infinite and $G/K = K\langle g\rangle/K \cong \langle g\rangle/(\langle g\rangle \cap K)$ we obtain that gK has infinite order. Let r_1, r_2 be two distinct primes. Then $K\langle g^{r_j}\rangle$ is a proper subgroup of G. Since it has finite index in G, $K\langle g^{r_j}\rangle$ is not finitely generated. It follows that $A/C_A(K\langle g^{r_j}\rangle)$ is minimax for every $j \in \{1,2\}$. Since $r_1 \neq r_2$ we have $\langle g\rangle = \langle g^{r_1}\rangle\langle g^{r_2}\rangle$. Corollary 1 shows that $A/C_A(\langle g\rangle)$ is minimax, that is $g \in D$, and we obtain a contradiction with the choice of g. It shows that $K\langle g\rangle$ is a proper subgroup of G.

Now let $Z/(K\langle g\rangle)$ be a proper subgroup of $G/(K\langle g\rangle)$. Then D does not include Z and hence Lemma 6 shows that Z is finitely generated. If we assume that Z has finite index in G, then G must be finitely generated, so we obtain a contradiction. This contradiction shows that the factor-group $G/(K\langle g\rangle)$ is $\mathfrak{F}$-perfect. Then $G/(K\langle g\rangle)$ includes a subgroup $P/(K\langle g\rangle)$ such that G/P is a quasicyclic d-group for some prime d. Since $g \in P$, we have that D does not include P. By Lemma 6, P is finitely generated. It follows that G/K is an abelian minimax group. Suppose that $\mathbf{Tor}(G/K) \neq G/K$. Then $T/K = \mathbf{Tor}(D/K) \neq D/K$. Put

$$\pi = \{r \mid r \text{ is a prime such that } D/T \neq (D/T)^r\}.$$

Since D/T is torsion-free and minimax, the set π is infinite. Therefore we can choose a prime r such that $r \neq q$ and $r \in \pi$. Let $L/T = (D/T)^r$, then D/L is a non-identity elementary abelian r- group. By the choice of L, $\Pi(G/L) = \{r, q\}$ and this contradicts Corollary 4. Hence we have that G/K is periodic. In this case, P/K is finite, so that G/K is a Chernikov group. Let Q/K be the divisible part of G/K. Since $Q/K \cong G/P$, Q/K is a quasicyclic q-subgroup. Since Q has finite index in G, Corollary 4 shows that $G = Q\langle g\rangle$ and G/Q is a q-group. It follows that $G/K = Q/K \times \langle gK\rangle$ (see [2, Theorem 21.2]). Moreover, by Lemma 4 Q is a p-group.

Suppose that $K \neq \langle 1\rangle$. Then $L = K' \neq K$. We have already proved above that $K\langle g\rangle$ is a proper subgroup of G. Since D does not include $K\langle g\rangle$, Lemma 6 shows that $K\langle g\rangle$ is finitely generated. The fact that G/K is periodic implies that K has finite index in $K\langle g\rangle$. Then K is finitely generated (see, for example, [5, Corollary 7.2.1]). Thus K/L is a finitely generated abelian group. Then K/L includes a proper G-invariant subgroup V/L of finite index in K/L (this subgroup can be identity). Then G/V is a Chernikov group with finite derived subgroup. Let Q_1/V be the divisible part of G/V, then $Q_1/V \cong Q/K$, so that Q_1/V is a quasicyclic q-subgroup. Since $[G/V, G/V]$ is finite, $Q_1/V \leq \zeta(G/V)$. Since index $|G : Q_1|$ is finite, $G = Q_1\langle g\rangle$ by Corollary 4. This equality together with the inclusion $Q_1/V \leq \zeta(G/V)$ implies that G/V is abelian. But in this case $K \leq V$, and this contradicts with the choice of V. Consequently we have $K = \langle 1\rangle$. So Q is a proper subgroup of G which is quasicyclic q-group, than by Lemma 4 Q is a p-group. □

Proposition 3. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. Suppose that $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, G is not finitely generated and $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is finitely generated. If G/G' is finite, then G includes a normal divisible Chernikov p-subgroup Q, such that $G = Q\langle g\rangle$ where g is a d-element, $g^d \in \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ and p, d are primes (not necessary different). Moreover, the subgroup Q is G-quasifinite.*

Proof. As usual we suppose that $C_G(A) = \langle 1 \rangle$. Let $D = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. By Lemma 7, G is soluble and G/D is a group of a prime order d. Choose an element g such that $G = \langle g, D \rangle$.

Put $K = G'$. Since G/K is finite, Corollary 4 shows that $G = K\langle g \rangle$ and G/K is a d-group. It follows that K is not finitely generated.

Since G is not finitely generated and soluble, $L = K'$ is a proper subgroup of K. If we suppose that $\langle g, L \rangle = G$, then $G/L = \langle g \rangle L/L \cong \langle g \rangle/(\langle g \rangle \cap L)$ is abelian. It follows that $K \leq L$, and we obtain a contradiction. Thus $\langle g, L \rangle$ is a proper subgroup of G. If we suppose that G/L is finite, then Corollary 4 shows that $G = L\langle g \rangle$. Hence G/L is infinite, i.e. K/L is infinite. As we noted above, $\langle L, g \rangle$ is a proper subgroup of G. Since D does not include $\langle L, g \rangle$, $\langle L, g \rangle$ is finitely generated by Lemma 6. A subgroup $\langle g \rangle \cap K$ is cyclic, so that $\langle g \rangle \cap K = \langle v \rangle$ for some $v \in \langle g \rangle$. Then we have

$$K \cap (L\langle g \rangle) = L(K \cap \langle g \rangle) = L\langle v \rangle.$$

Clearly $L\langle v \rangle$ is a G-invariant subgroup of K. Furthermore, $|\langle L, g \rangle : L\langle v \rangle| \leq |G : D| = d$. It follows that $\langle L, v \rangle$ is finitely generated (see, for example, [5, Corollary 7.2.1]). If we suppose that $K/(L\langle v \rangle)$ is finitely generated, then K is finitely generated. This contradiction shows that $K/(L\langle v \rangle)$ is not finitely generated.

Let $Z/(L\langle v \rangle)$ be a proper G-invariant subgroup of $K/(L\langle v \rangle)$, then we have $Z\langle g \rangle \cap K = X(\langle g \rangle \cap K) = Z\langle v \rangle = Z$. It follows that $Z\langle g \rangle$ is a proper subgroup of G. Since D does not include $Z\langle g \rangle$, $Z\langle g \rangle$ is finitely generated by Lemma 6.

Assume that $K/(L\langle v \rangle)$ includes a proper subgroup $U/\langle L, v \rangle$ of finite index. Then $|G : U|$ is finite, so that $U_1 = \mathbf{Core}_G(U)$ has finite index in G. By above proved $U_1\langle g \rangle$ is finitely generated. Finiteness of $|G : U_1|$ implies that G is finitely generated. This contradiction shows that $K/(L\langle v \rangle)$ is $\mathfrak{F}$-perfect. Then $K/(L\langle v \rangle)$ includes a subgroup $P/(L\langle v \rangle)$ such that K/P is a quasicyclic q-group for some prime q. We remark that $K/P^x = K^x/P^x \cong K/P$, i.e. K/P^x is a quasicyclic q- group for all $x \in G$. Finiteness of G/K implies that the family $\{P^x \mid x \in G\}$ is finite. Let $\{P^x \mid x \in G\} = \{P_1, P_2, \ldots, P_m\}$ where $P_1 = P$. Then the embedding

$$K/\mathbf{Core}_G(P) \hookrightarrow G/P_1 \times G/P_2 \times \ldots \times G/P_m,$$

shows that $K/\mathbf{Core}_G(P)$ is a Chernikov q-group. Since $K/\mathbf{Core}_G(P)$ is $\mathfrak{F}$- perfect, it is divisible. Since $\langle L, v \rangle \leq P$ and $\langle L, v \rangle$ is G-invariant, $\langle L, v \rangle \leq C = \mathbf{Core}_G(P)$. By proved above, C is finitely generated. In particular, C/L is an abelian finitely generated group, so that K/L is an abelian minimax group. Suppose that $\mathbf{Tor}(K/L) = T/L \neq K/L$. Put

$$\pi = \{r \mid r \text{ is a prime such that } K/T \neq (K/T)^r\}.$$

Since K/T is torsion-free and minimax, the set π is infinite. Therefore we can choose a prime r such that $r \neq d$ and $r \in \pi$. Let $M/T = (K/T)^r$, then K/M is a non-identity elementary abelian r- group. Clearly a subgroup M is G-invariant. By the choice of M, $\Pi(G/M) = \{r, d\}$. This contradiction with Corollary 4 shows that K/L is periodic. In this case, C/L is finite, so that K/L is Chernikov. Let Q/L be a divisible part of K/L. The isomorphism $Q/L \cong K/C$ shows that Q/L is a q-subgroup. Since Q has finite index, an application of Corollary 4 shows that $G = Q\langle g \rangle$ and G/Q is a d-group.

Suppose that Q/L includes an infinite G-invariant subgroup Q_1/L and that $Q_1\langle g \rangle$ is finitely generated. Then $Q_1\langle g \rangle/L = (Q_1/L)\langle gL \rangle$ is also finitely generated. Now G/L is periodic, in particular, $\langle gL\rangle$ is finite. It follows that Q_1/L is finitely generated. On the other hand, Q_1/L is an infinite Chernikov group, therefore it cannot be finitely generated. This contradiction shows that $Q_1\langle g\rangle$ is not finitely generated. Then $A/C_A(Q_1\langle g\rangle)$ is minimax. Corollary 1 shows that $g \in D$. This contradiction shows that Q/L is G-quasifinite.

Suppose that $L \neq \langle 1\rangle$. Then $V = L' \neq L$. We have already proved that $L\langle g\rangle$ is is finitely generated. The fact that G/L is periodic implies that L has finite index in $L\langle g\rangle$. Then L is finitely generated (see, for example, [5, Corollary 7.2.1]). Thus L/V is a finitely generated abelian group. Then L/V includes a proper G-invariant subgroup W/V of finite index in L/V (this subgroup can be identity). Then K/W is a Chernikov group, having finite derived subgroup. Let Q_2/W be the divisible part of K/W, then $Q_2/W \cong Q/L$, so that Q_2/W is a divisible Chernikov q-subgroup. Since $(K/W)'$ is finite, $Q_2/W \leq \zeta(K/W)$. Since index $|G : Q_2|$ is finite, $G = Q_2\langle g\rangle$ by Corollary 4. Then

$$K = K \cap (Q_2\langle g\rangle) = Q_2(K \cap \langle g\rangle) = Q_2\langle v\rangle.$$

It follows that K/Q_2 is cyclic. Then the inclusion $Q_2/W \leq \zeta(K/W)$ implies that K/W is abelian. But $L \leq W$, and this contradiction the choice of W. Consequently L is abelian. So Q is a proper subgroup of G which is Chernikov q-group, than by Lemma 4 Q is a p-group. $\square$

Recall that a group G have *finite special rank* $\mathbf{r}(G) = r$ if every finitely generated sub group of G has at most r generators and there exists a finitely generated subgroup H of G such that H has exactly r generators. Therefore every abelian minimax group has finite special rank.

3 PROOF OF THE MAIN THEOREM

If G/D is not finitely generated, then Proposition 1 shows that G is a group of type (1).

Suppose now that G/D is finitely generated. Then Lemma 7 proves that G is soluble and G/D is a group of a prime order q. If we assume that G/G' is infinite, then Proposition 2 shows that G is a group of type (2).

Finally suppose that G/G' is finite. Then Proposition 3 shows that G includes a normal divisible Chernikov p-subgroup Q, such that $G = Q\langle g\rangle$ where g is a d-element, p, d are primes (not necessary different). Moreover, $g^d \in D$ and Q is G-quasi-finite. Finally, the assertion 3c follows from the results of Section 3 of the paper [21], and the assertion 3d follows from Theorem 3.4 of the paper [6].

Let G be a group of the type (2) or (3). Then $D = Q\langle g^d\rangle$ is a proper Chernikov subgroup of G, and hence it is not finitely generated. Then $A/C_A(D)$ is minimax and Lemma 4 proves the final assertion.

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Нехай R — кільце, G — група. R-модуль A називається *мінімаксним* якщо A містить нетеровий підмодуль B такий, що A/B артіновий. Вивчаються $\mathbb{Z}_{p^\infty}G$-модулі A такі, що $A/C_A(H)$ є мінімаксним як $\mathbb{Z}_{p^\infty}$-модуль, для кожної власної підгрупи H, яка не є скінченно породженою.

Ключові слова і фрази: мінімаксний модуль, коцентралізатор, модуль над груповим кільцем, мінімаксно-антифінітарний RG-модуль, узагальнено радикальна група.