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LATERAL CONTINUITY AND ORTHOGONALLY ADDITIVE OPERATORS

We generalize the notion of a laterally convergent net from increasing nets to general ones and study the corresponding lateral continuity of maps. The main result asserts that, the lateral continuity of an orthogonally additive operator is equivalent to its continuity at zero. This theorem holds for operators that send laterally convergent nets to any type convergent nets (laterally, order or norm convergent).

Key words and phrases: orthogonally additive operator, lateral continuity.

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1 INTRODUCTION

Some versions of laterally (i.e., horizontally) continuous maps acting between vector lattices were considered in [4] and [6]. A net (x_α) in a vector lattice E in the mentioned above papers is called laterally convergent to $x \in E$ if $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ as $\alpha < \beta$ and $x_\alpha \xrightarrow{\mathrm{o}} x$. Here and in the sequel the relation $u \sqsubseteq v$ means that u is a fragment (component, in another terminology) of v, that is, $u \perp (v-u)$, and the notation $x_\alpha \xrightarrow{\mathrm{o}} x$ means that the net (x_α) order converges to x, i.e. there is a net (u_α) in E with the same index set such that $|x_\alpha - x| \le u_\alpha$ for all α, and $u_\alpha \downarrow 0$, that is, (u_α) is a decreasing (in the non-strict sense) net with zero infimum. In our opinion, the assumption $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ on the net in the above definition of the lateral convergence is too restrictive and unjustified. One of the tasks of the present note is to generalize the lateral convergence to not necessarily laterally increasing nets.

In [4] the authors considered maps that laterally convergent nets send to order convergent nets (such maps were called disjointly continuous). In [6] the maps that laterally convergent nets send to norm convergent nets in a normed space were called laterally-to-norm continuous. In both papers [4] and [6] laterally convergent nets were considered to be laterally increasing. Another task of the present paper is to analyze the relationships between different versions of lateral continuity. We provide an example of a disjointly continuous map which is not laterally continuous in the sense of new (generalized) definition of the lateral continuity. However, we do not know if there exists an orthogonally additive operator of the kind.

Due to the generalized definition of the lateral continuity, there are nontrivial nets laterally converging to zero. So, it is naturally to ask, whether the lateral continuity of a linear (or, more general, orthogonally additive) operator can be reduced to the same continuity at zero. Our mail result answer this in the affirmative.

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1.1 Terminology and notation

Terminology, notation and facts on vector lattices, that are familiarly used in the paper were taken from [1]. The equality $z = x \sqcup y$ for elements x, y, z of a vector lattice E means that $z = x + y$ and $x \perp y$, that is, $|x| \wedge |y| = 0$. All vector lattices considered in the paper are assumed to be Archimedean.

For the first time orthogonally additive operators on vector lattices were considered and investigated in [4] and [5]. Let E be a vector lattice and X be a vector space. A function $T : E \to X$ is called an *orthogonally additive operator* if $T(x \sqcup y) = T(x) + T(y)$. In other words, orthogonally additive operators the sum of two disjoint elements send to the sum of their images.

An important example of a nonlinear orthogonally additive operator is the positive part x^+ of an element x in a vector lattice E. Show that, if $x \perp y$ then $(x + y)^+ = x^+ + y^+$. Using the well known properties $(u + v) \vee (u + w) = u + (v \vee w)$ [1, Theorem 2.1] and $\sup(-A) = -\inf A$ [1, p. 3] for $u, v, w \in E$ and $A \subseteq E$, taking into account that $x^+ \perp y^-$, $y^+ \perp x^-$, and that the disjoint (orthogonal) complement is a linear space [1, Theorem 3.3], we obtain $(x^+ + y^+) \wedge (x^- + y^-) = 0$, and hence

$$\begin{aligned}(x+y)^+ &= (x+y) \vee 0 = (x^+ + y^+ - x^- - y^-) \vee (x^+ + y^+ - x^+ - y^+) \\ &= x^+ + y^+ + \big(-(x^- + y^-) \vee -(x^+ + y^+)\big) \\ &= x^+ + y^+ - (x^+ + y^+) \wedge (x^- + y^-) = x^+ + y^+.\end{aligned}$$

We use several times the example of a vector lattice $\mathbb{R}^\Omega$ of all functions $x : \Omega \to \mathbb{R}$ with respect to the pointwise linear operations of taking the sum and the multiplication by a scalar, and with the pointwise order: $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in \Omega$. Given a subset $A \subseteq \Omega$, the symbol $\mathbf{1}_A$ denotes the characteristic function of A, that is, the function $\mathbf{1}_A : \Omega \to \mathbb{R}$ given by

$$\mathbf{1}_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \in \Omega \setminus A. \end{cases}$$

Definitions and necessary properties of Boolean algebras see in [2, Definition 7.9].

1.2 The lateral order

For the first time the lateral order and its properties were considered in [3]. But, as far as we know, the cited paper is not yet published. So, for convenience of the reader, propositions that we took from [3], we provide with complete proofs and citation.

Proposition 1 ([3]). *Let E be a vector lattice and $x, y \in E$.*

(1) If $x \sqsubseteq y$ then

(a) $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$,

(b) $x^+ \leq y^+$ and $x^- \leq y^-$,

(c) $x^- \perp y^+$ and $x^+ \perp y^-$,

(d) $|x| \sqsubseteq |y|$.

(2) $x \sqsubseteq y$ if and only if $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$.

Proof. Assume $x \sqsubseteq y$, that is, $y = x \sqcup (y-x)$. Then $y^+ = x^+ \sqcup (y-x)^+$, which implies $x^+ \le y^+$ and $(y-x)^+ = y^+ - x^+$. Hence $y^+ = x^+ \sqcup (y^+ - x^+)$, i.e., $x^+ \sqsubseteq y^+$. Analogously, $x^- \le y^-$ and $x^- \sqsubseteq y^-$. Thus, *(a)*, *(b)* and the "only if" part of item *(2)* is proved.

(c) By *(b)*, $0 \le x^- \wedge y^+ \le y^- \wedge y^+ = 0$. The second part of *(c)* is proved analogously.

(d) By *(a)*, $x^+ \perp y^+ - x^+$, and by *(c)*, $x^+ \perp y^-$. Moreover, $x^+ \perp x^-$. Hence $x^+ \perp |y| - |x|$. Analogously, $x^- \perp |y| - |x|$. The latter two relations yield $|x| \perp |y| - |x|$, that is, $|x| \sqsubseteq |y|$.

The "if" part of *(2)*. Suppose $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$. Then the first relation implies $x^+ \le y^+$. Then $0 \le x^+ \wedge y^- \le y^+ \wedge y^- = 0$, and hence $x^+ \perp y^-$. Taking into account $x^+ \perp (y^+ - x^+)$ and $x^+ \perp x^-$, one gets $x^+ \perp (y^+ - x^+ - y^- + x^-)$, i.e., $x^+ \perp (y-x)$. Analogously, $x^- \perp (y-x)$, and thus, $x \perp (y-x)$. ☐

Proposition 2 ([3])**.** *Let E be a vector lattice. Then the binary relation $\sqsubseteq$ is a partial order on E.*

Proof. For every $x \in E$ the relation $x \sqsubseteq x$ means that $x \perp 0$, which is obviously valid.

Assume $x, y \in E$ and $x \sqsubseteq y \sqsubseteq x$. Since $x \perp (y-x)$ and $y \perp (y-x)$, one has $(y-x) \perp (y-x)$, that is, $y - x = 0$. Let $x, y, z \in E$ and $x \sqsubseteq y \sqsubseteq z$. Then $x \perp (y-x)$. Moreover, by *(1) (b)* of Proposition 1 one has $|x| \le |y|$. The latter inequality together with $y \perp (z-y)$ gives $x \perp (z-y)$. Since the orthogonal complement is a linear space [1, Theorem 3.3], we obtain $x \perp (y-x) + (z-y) = z - x$, that is $x \sqsubseteq z$. ☐

Given any $e \in E$, by $\mathfrak{F}_e$ we denote the set of all fragments of e, $\mathfrak{F}_e = \{x \in E : x \sqsubseteq e\}$. Item *(1)* of the following proposition is very known for $e \ge 0$ [1, Theorem 3.15].

Proposition 3 ([3])**.** *Let E be a vector lattice and $e \in E$. Then*

(1) the set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with zero 0, unit e with respect to the operations $x \cup y = (x^+ \vee y^+) - (x^- \vee y^-)$ and $x \cap y = (x^+ \wedge y^+) - (x^- \wedge y^-)$;

(2) if $e \ge 0$ then the lateral order $\sqsubseteq$ on $\mathfrak{F}_e$ coincides with the lattice order $\le$, and hence the lateral supremum (infimum) of an arbitrary set $A \subseteq \mathfrak{F}_e$ equals its lattice supremum;

(3) $x \cup y$ equals the supremum, and $x \cap y$ equals the infimum of a two-point set $\{x, y\} \subseteq \mathfrak{F}_e$ with respect to the lateral order $\sqsubseteq$ both in $\mathfrak{F}_e$ and E.

Proof. *(1)* By [1, Theorem 3.15], $\mathfrak{F}_{e^+}$ and $\mathfrak{F}_{e^-}$ are Boolean algebras with zero 0, units e^+ and e^- respectively and operations $\vee$ and $\wedge$, that coincide with the lattice operations on E. Consider the direct sum $\mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$, that is, the Cartesian product $\mathfrak{F}_{e^+} \times \mathfrak{F}_{e^-}$ with zero $(0,0)$, unit (e^+, e^-) and operations $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$ and $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$. Obviously, $\mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$ is a Boolean algebra. Then the bijection $\tau : \mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-} \to \mathfrak{F}_e$ given by $\tau(x, y) = x - y$ for any $(x, y) \in \mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$ (the facts that $\tau(x, y) \in \mathfrak{F}_e$, and that τ is one-to-one follow from Proposition 1) induces the Boolean algebra structure on $\mathfrak{F}_e$. It remains to observe that τ sends $(0,0)$ to 0, (e^+, e^-) to $e^+ - e^- = e$, and the induces operations are given by the formulas given in the statement of *(1)*.

(2) Assume $e \ge 0$ and $x, y \in \mathfrak{F}_e$. By Proposition 1, $x, y \ge 0$.

Let $x \sqsubseteq y$. By *(1) (b)* of Proposition 1, we get $x \le y$.

Let $x \le y$. Then $0 \le x \wedge (e-y) \le x \wedge (e-x) = 0$, and hence $x \perp (e-y)$. Since $x \perp (e-x)$ and the disjoint complement is a linear subspace [1, Theorem 3.3], we obtain $x \perp (y-x)$, and hence $x \sqsubseteq y$.

(3) follows from *(2)* and Proposition 1. ☐

By Proposition 3, using the well known equality $x + y = x \vee y + x \wedge y$ [1, Theorem 1.2], we obtain the following consequence.

Corollary 1 ([3]). *Let E be a vector lattice, $e \in E$ and $x, y \sqsubseteq e$. Then $x + y = x \cup y + x \cap y$.*

Proof. The proof follows from equalities:

$$\begin{aligned} x + y &= x^+ + y^+ - (x^- + y^-) \\ &= x^+ \vee y^+ + x^+ \wedge y^+ - (x^- \vee y^- + x^- \wedge y^-) = x \cup y + x \cap y. \end{aligned}$$

□

In the sequel, on the Boolean algebra $\mathfrak{F}_e$ we will consider the set-theoretical operations $x \setminus y = x \cap (e - y) = x - x \cap y$ and $x \Delta y = (x \setminus y) \cup (y \setminus x) = (x \setminus y) \sqcup (y \setminus x)$.

Definition 1. *A subset A of a vector lattice E is said to be* laterally bounded *if $A \subseteq \mathfrak{F}_e$ for some $e \in E$.*

2 LATERAL CONVERGENCE

In this section, we generalize the lateral convergence from laterally increasing nets to arbitrary ones. All statements that are used to prove the main result are given as lemmas, however they could be of their own interest. By a laterally converging net in a vector lattice we mean any laterally bounded order converging net. But not only such nets. The point is that, by attaching of several new elements to a laterally bounded net, one can spoil the lateral boundedness, however, by the idea of convergence, this should not affect the lateral convergence. Taking this into account, we give the next definition.

Definition 2. *An order converging net (x_α) to an element x of a vector lattice E, so that there is an index α_0 such that the net $(x_\alpha)_{\alpha \geq \alpha_0}$ is laterally bounded, is said to be* laterally converging *to x, and the element x is called the* lateral limit *of (x_α). The notation $x_\alpha \xrightarrow{\text{lat}} x$ means that the net (x_α) laterally converges to x. In the particular case, where $x_\alpha \sqsubseteq x_\beta$ for any $\alpha < \beta$, the laterally convergent net (x_α) is called* up-laterally convergent *to its lateral limit*[1].

It is interesting to observe that the lateral limit is laterally bounded by the same element as the net itself. This follows from the next statement.

Lemma 1. *Let E be a vector lattice and $e \in E$. Then the set $\mathfrak{F}_e$ is order closed.*

Proof. Let $x_\alpha \xrightarrow{\text{o}} x$, where $x_\alpha \in \mathfrak{F}_e$ and $x \in E$. Show that $x \sqsubseteq e$. By the order continuity of the lattice operations, $0 = |x_\alpha| \wedge |e - x_\alpha| \xrightarrow{\text{o}} |x| \wedge |e - x|$, and hence, $|x| \wedge |e - x| = 0$. □

As an immediate consequence of Lemma 1 we obtain the following fact.

Lemma 2. *Let E be a vector lattice, $e \in E$ and $x_\alpha \xrightarrow{\text{lat}} x$, where $x \in E$ and $x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. Then $x \sqsubseteq e$.*

[1]Recall that exactly these nets in [4] and [6] were said to be laterally convergent.

We say that a subset A of a vector lattice E is *laterally closed* if the lateral limit of any net from A belongs to A. Using this terminology, Lemma 2 asserts that, for any $e \in E$ the set $\mathfrak{F}_e$ is laterally closed.

Next we show that, in the definition of the lateral convergence, one can choose a majorizing net to be laterally bounded.

Proposition 4. *Let E be a Dedekind complete vector lattice, $e \in E$, $x_\alpha \xrightarrow{\text{lat}} x$, where $x \in E$ and $x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. Then there is a net (v_α) with the same index set such that $v_\alpha \sqsubseteq |e|$ and $|x_\alpha - x| \sqsubseteq v_\alpha$ for all $\alpha \geq \alpha_0$ and $v_\alpha \downarrow 0$.*

For the proof, we need the following lemma.

Lemma 3. *Let E be a vector lattice, $e \in E$ and $x, y \sqsubseteq e$. Then $|x - y| = |x \Delta y| \sqsubseteq |e|$.*

Proof of Lemma 3. Subtracting from the equality $x = (x \setminus y) \sqcup (x \cap y)$ the equality $y = (y \setminus x) \sqcup (x \cap y)$, we obtain $x - y = (x \setminus y) - (y \setminus x)$. Since $(x \setminus y) \perp (y \setminus x)$, by the orthogonal additivity of the positive part of an element and Corollary 1, we obtain

$$|x - y| = |x \setminus y| + |y \setminus x| = |(x \setminus y) + (y \setminus x)| = |(x \setminus y) \cup (y \setminus x)| = |x \Delta y|.$$

Since $x \Delta y \sqsubseteq e$, by item (1)(d) of Proposition 1 we get $|x \Delta y| \sqsubseteq |e|$. □

Proof of Proposition 4. Let (u_α) be a net in E such that $|x_\alpha - x| \leq u_\alpha \downarrow 0$. For every α we set $v_\alpha = \bigvee_{\beta \geq \alpha} |x_\beta - x|$. The supremum exists because $|x_\beta - x| \leq 2e$ for all β and E is Dedeking complete. By Lemma 3, $|x_\beta - x| \sqsubseteq |e|$ for all β. By (2) of Proposition 3, v_α equals the lateral supremum of the net $(|x_\beta - x|)_{\beta \geq \alpha}$. Hence $v_\alpha \sqsubseteq |e|$. The inequality $|x_\alpha - x| \leq v_\alpha$ for all α follows from the construction of v_α. Finally, the condition $v_\alpha \downarrow 0$ follows from

$$0 \leq v_\alpha \leq \bigvee_{\beta \geq \alpha} u_\beta = u_\alpha \downarrow 0.$$

□

Lemma 4. *Let E be a vector lattice, (x_α) a net in E and $x \in E$. Then the following assertions are equivalent:*

(i) $x_\alpha \xrightarrow{\text{lat}} x$;

(ii) $x_\alpha^+ \xrightarrow{\text{lat}} x^+$, $x_\alpha^- \xrightarrow{\text{lat}} x^-$ *and* $(x_\alpha)_{\alpha \geq \alpha_0}$ *is laterally bounded for some* α_0;

(iii) The set $\{x\} \cup \{x_\alpha : \alpha \geq \alpha_0\}$ *is laterally bounded and* $x_\alpha \Delta x \xrightarrow{\text{lat}} 0$.

Moreover, each of (i)–(iii) implies $|x_\alpha| \xrightarrow{\text{lat}} |x|$.

Proof. (i) ⇔ (ii) The equivalence of $x_\alpha \xrightarrow{\text{o}} x$ and the conditions $x_\alpha^+ \xrightarrow{\text{o}} x^+$, $x_\alpha^- \xrightarrow{\text{o}} x^-$ is easily seen. It remains to observe that, the lateral boundedness of (x_α) implies that of the nets (x_α) and (x_α) by Proposition 1.

(i) ⇒ (iii) Assume $x_\alpha \xrightarrow{\text{lat}} x$. By Lemma 2, there is $e \in E$ such that $x, x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. Then the net $(x_\alpha \Delta x)_{\alpha \geq \alpha_0}$ is laterally bounded by e. Moreover, by Lemma 3, $|x_\alpha \Delta x| = |x_\alpha - x|$, and hence $x_\alpha \Delta x \xrightarrow{\text{o}} 0$.

(iii) ⇒ (i) directly follows from Lemma 3.

It remains to observe that the condition $|x_\alpha| \xrightarrow{\text{lat}} |x|$ follows from *(1) (d)* of Proposition 1. □

Remark that the assumption of lateral boundedness of the set $\{x\} \cup \{x_\alpha : \alpha \geq \alpha_0\}$ in *(iii)* serves for the elements $x_\alpha \Delta x$ to be well defined, and the implication *(ii)* $\Rightarrow$ *(i)* may fail to be valid if one removes the assumption of lateral boundedness of the net (x_α) in *(ii)*, as the following example shows.

Example 1. *There exist a vector lattice E, a sequence (x_n) in E and an element $x \in E$ such that $x_n^+ \xrightarrow{\text{lat}} x^+$, $x_n^- \xrightarrow{\text{lat}} x^-$, but for every $n_0 \in \mathbb{N}$ the sequence $(x_n)_{n \geq n_0}$ is not laterally bounded, and hence, (x_n) laterally diverges.*

Proof. Indeed, consider the vector lattice $E = \mathbb{R}^{\mathbb{R}}$ with the pointwise order and the sequence (x_n) in E, given by

$$x_n(t) = \begin{cases} 1, & \text{if } t \in (-\infty, \frac{1}{n}], \\ -1, & \text{if } t \in (\frac{1}{n}, +\infty). \end{cases}$$

It is a simple technical exercise to show that the sequence (x_n) order converges to

$$x(t) = \begin{cases} 1, & \text{if } t \in (-\infty, 0], \\ -1, & \text{if } t \in (0, +\infty), \end{cases}$$

however, the sequence $(x_n)_{n \geq n_0}$ is not laterally bounded for all $n_0 \in \mathbb{N}$. On the other hand, $x_n^+ = \mathbf{1}_{(-\infty, \frac{1}{n}]} \xrightarrow{\text{o}} \mathbf{1}_{(-\infty, 0]}$. Since $x_n^+ \sqsubseteq \mathbf{1}_{(-\infty, 1]}$ for al $n \in \mathbb{N}$, one has that $x_n^+ = \mathbf{1}_{(-\infty, \frac{1}{n}]} \xrightarrow{\text{lat}} \mathbf{1}_{(-\infty, 0]}$. Analogously, $x_n^- = \mathbf{1}_{(\frac{1}{n}, +\infty)} \xrightarrow{\text{lat}} \mathbf{1}_{(0, +\infty)}$. $\square$

3 Lateral continuity

In this section we study versions of continuity connected to the lateral convergence.

Definition 3. *Let E, F be vector lattices. A function $f : E \to F$ is said to be:*

(L-L) laterally continuous *at a point $x \in E$ if for any net (x_α) in E the relation $x_\alpha \xrightarrow{\text{lat}} x$ implies $f(x_\alpha) \xrightarrow{\text{lat}} f(x)$;*

(L-O) laterally-to-order continuous *at a point $x \in E$ if for any net (x_α) in E the relation $x_\alpha \xrightarrow{\text{lat}} x$ implies $f(x_\alpha) \xrightarrow{\text{o}} f(x)$.*

Definition 4. *Let E be a vector lattice and F a normed space. A function $f : E \to F$ is said to be*

(L-N) laterally-to-norm continuous *at a point $x \in E$ if for any net (x_α) in E the condition $x_\alpha \xrightarrow{\text{lat}} x$ yields $\|f(x_\alpha) - f(x)\| \to 0$.*

Following the terminology of [4], a map $f : E \to F$ acting from a vector lattice E to a vector lattice or a normed space F is said to be *disjointly laterally* (*disjointly order* or *disjointly norm*) *continuous* at a point $x \in E$ if for every net (x_α) in E up- laterally converging to x the net $(f(x_\alpha))$ laterally (order or norm, respectively) converges to $f(x)$ in F. The corresponding type of convergence we denote by *(DL-L)*, *(DL-O)* or *(DL-N)*.

We say that a function $f : E \to F$ is *continuous* in some of the senses (*(L-L)*, *(L-O)*, *(L-N)*, *(DL-L)*, *(DL-O)* or *(DL-N)*), if f is continuous in the same sense at any point $x \in E$.

Notice that the generalization of the notion of a laterally convergent net from up-laterally convergent nets to arbitrary nets may affect the lateral continuity at a fixed point. Indeed, if a net (x_α) in a vector lattice E up-laterally converges to zero then $x_\alpha = 0$ for all α. Hence, an arbitrary map $f : E \to F$ up-laterally convergent to zero nets sends to convergent nets in any sense. So, it is not a big deal to provide an example where the same happen at a nonzero point $x_0 \in E$ (say, at a point x_0 which is an atom in E, that is, the only fragments of x_0 are 0 and x_0 itself). It is clear that not every map acting from $E = \mathbb{R}^{\mathbb{R}}$ to a nontrivial vector lattice or a normed space is continuous in any of the senses *(L-L)*, *(L-O)* or *(L-N)* at x_0. For instance, the one given by $f(x_0) = 0$ and $f(x) = y_0 \neq 0$ for all $x \in E \setminus \{x_0\}$. Indeed, the sequence $x_n = \mathbf{1}_{[0,\frac{1}{n}]}$ laterally converges to x_0, however $f(x_n) = y_0 \not\to 0$ in any of the senses *(L-L)*, *(L-O)* or *(L-N)*.

The following theorem, which is the main result, in particular, asserts that the lateral continuity of an orthogonally additive operator is equivalent to its lateral continuity just at zero.

Theorem 1. *Let E be a vector lattice, F a vector lattice or a normed space, $T : E \to F$ an orthogonally additive operator. Let X be one of the letters L, O or N. Then the following assertions are equivalent:*

(1) T is (L-X) *continuous;*

(2) T is (L-X) *continuous at zero.*

Proof. The implication *(1)* $\Rightarrow$ *(2)* is obvious. Prove *(2)* $\Rightarrow$ *(1)*. Let (x_α) be a net in E, $x \in E$ and $x_\alpha \xrightarrow{\text{lat}} x$. Choose $e \in E$ and an index α_0 so that $x_\alpha \sqsubseteq e$ as $\alpha \geq \alpha_0$. Then, by Lemma 2, $x \sqsubseteq e$. Next, Lemma 4 implies that $x_\alpha \Delta x \xrightarrow{\text{lat}} 0$. Let (u_α) be a net in E such that $|x_\alpha \Delta x| \leq u_\alpha \downarrow 0$. Taking into account that $x_\alpha \Delta x = (x_\alpha \setminus x) \cup (x \setminus x_\alpha)$, we obtain $|x_\alpha \setminus x| \leq u_\alpha \downarrow 0$ and $|x \setminus x_\alpha| \leq u_\alpha \downarrow 0$. Then $x_\alpha \setminus x \xrightarrow{\text{o}} 0$ and $x \setminus x_\alpha \xrightarrow{\text{o}} 0$, and hence, $x_\alpha \setminus x \xrightarrow{\text{lat}} 0$ and $x \setminus x_\alpha \xrightarrow{\text{lat}} 0$. By the *(L-X)*-continuity at zero, $T(x_\alpha \setminus x) \to 0$ and $T(x \setminus x_\alpha) \to 0$ in the sense of X-convergence, because $T(0) = 0$ (as T is orthogonally additive). Since $x_\alpha = (x_\alpha \setminus x) \sqcup (x_\alpha \cap x)$, by the orthogonal additivity of T,

$$T(x_\alpha) = T(x_\alpha \setminus x) + T(x_\alpha \cap x). \tag{1}$$

Analogously,

$$T(x) = T(x \setminus x_\alpha) + T(x \cap x_\alpha). \tag{2}$$

Subtracting from (1) the equality (2), we obtain $T(x_\alpha) - T(x) = T(x_\alpha \setminus x) - T(x \setminus x_\alpha) \to 0$ in the sense of X. $\square$

The following example shows that, the notion of lateral continuity changes when replacing the up-laterally convergent nets with arbitrary lateral converging nets.

Recall that, following [4], a map $f : E \to F$ between vector lattices E and F is called disjointly continuous if for every $x \in E$ and every up-laterally convergent net (x_α) the condition $x_\alpha \xrightarrow{\text{lat}} x$ implies $f(x_\alpha) \xrightarrow{\text{o}} f(x)$.

Example 2. *There exist vector lattices E, F and a disjointly continuous map $f : E \to F$ which is not laterally-to-order continuous.*

Proof. Set $E = \mathbb{R}^{[0,1]}$, $F = \mathbb{R}^{[0,2]}$ and define a map $f : E \to F$ by $f(0) = 0$ and $f(x) = x + \mathbf{1}_{(1,2]}$ for $x \in E \setminus \{0\}$. Then f is disjointly continuous at zero, because all up-laterally convergent to zero nets consist of zero elements, and the disjoint continuity of f at any nonzero point is obvious. Show that f is not laterally continuous at zero. Indeed, for the sequence $x_n = \mathbf{1}_{(0,\frac{1}{n})}$, $n = 1, 2, \dots$ one has $x_n \xrightarrow{\text{lat}} 0$, and nevertheless, $f(x_n) = x_n + \mathbf{1}_{(1,2]} \xrightarrow{\text{lat}} \mathbf{1}_{(1,2]} \neq 0 = f(0)$. □

We do not know if there is an orthogonally additive operator with the same properties.

Problem. *Do there exist vector lattices E, F and an orthogonally additive operator $T : E \to F$ which is not laterally-to-order continuous?*

Remark that any other version of Theorem 1 holds true in which instead of the convergence in the sense X one considers another convergence (say, topological), which has the property of uniqueness of limit and such that the sum of two convergent nets converges to the sum of their limits.

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Ми узагальнюємо поняття латерально збіжної сітки зі зростаючих сіток на довільні та вивчаємо відповідну латеральну неперервність відображень. Основний результат стверджує, що латеральна неперервність ортогонально адитивного оператора еквівалентна до його латеральної неперервності в нулі. Ця теорема має місце для операторів, що переводять латерально збіжні сітки у сітки, які збігаються в будь-якому розумінні (латерально, порядково чи за нормою).

Ключові слова і фрази: ортогонально адитивний оператор, латеральна неперервність.