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MULTIPOINT NONLOCAL PROBLEM FOR FACTORIZED EQUATION WITH DEPENDENT COEFFICIENTS IN CONDITIONS

The conditions of correct solvability of multipoint nonlocal problem for factorized PDE with coefficients in conditions, which depend on one real parameter, are established. It is shown that these conditions on the set of full Lebesgue measure of the interval parameters are fulfilled.

Key words and phrases: differential equations, multipoint nonlocal problem, dependent coefficients, small denominators, diophantine approximation, metric estimations.

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Let $\mathbb{T}^p$ denote the p-dimensional torus $(\mathbb{R}/2\pi\mathbb{Z})^p$, $T>0$, $Q_p^T = (0,T)\times\mathbb{T}^p$, $\Pi_n = \{\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n : \lambda_i \neq \lambda_j \text{ if } i \neq j\}$, $D_x = (-i\partial/\partial x_1,\ldots,-i\partial/\partial x_p)$, $k=(k_1,\ldots,k_p)\in\mathbb{Z}^p$, $|k| = |k_1|+\ldots+|k_p|$; $B(D_x)$ is differential expression such that

$$\exists\, N_1, N_2 \in \mathbb{R},\ C_1, C_2 > 0: \quad (\forall\, k \in \mathbb{Z}^p) \quad C_1(1+|k|)^{N_1} \le |B(k)| \le C_2(1+|k|)^{N_2}. \tag{1}$$

We use the following functional spaces: $\mathbf{H}_q = \mathbf{H}_q(\mathbb{T}^p)$, $q\in\mathbb{R}$, is Sobolev space obtained by completing the space of all finite trigonometric polynomials

$$\varphi(x) = \sum_k \varphi_k \exp(ik,x)$$

by the norm

$$\|\varphi;\mathbf{H}_q\| = \left(\sum_{k\in\mathbb{Z}^p} \left(1+|k|^2\right)^q |\varphi_k|^2\right)^{1/2}.$$

Let us denote by $\mathbf{C}^n_\theta([0,T];\mathbf{H}_q)$, $n\in\mathbb{Z}_+$, $\theta\in\mathbb{R}$, space of functions

$$u(t,x) = \sum_{k\in\mathbb{Z}^p} u_k(t)\exp(ik,x)$$

such that for any fixed point $t\in[0,T]$ function

$$\partial^j u(t,x)/\partial t^j \equiv \sum_{k\in\mathbb{Z}^p} u_k^{(j)}(t)\exp(ik,x)$$

belong to the space $\mathbf{H}_{q-j\theta}$, $j=0,1,\ldots,n$, and it, as an element of this space, is continuous in t on $[0,T]$; the norm in $\mathbf{C}^n_\theta([0,T];\mathbf{H}_q)$ is defined as follows

$$\|u;\mathbf{C}^n_\theta([0,T];\mathbf{H}_q)\|^2 = \sum_{j=0}^{n} \max_{t\in[0,T]} \|\partial^j u(t,x)/\partial t^j;\mathbf{H}_{q-j\theta}\|^2.$$

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In the domain Q_p^T we consider the following problem:

$$L(\partial/\partial t, D_x)u \equiv \prod_{j=1}^{n} \left(\frac{\partial}{\partial t} - \lambda_j B\left(D_x\right) \right) u(t,x) = 0, \quad (t,x) \in Q_p^T, \tag{2}$$

$$L_j u \equiv \sum_{r=1}^{m} \mu_r(\tau) \frac{\partial^{j-1} u(t,x)}{\partial t^{j-1}} \Big|_{t=t_r} = \varphi_j(x), \quad x \in \mathbb{T}^p, \ j = 1, \dots, n, \tag{3}$$

where $(\lambda_1, \dots, \lambda_n) \in \Pi_n$, the real-valued coefficients $\mu_1, \dots, \mu_m$ depend on parameters τ, $\tau \in I$, where I is an arbitrary fixed segment of the line $\mathbb{R}$, $t_1, \dots, t_m$ are the points of the interval $[0, T]$, and $0 = t_1 < t_2 < \ldots < t_{m-1} < t_m = T$.

Solvability of boundary value problems with multipoint nonlocal conditions for parabolic, strictly hyperbolic, typeless and pseudodifferential equations studied in works [1–4, 6–10].

The problem (2), (3) belong to a class of incorrect problems by Hadamard and its solvability related to the problem of small denominators. In the assumption when the coefficients $\mu_1, \dots, \mu_m$ are independent correct solvability of the problem (2), (3) follows from the results of [10, §14]. If the coefficients $\mu_1, \dots, \mu_m$ are dependent, then these results will not be used to proving solvability of the problem (2), (3). It shoud be noted that two-point nonlocal problem for partial differential equation of the n-th order with conditions (3) was investigated in [11] for the case of two-point nonlocal conditions ($m = 2$).

In the paper we found that the conditions of correct solvability of the multipoint nonlocal problem (2), (3) in the scale of Sobolev spaces are fulfilled for almost all (with respect to Lebesgue measure in the space $\mathbb{R}$) numbers $\tau \in I$.

The solution u to the problem (2), (3) has the form of a Fourier series

$$u(t,x) = \sum_{k \in \mathbb{Z}^p} u_k(t) \exp(ik, x), \tag{4}$$

where function $u_k(t)$, $k \in \mathbb{Z}^p$, is a solution of multipoint nonlocal problem of ordinary differential equations:

$$L(d/dt, k) u_k(t) = 0, \tag{5}$$

$$L_j u_k(t) = \varphi_{jk}, \qquad j = 1, \dots, n. \tag{6}$$

Here, φ_{jk} are Fourier coefficients of the function $\varphi_j(x)$, $j = 1, \dots, n$.

For each fixed $k \in \mathbb{Z}^p$ let us construct a solution to problem (5), (6). Since the $(\lambda_1, \dots, \lambda_n) \in \Pi_n$ and coefficients $B(k)$ satisfy the condition (1), the equation (5) for each $k \in \mathbb{Z}^p$ has the fundamental system of solutions $\{e^{\lambda_1 B(k)t}, \dots, e^{\lambda_n B(k)t}\}$. Then the general solution of the equation (5) has the form

$$u_k(t) = c_{1k} \exp(\lambda_1 B(k)t) + \ldots + c_{nk} \exp(\lambda_n B(k)t),$$

where constants $c_{1k}, \dots, c_{nk}$ are determined from conditions (6) with the help of the system of linear equations

$$\begin{gathered} c_{1k}\lambda_1^{j-1} \sum_{r=1}^{m} \mu_r(\tau) \exp(\lambda_1 B(k) t_r) + c_{2k}\lambda_2^{j-1} \sum_{r=1}^{m} \mu_r(\tau) \exp(\lambda_2 B(k) t_r) \\ + c_{nk}\lambda_n^{j-1} \sum_{r=1}^{m} \mu_r(\tau) \exp(\lambda_n B(k) t_r) = \varphi_{jk} B^{1-j}(k), \qquad j = 1, \dots, n. \end{gathered} \tag{7}$$

Determinant of the system (7) is factorized and represented by the formula

$$\Delta_k(\tau) = W(\lambda) \prod_{s=1}^{n} \Phi_{sk}(\tau),$$

where

$$\Phi_{sk}(\tau) = \sum_{r=1}^{m} \mu_r(\tau) \exp(\lambda_s B(k) t_r),$$

$W(\lambda) = \prod\limits_{1 \le \alpha < \beta \le n} (\lambda_\alpha - \lambda_\beta)$ is Vandermonde determinant constructed from different numbers $\lambda_1, \dots, \lambda_n$, hense $W(\lambda) \neq 0$.

If $\prod\limits_{s=1}^{n} \Phi_{sk}(\tau) \neq 0$, then

$$c_{sk} = \frac{1}{W(\lambda)\Phi_{sk}(\tau)} \sum_{j=1}^{n} (-1)^{s+j} W_{js}(\lambda) B^{1-j}(k) \varphi_{jk}, \quad s = 1, \dots, n.$$

Here we denote by $W_{js}(\lambda)$ the Vandermonde-type determinant obtained from the determinant $W(\lambda)$ by crossing out j-row and s-column.

Thus, the solution to the problem (5), (6) under the condition $\prod\limits_{s=1}^{n} \Phi_{sk}(\tau) \neq 0$ is unique and has the following form

$$u_k(t) = \sum_{s,j=1}^{n} \frac{(-1)^{s+j} W_{js}(\lambda) B^{1-j}(k)}{W(\lambda)\Phi_{sk}(\tau)} \varphi_{jk} \exp(\lambda_s B(k) t), \qquad k \in \mathbb{Z}^p. \tag{8}$$

Conditions for uniqueness of the solution u of the problem (2), (3) follows from the theorem on uniqueness of Fourier expansion of a periodic function and from conditions of uniqueness of the solution $u_k(t)$ of the problem (5), (6) for each $k \in \mathbb{Z}^p$.

Theorem 1. *For uniqueness (at fixed parameter τ) of the solution of the problem* (2), (3) *in the space $\mathbf{C}_\theta^n([0,T]; \mathbf{H}_q)$ it is necessary and sufficient that for all $k \in \mathbb{Z}^p$*

$$\Phi_{1k}(\tau)\Phi_{2k}(\tau) \cdot \ldots \cdot \Phi_{nk}(\tau) \neq 0. \tag{9}$$

If the condition (9) holds, then the formal solution of the problem (2), (3) is represented by the formula

$$u(t,x) = \sum_{k \in \mathbb{Z}^p} \sum_{s,j=1}^{n} \frac{(-1)^{s+j} W_{js}(\lambda) B^{1-j}(k)}{W(\lambda)\Phi_{sk}(\tau)} \varphi_{jk} \exp\big(\lambda_s B(k) t + (ik, x)\big). \tag{10}$$

Expressions $\Phi_{1k}(\tau), \Phi_{2k}(\tau), \dots, \Phi_{nk}(\tau)$ influence the convergence of the series (10), which determines the norm of the solution of the problem (2), (3) in the space $\mathbf{C}_\theta^n([0,T]; \mathbf{H}_q)$. This is explained by the fact that the denominators $\Phi_{1k}(\tau), \Phi_{2k}(\tau), \dots, \Phi_{nk}(\tau)$, $k \in \mathbb{Z}^p$, although non vanishing by the condition above, can arbitrarily rapidly approach to zero for infinite set of vectors $k \in \mathbb{Z}^p$. Therefore, the existence of the solution u of the problem related to the so called problem of small denominators.

To solve this problem we use the metric approach [5] to estimations from below of small denominators.

At first, we formulate the corresponding theorem from the work [12].

Theorem 2. *Let*

$$F(\tau, z) = f_1(\tau)z_1 + \ldots + f_m(\tau)z_m,$$

where $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$*, and* $\{f_1, \ldots, f_m\} \subset C^m(I; \mathbb{R})$*. If the Wronskian* $W[f_1, \ldots, f_m]$ *of the functions* $f_1, \ldots, f_m$ *is not equal to zero on the interval* $I \subset \mathbb{R}$*, then for all* $z \in \mathbb{C}^m \setminus \{0\}$ *and an arbitrary* $\varepsilon \in (0, C_1|z|/2)$*, the following evaluation is valid*

$$\operatorname{meas}\{\tau \in I : |F(\tau, z)| < \varepsilon\} \le C_2 \sqrt[m-1]{\varepsilon/|z|},$$

where $|z| = |z_1| + \ldots + |z_m|$*, positive constants* C_1 *and* C_2 *are defined by formulas*

$$C_1 = \frac{1}{m} \min_{\tau \in I} \left| W[f_1, \ldots, f_m](\tau) \right| \Big(\prod_{j=1}^{m} \|f_j\|_{C^{(m-1)}(I;\mathbb{R})} \sum_{j=1}^{m} \|f_j\|^{-1}_{C^{(m-1)}(I;\mathbb{R})} \Big)^{-1},$$
$$C_2 = 4(\sqrt{2} + 1)(m-1)C_1^{m/(1-m)} \big(\operatorname{meas} I \max_{1 \le j,q \le m} ||f_j^{(q)}||_{C(I;\mathbb{R})} + C_1 \big).$$

Theorem 3. *If* $\mu_r \in C^m(I)$*,* $r = 1, \ldots, m$*, and Wronskian* $W[\mu_1, \ldots, \mu_m]$ *of functions* $\mu_1, \ldots, \mu_m$ *is not equal to zero on the interval* I*, then for almost all (with respect to Lebesgue measure in the space* $\mathbb{R}$*) numbers* $\tau \in I$ *evaluations*

$$|\Phi_{sk}(\tau)| \ge |k|^{-\gamma} \max\big(1, \exp(\operatorname{Re}\lambda_s B(k)T)\big), \quad s = 1, \ldots, n, \tag{11}$$

are satisfied for all (except perhaps a finite number) vectors $k \in \mathbb{Z}^p$ *for* $\gamma > p(m-1)$*.*

Proof. For fixed s we introduce the sets

$$B_k^s = \{\tau \in I \colon |\Phi_{sk}(\tau)| < \varepsilon_k\}, \qquad k \in \mathbb{Z}^p,$$

and the set B^s of such points $\tau \in I$, for which infinite times on $\mathbb{Z}^p$ the estimate is true

$$|\Phi_{sk}(\tau)| < \varepsilon_k = \frac{C_1|k|^{-\gamma}}{2} \max\big(1, \exp(\operatorname{Re}\lambda_s B(k)T)\big), \quad \delta > 0.$$

If $z(s,k) = \big(e^{\lambda_s B(k)t_1}, \ldots, e^{\lambda_s B(k)t_m}\big)$, $f_j(\tau) = \mu_j(\tau)$ for $j = 1, \ldots, m$, then from Theorem 2 follow the equalities:

$$F(\tau, z(s,k)) = \Phi_{sk}(\tau), \qquad W[f_1, \ldots, f_m] = W[\mu_1, \ldots, \mu_m].$$

Since

$$|z(s,k)| = 1 + |e^{\lambda_s B(k)t_2}| + \ldots + |e^{\lambda_s B(k)t_{m-1}}| + +|e^{\lambda_s B(k)T}| \ge \max\big(1, \exp(\operatorname{Re}\lambda_s B(k)T)\big)$$

for all $k \in \mathbb{Z}^p \setminus \{0\}$ and the inequalities are fulfilled

$$0 < \varepsilon_k < \frac{C_1}{2} \max\big(1, \exp(\operatorname{Re}\lambda_s B(k)T)\big) < \frac{C_1}{2}|z(s,k)|$$

then for each $k \neq 0$ by conditions of Theorem 2 we have the following estimation for the measure B_k^s

$$\operatorname{meas} B_k^s \le C_2 \sqrt[m-1]{\varepsilon_k/|z(s,k)|} \le C_3|k|^{-\gamma/(m-1)}, \qquad C_3 = C_2 \Big(\tfrac{C_1}{2}\Big)^{1/(m-1)}.$$

For selected $\gamma > p(m-1)$ series $\sum\limits_{k\in\mathbb{Z}^p\setminus\{0\}} \operatorname{meas} B_k^s$ is majorized by the convergent series $C_3 \sum\limits_{k\in\mathbb{Z}^p} |k|^{-\delta/(p-1)}$. Then from the Borel–Cantelli lemma follows that Lebesgue measure of the set of points τ from I, which contained into the infinite number of sets B_k^s, is equal to zero for fixed s. Thus, $\operatorname{meas} B^s = 0$ for all $s = 1, \dots, n$.

Therefore, when $\gamma > p(m-1)$ for almost all (with respect to Lebesgue measure in $\mathbb{R}$) numbers $\tau \in I$ inequality $|\Phi_{sk}(\tau)| \geq \varepsilon_k$, $s = 1, \dots, n$, is satisfied for all (except for a finite number of) vectors k. The theorem is proved. □

Theorem 4. *Let the condition* (9) *is valid,* $\min\limits_{\tau\in I} |W[\mu_1, \dots, \mu_m](\tau)| > 0$, $\mu_r \in C^m(I)$, $r = 1, \dots, m$, *and* $\varphi_j \in \mathbf{H}_{q+N_1(1-j)+\gamma}$, *where* $\gamma > p(m-1)$, $j = 1, \dots, n$. *Then for almost all (with respect to Lebesgue measure in the space* $\mathbb{R}$*) numbers* $\tau \in I$ *there exists a unique solution of the problem* (2), (3) *in the space* $\mathbf{C}^n_{N_2}([0, T]; \mathbf{H}_q)$, *which is represented by a series* (10) *and continuously depends on the functions* φ_j, $j = 1, \dots, n$.

Proof. Taking into account, that

$$\left|\frac{W_{js}(\lambda)}{W(\lambda)}\right| \leq M_1,\ M_1 = M_1(\lambda),$$

on the basis of formula (10) and estimations (1), (11) we obtain the inequality

$$\|u; \mathbf{C}^n_{N_2}([0, T]; \mathbf{H}_q)\|^2 \leq M_2 \sum_{j=1}^{n} \|\varphi_j; \mathbf{H}_{q+\gamma+N_1(1-j)}\|^2,$$

where $M_2 = 2^{N_2 n+\gamma} n^3 (n+1) M_1^2 C_2^{2n} |\lambda|^{2n}$, $|\lambda| = \max\limits_{1\leq s\leq n} |\lambda_s|$. The proof of the theorem is complete. □

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Василишин П.Б., Савка І.Я., Клюс І.С. *Багатоточкова нелокальна задача для факторизованого рівняння з залежними коефіцієнтами в умовах* // Карпатські матем. публ. — 2015. — Т.7, №1. — С. 22–27.

Встановлено умови коректної розв'язності нелокальної багатоточкової задачі для факторизованого рівняння з коефіцієнтами в умовах, що залежать від одного дійсного параметра. Показано, що ці умови виконуються на множині повної міри Лебеґа відрізка параметрів.

Ключові слова і фрази: диференціальні рівняння, багатоточкова нелокальна задача, залежні коефіцієнти, малі знаменники, діофантові наближення, метричні оцінки.