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ON THE PRIMITIVE REPRESENTATIONS OF FINITELY GENERATED METABELIAN GROUPS OF FINITE RANK OVER A FIELD OF NON-ZERO CHARACTERISTIC

We consider some conditions for imprimitivity of irreducible representations of a metebelian group G of finite rank over a field k. We shoved that in the case where $char k = p > 0$ these conditions strongly depend on existence of infinite p-sections in G.

Key words and phrases: primitive representations, metabelian groups, rank of groups.

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We recall that a group G has finite (Prufer) rank if there is an integer r such that each finitely generated subgroup of G can be generated by r elements; its rank $r(G)$ is then the least integer r with this property. A group G is said to have finite torsion-free rank if it has a finite series in which each factor is either infinite cyclic or locally finite; its torsion-free rank $r_0(G)$ is then defined to be the number of infinite cyclic factors in such a series. The set SpG of all prime numbers p such that a soluble group G of finite rank has a p-quasicyclic factor is said to be the spectrum of the group G.

A group G is said to be minimax if it has a finite series each of whose factor is either cyclic or quasicyclic. It follows from results of [3] that any finitely generated metabelian group of finite rank is minimax.

Let R be a ring and let G be a group. Let H be a subgroup of the group G and let U be a right RH-module. Since the group ring RG can be considered as a left RH-module, we can define the tensor product $U \otimes_{RH} RG$ which is a right RG-module named as the RG-module induced from the RH-module U.

If M is an RG-module and

$$M = U \otimes_{RH} RG \tag{1}$$

for some subgroup $H \leq G$ and some RH-submodule U of M, then the module M is said to be induced from the RH-submodule U.

An RG-module M is said to be primitive if for any subgroup $H < G$ and any RH-submodule $U < M$ the identity (1) does not hold. If the group G has finite torsion-free rank and for any subgroup $H < G$ such that $r_0(H) < r_0(G)$ and any RH-submodule the identity (1) does not hold, then the module M is said to be semi-primitive. A representation of the group G is said to be primitive (semi-primitive) if the module of the representation is primitive (semi-primitive). Certainly, primitive irreducible modules are a basic subject for investigations when we are dealing with induced modules and, naturally, the following question appears: what can be said on the construction of a group if it has a faithful primitive irreducible representation over a field?

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In [4] Harper proved that any not abelian-by-finite finitely generated nilpotent group has an irreducible primitive representation over a not locally finite field. In [11] we proved that if a minimax nilpotent group of class 2 has a faithful irreducible primitive representation over a finitely generated field of characteristic zero then the group is finitely generated. In [5] Harper studied polycyclic groups which have faithful irreducible representations. It is well known (see [14]) that any polycyclic group is finitely generated soluble of finite rank and meets the maximal condition for subgroups (in particular, for normal subgroups). In [10] we showed that in the class of soluble groups of finite rank with the maximal condition for normal subgroups only polycyclic groups may have faithful irreducible primitive representations over a field of characteristic zero. In [7–9] we studied irreducible primitive representations of metabelian groups of finite rank over a field of characteristic zero. In the presented paper we consider the case of a field of positive characteristic.

Let A be a torsion-free abelian group of finite rank acted by a group Γ. Elements of the group A, which have finite orbits under action of the group Γ, form a Γ-invariant subgroup $\Delta_\Gamma(A)$ of the group A.

Let k be a field and I be an ideal of the group algebra kA. We put $I^\dagger = (I+1)\bigcap A$. The ideal I is said to be locally prime if $kB\bigcap I$ is a prime ideal of kB for some finitely generated dense subgroup $B \leq A$. Elements γ of the group Γ such that $I^\gamma \cap kB = I \cap kB$ for some finitely generated dense subgroup $B \leq A$ form a subgroup $S_\Gamma(I) \leq \Gamma$ (see [1]). We also put $Sep_\Gamma(I) = \langle \gamma \in S_\Gamma(I) | Sp(I) \bigcap Sp(I^\gamma) \neq \varnothing \rangle$, where $Sp(I)$ is the prime specter of the ideal I. The subgroup $Sep_\Gamma(I)$ is said to be the separator of the ideal I in the group Γ (see [8]).

An R-module is said to be Chernikov if its additive group is Chernikov.

Proposition 1. *Let $A = \oplus_{i=1}^n A_i$ be a Chernikov $\mathbb{Z}[g]$-module such that $Soc(A_i)$ is a cyclic $\mathbb{Z}[g]$-module for each i. Let k be a field such that $char k \notin \pi(A)$ and let M be a kA-module. Then there is an element $a \in M \backslash \{0\}$ such that $kC_i \cap Ann_{kA}(x) = P_i$ is a maximal ideal of kC_i for any $x \in akA$ and for each $1 \leq i \leq n$, where $C_i/H_i = Soc(A_i/H_i)$ and H_i is a maximal g-invariant subgroup of $Ann^\dagger_{kA}(x) \cap A_i$.*

Proof. We can repeat the argument of the proof of proposition 2.6 of [8] noting out that lemma 2.5 of [8] remains true because the condition $char k \notin \pi(A)$ allows us to apply Maschke's theorem. □

Theorem 1. *Let A be an abelian torsion-free group of finite rank acted by a group of operators Γ of finite torsion-free rank. Let k be a field such that $char k \notin Sp(A)$, let M be a kA-module and let $x \neq 0$ be an element of M such that $Ann^\dagger_{kA}(x)$ is a dense subgroup of A. Then there is an element $y \in M \backslash \{0\}$ such that $Ann^\dagger_{kA}(y)$ has a non- trivial subgroup W such that $Sp(Ann_{kA}(y)) \cap Sp(Ann_{kA}(y)^\gamma) = \varnothing$ for any $\gamma \in \Gamma \backslash N_\Gamma(W)$, where $N_\Gamma(W)$ is the normalizer of the subgroup W in Γ.*

Proof. If $char k = 0$, then the assertion is proved in theorem 3.5 of [8]. Suppose that $char k = p > 0$, then $p \notin Sp(A)$ and hence the Sylow p-subgroup $B/Ann^\dagger_{kA}(x)$ of the quotient group $A/Ann^\dagger_{kA}(x)$ is finite. Then xkB is a finite k-dimensional and hence Artenian kB- module. Therefore, there is an element $z \in xkB$ such that $Ann_{kB}(z)$ is a maximal ideal of kB and, evidently, $Ann_{kB}(x) \leq Ann_{kB}(z)$. As $B/Ann^\dagger_{kA}(x)$ is a p-group and $char k = p$, it is well known that $Ann_{kB}(z)$ is the augmentation ideal of kB and hence, as $Ann_{kB}(z) \leq Ann_{kA}(z)$, we can conclude that $B \leq Ann^\dagger_{kA}(z)$. Since $Ann_{kA}(x) \leq Ann_{kA}(z)$, $B \leq Ann^\dagger_{kA}(z)$ and

$B/Ann^{\dagger}_{kA}(x)$ is the Sylow p-subgroup of the quotient group $A/Ann^{\dagger}_{kA}(x)$, it is easy to show that $p \notin \pi(A/Ann^{\dagger}_{kA}(z))$. Thus, changing x by z we can assume that $char k \notin \pi(A/Ann^{\dagger}_{kA}(x))$. Now, we can repeat the argument of the proof of theorem 3.5 of [8] applying proposition 1 instead of proposition 2.6 of [8]. □

Theorem 2. *Let A be an abelian torsion-free group of finite rank acted by a soluble group Γ of finite torsion-free rank such that $\Delta_{\Gamma}(A) = 1$. Let k be a field such that $char k \notin Sp(A)$ and let M be a kA-module. Suppose that there is an element $x \in M\setminus\{0\}$ such that $Ann_{kA}(x)$ is a non-zero locally prime ideal of kA and $r_0(Sep_{\Gamma}(Ann_{kA}(x))) = r_0(\Gamma)$. Then there is an element $y \in M\setminus\{0\}$ such that $Ann^{\dagger}_{kA}(y)$ contains a non-trivial $Sep_{\Gamma}(Ann_{kA}(y))$-invariant subgroup.*

Proof. We can repeat the arguments of the proof of theorem 3.8 of [8] applying theorem 1 instead of theorem 3.5 of [8]. □

Theorem 3. *Let G be a soluble group of finite torsion-free rank and let A be an abelian normal torsion-free subgroup of G such that $\Delta_G(A) = 1$. Let k be a field such that $char k \notin Sp(A)$ and let M be a kG-module. If the module M is not kA-torsion-free then there is an element $a \in M\setminus\{0\}$ such that*

$$akG = akH \otimes_{kH} kG \qquad \text{and} \qquad r_0(H/C_H(akH)) < r_0(G),$$

where $H = Sep_G(Ann_{kA}(a))$.

Proof. We can repeat the arguments of the proof of theorem 4.2 of [8] applying theorem 2 instead of theorem 3.8 of [8]. □

Lemma 1. *Let A be a torsion-free abelian minimax group acted by a soluble group Γ, let k be a field such that $char k \notin SpA$ and let $0 \neq \alpha \in kA$. Then there is a maximal ideal L of kA such that $|A : L^{\dagger}| < \infty$ and $\alpha^{\gamma} \notin L$ for any $\gamma \in \Gamma$.*

Proof. Evidently, there is a finitely generated subring $R \leq k$ such that $\alpha \in RA$ then, by theorem 2.1 of [6], there is a maximal ideal $I \leq RA$ such that $|RA : I| < \infty$ and $\alpha^{\gamma} \notin I$ for any $\gamma \in \Gamma$. Then RA/I is a finite field and hence $A/I^{\dagger} = \langle g \rangle$ is a finite cyclic group such that $char k \notin \pi(\langle g \rangle)$. Let f be the field of fractions of the domain R then, by Maschke's theorem , $f\langle g \rangle \cong fA/(1 - I^{\dagger})fA$ is a semi-prime ring. Then there are elements $\beta_i, \gamma_i \in fA$, where $1 \leq i \leq n$, such that $\beta_i f\langle g \rangle$ is a maximal ideal of $f\langle g \rangle$, $\prod_{i=1}^{n} \beta_i = 0$ and $\sum_{i=1}^{n} \beta_i\gamma_i = 1$. Evidently, there is a finitely generated subring $S \leq f$ such that $R \leq S$ and $\beta_i, \gamma_i \in SA$. Let J be a maximal ideal of SA such that $J \cap RA = I$. Since $\alpha^{\gamma} \in RA\setminus I$ for any $\gamma \in \Gamma$ and $J \cap RA = I$, we can conclude that $\alpha^{\gamma} \notin J$ for any $\gamma \in \Gamma$. As $\prod_{i=1}^{n} \beta_i = 0$ and the ideal J is maximal, we see that $\beta_i \in J$ for some i. Therefore,

$$\beta_i f\langle g \rangle \cap S\langle g \rangle = \beta_i S\langle g \rangle \leq J/(1 - I^{\dagger})SA.$$

Put $\beta_i f\langle g \rangle = X/(1 - I^{\dagger})fA$ then X is a maximal ideal of fA such that $X \cap SA \leq J$. As $\alpha^{\gamma} \in SA\setminus J$ for any $\gamma \in \Gamma$, we can conclude that $\alpha^{\gamma} \notin X$ for any $\gamma \in \Gamma$. Let L be a maximal ideal of kA such that $X \leq L$ then $L \cap fA = X$ and as $\alpha^{\gamma} \in fA\setminus X$ for any $\gamma \in \Gamma$, we can conclude that $\alpha^{\gamma} \notin L$ for any $\gamma \in \Gamma$. □

Lemma 2. *Let G be a finitely generated metabelian group of finite Prufer rank, let k be a field such that $char k \notin SpG$ and let M be a simple kG-module. Let A be an abelian torsion-free normal subgroup of G such that A is contained in the derived subgroup of G and the quotient group G/A is polycyclic. Then the module M is not kA-torsion-free.*

Proof. By corollary 2.1 of [2], there are a free kA-submodule F of M and a non-zero element $\alpha \in kA$ such that each element of M/F is annihilated by some product $\alpha^{g_1} \dots \alpha^{g_m}$ of conjugates of α by elements of G. By lemma 1, there is a maximal ideal L of kC such $\left|A : L^{\dagger}\right| < \infty$ and L contains no conjugates of α by elements of G. Since $\left|A : L^{\dagger}\right| < \infty$, it is not difficult to show that L contains a non-zero G-invariant ideal I. As the ideal I is G-invariant, it is not difficult to show that MI is a submodule of M and hence, as the module M is simple, either $MI = 0$ or $MI = M$. If $MI = 0$, then the lemma holds. Thus we may assume that $MI = M$ and hence $ML = M$. Then, by lemma 5.2 of [8], each element of F/FL is annihilated by some product $\alpha^{g_1} \dots \alpha^{g_m}$ of conjugates of α by elements of G. As F is a free kA-module $\oplus_i (kA/kAL)_i \simeq F/FL$ and hence some such a product $\alpha^{g_1} \dots \alpha^{g_m}$ is contained in L. But it is a contradiction, because the maximal ideal L contains no conjugates of α by elements of G. □

Theorem 4. *Let G be a finitely generated metabelian group of finite Prufer rank, let k be a field such that $char k \notin Sp(A)$ and let M be an irreducible kG-module such that $C_G(M) = 1$. If the group G is not nilpotent-by-finite, then there are a subgroup $H \leq G$ and an irreducible kH-submodule $U \leq M$ such that $M = U \otimes_{kH} kG$ and $r_0(H/C_H(U)) < r_0(G)$.*

Proof. We can repeat the arguments of the proof of theorem 5.5 of [8] applying lemma 2 instead of lemma 5.4 of [8] and theorem 3 instead of theorem 4.2 of [8]. □

Corollary 1. *Let G be finitely generated group of finite Prufer rank which is an extension of an abelian group A by a cyclic group $< g >$ and such that G is not nilpotent-by-finite. Let k be a field such that $char k \notin Sp(A)$, then every faithful irreducible representation of G over k is induced from an irreducible representation of the group A.*

Proof. It is not difficult to note that the subgroup H in the proof of theorem 3 contains A. As $r_0(H/C_H(U)) < r_0(G)$, it implies that $A = H$. □

The corollary generalizes some results of [8] to the case of fields of nonzero characteristic. As it was proved in [12], an example constructed by Wehrfritz in [13] shows that the restriction on characteristic $p > 0$ of the field k ($p \notin SpG$) is essential.

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Тушев А.В. *Про примітивні зображення скінченно породжених метабелевих груп скінченного рангу над полем ненульової характеристики* // Карпатські матем. публ. — 2014. — Т.6, №2. — С. 389–393.

Розглядаються деякі умови імпримітивності незвідних зображень метабелевої групи G скінченного рангу над полем k. Показано, що у випадку $char k = p > 0$ ці умови суттєво залежать від існування нескінченних p-секцій у групі G.

Ключові слова і фрази: примітивні зображення, метабелеві групи, ранг груп.

Тушев А.В. *О примитивных представлениях конечно-порожденных метабелевых групп конечного ранга над полем ненулевой характеристики* // Карпатские матем. публ. — 2014. — Т.6, №2. — С. 389–393.

Рассматриваются некоторые условия импримитивности неприводимых представлений метабелевой группы G конечного ранга над полем k. Показано, что в случае $char k = p > 0$ эти условия существенно зависят от существования бесконечных p-секций в группе G.

Ключевые слова и фразы: примитивные представления, метабелевы группы, ранг групп.