ISSN 2075-9827
Carpathian Math. Publ. 2014, **6** (2), 337–341
doi:10.15330/cmp.6.2.337-341

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2014, Т.6, №2, С.337–341

PROTASOV I.V., PROTASOVA K.D.

AROUND P-SMALL SUBSETS OF GROUPS

A subset X of a group G is called P-small (almost P-small) if there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that the subsets $(g_nX)_{n\in\omega}$ are pairwise disjoint ($g_nX \cap g_mX$ is finite for all distinct n, m), and weakly P-small if, for every $n \in \omega$, there exist $g_0, \dots, g_n \in G$ such that the subsets $g_0X, \dots, g_nX$ are pairwise disjoint. We generalize these notions and say that X is near P-small if, for every $n \in \omega$, there exist $g_0, \dots, g_n \in G$ such that $g_iX \cap g_jX$ is finite for all distinct $i, j \in \{0, \dots, n\}$. We study the relationships between near P-small subsets and known types of subsets of a group, and the behavior of near P-small subsets under the action of the combinatorial derivation and its inverse mapping.

Key words and phrases: P-small, almost P-small, weakly P-small, near P-small subsets of a group; the combinatorial derivation.

Taras Shevchenko National University, 64/13 Volodymyrska str., 01601, Kyiv, Ukraine
E-mail: i.v.protasov@gmail.com (Protasov I.V.)

INTRODUCTION

Let G be a group with the identity e, $[G]^{<\omega}$ denotes the family of all finite subsets of G. A subset X of G is called

- *large* if $G = FX$ for some $F \in [G]^{<\omega}$;
- *small* if $L \setminus X$ is large for each large subset L of G;
- *P-small* if there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that the subsets $(g_nX)_{n\in\omega}$ are pairwise disjoint;
- *weakly P-small* if, for every $n \in \omega$, there exist $g_0, \dots, g_n \in G$ such that the subsets $g_0X, \dots, g_nX$ are pairwise disjoint;
- *almost P-small* if there exists an injective sequence $(g_n)_{n\in\omega}$ in G such that $g_nX, \dots,$ g_mX is finite for all distinct m, n;
- *near P-small* if, for every $n \in \omega$, there exist $g_0, \dots, g_n \in G$ such that $g_iX \cap g_jX$ is finite for all distinct $i, j \in \{0, \dots, n\}$;
- *thin* if $gA \cap A$ is finite for for every $g \in G\setminus\{e\}$;
- *sparse* if, for every infinite subset Y of G, there exists a non-empty finite subset $F \subset Y$ such that $\bigcap_{g\in F} gX$ is finite.

УДК 519.51
2010 *Mathematics Subject Classification:* 20D30, 20F05.

The terms large and small subsets appeared in [2], P-small subsets were introduced unexplicitly by Prodanov [7] and explicitly in [3, § 2.1]. Every infinite group G can be generated by some small and P-small subset [4] and contains weakly P-small not P-small subset [1]. Each almost P-small subset of a group can be partitioned into two P-small subsets [6]. For thin and sparse subsets see [6]. We recall that a subset X of an amenable group G is *absolute zero* if $\mu(X) = 0$ for each left invariant Banach measure μ on G. It is easy to see that each near P-small subset of an amenable group G is absolute zero. By [6, Corollary 5.1], each absolute zero is small. Hence, each near P-small subset of an amenable group is small. By [6, Theorem 5.3], each countable amenable groups contains a small subset which is not absolute zero. On the other hand, we take a free group F_A in the alphabet A, $\mid A \mid > 1$, choose $a \in A$ and consider a subset X of all group words in A starting with a or a^{-1}. Then X is P-small but X is large, so X is not small.

In this note we introduce near P-small subsets generalizing weakly and almost P-small subsets. All results are exposed in section 2. We study the relationships between modified P-small subsets and thin subsets (Theorems 1 and 2), and the behavior of near P-small subsets under the action of the combinatorial derivation and its inverse mapping (Theorems 3 and 4). The combinatorial derivation, the main tool in this note, was introduced in [9] and studied in [5], [10], [11]. Some necessary auxiliary statements on the combinatorial derivation are arranged in section 1. In section 2 we also show that a near P-small subset needs not to be neither weakly nor almost P-small (Theorem 5) and partition every infinite group G into $\aleph_0$ P-small subsets (Theorem 6). For partition of a group into $\aleph_0$ small subsets see [8].

1 THE COMBINATORIAL DERIVATION

For a group G, $\mathcal{P}_G$ denotes the family of all subsets of G. A mapping $\Delta : \mathcal{P}_G \to \mathcal{P}_G$ defined by $\Delta(A) = \{g \in G : gA \cap A \ \ is\ infinite\}$ is called the *combinatorial derivation*. Clearly, $\Delta(A) = \varnothing$ if A is finite and $e \in \Delta(A)$, $(\Delta(A))^{-1} = \Delta(A)$ for each infinite subset A of G. An infinite subset A is thin if and only if $\Delta(A) = \{e\}$. We denote $Sym_G = \{X \subseteq G : X = X^{-1},\ \ e \in X\}$ and use the following auxiliary statement [10, Lemma 2.6].

Lemma 1. *For every subset $A \in Sym_G$ there exist two thin subsets X, Y such that*

$$\Delta(X \cup Y) = A.$$

Lemma 2. *For every countable group G and every non-empty subset $A \in Sym_G$, there exists a subset X of G such that $\Delta(X) = A$ and $G = XX^{-1}$.*

Proof. We enumerate $G = \{g_n : n \in \omega\}$, put $F_n = \{g_0, \dots, g_n\}$ and write the elements of A in a sequence $(a_n)_{n\in\omega}$ (if A is finite, all but finitely many a_n are equal to e). Then we choose inductively a sequence $(X_n)_{n\in\omega}$ of finite subsets of G of the form

$$X_n = \{y_n, g_n, x_{n0}, a_0x_{n0}, x_{n1}, a_1x_{n1}, \dots, x_{nn}, a_nx_{nn}\}$$

such that, for each $n \in \omega$,

(a) $F_{n+1}X_{n+1} \bigcap F_{n+1}(X_0 \bigcup \dots \bigcup X_n) = \varnothing;$

(b) $F_n\{y_n, g_ny_n\} \bigcap F_n\{(x_{ni}, a_ix_{ni}\} = \varnothing, i \in \{0, \dots, n\};$

(c) $F_n\{x_{ni}, a_i x_{ni}\} \bigcap F_n\{(x_{nj}, a_j x_{nj}\} = \varnothing$, for all distinct $i, j \in \{0, \dots, n\}$.

After ω steps, we denote $X = \bigcup_{n\in\omega} X_n$. By the choice of $(X_n)_{n\in\omega}$, we have $G = XX^{-1}$ and $A \subseteq \Delta(X)$. The conditions (a), (b), (c) guarantee $\Delta(X) \subseteq A$. □

2 RESULTS

Theorem 1. *For every infinite group G, the following statements hold*
(i) *every thin subset of G is almost P-small;*
(ii) *there exists a thin but not weakly P-small subset of G.*

Proof. The statement (i) follows directly from corresponding definitions. To prove (ii), we consider two cases: $\mid G \mid = \aleph_0$ and $\mid G \mid > \aleph_0$. If G is countable, we enumerate $G = \{g_n : n \in \omega\}$, put $F_n = \{g_{0n}, \dots, g_n\}$ and choose inductively a sequence $(x_n)_{n\in\omega}$ in G such that, for each $n \in \omega$, $K_{n+1}\{x_{n+1}, g_{n+1}x_{n+1}\} \bigcap K_n\{x_i, g_i x_i : i \le n\} = \varnothing$. Then the subset $X = \{x_n, g_n x_n : n \in \omega\}$ is thin but $gX \bigcap X \ne \varnothing$ for each $g \in G$ so X is not weakly P-small. If $\mid G \mid > \aleph_0$, we denote $\kappa = \mid G \mid$, enumerate $G\backslash\{e\} = \{g_\alpha : \alpha \in \kappa\}$, put $X_0 = \{e\}$, choose $x_0 \in G$ such that $x_0, g_0 x_0 \notin X_0$ and construct inductively a κ-sequence $\{x_\alpha : \alpha \in \kappa\}$ in G and a κ-sequence $\{X_\alpha : \alpha \in \kappa\}$ of subgroups of G such that, for each $\alpha \in \kappa$,
(a) $x_\alpha, g_\alpha x_\alpha \notin X_\alpha$;
(b) $X_{\alpha+1}$ is a subgroup generating by X_α and $\{g_\alpha, x_\alpha\}$;
(c) $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for each limit ordinal $\alpha \in \kappa$.

After κ steps, we denote $X = \{x_\alpha, g_\alpha x_\alpha : \alpha \in \kappa\}$. By (b), $gX \bigcap X \ne \varnothing$ for each $g \in G$, so G is not weakly P-small. To verify that X is thin, we take an arbitrary $g \in G \setminus \{e\}$ and use (b), (c) to choose $\gamma \in \kappa$ such that $g \in X_{\gamma+1}\backslash X_\gamma$. Since $g(X \cap X_\gamma) \subseteq X_{\gamma+1}\backslash X_\gamma$, by (a) and (b), we have $|g(X \cap X_\gamma) \bigcap X| \le 2$. If $y \in g(X\backslash X_{\gamma+1}) \bigcap X$ then $y \in \{x_\lambda, gx_\lambda, x_\mu, g^{-1}x_\mu\}$ for some $\lambda, \mu \ge \gamma + 1$. Thus, $\mid g(X \setminus X_\gamma) \cap X \mid \le 4$. By (a) and (b), $\mid X \cap (X_{\gamma+1} \setminus X_\gamma) \mid = 2$. Hence, $\mid gX \cap X \mid \le 8$ and X is thin. □

Theorem 2. *For every infinite group G, there exist two thin subsets X, Y of G such that X, Y is not near P-small.*

Proof. We use Lemma 1 to find thin subsets X, Y of G such that $\triangle(X \cup Y) = G$, so $X \cup Y$ is not near P-small. □

By [6, Lemma 2] , the family of all sparse subsets of G is closed under finite unions. Since each thin subset is sparse, Theorem 2 gives a sparse but not near P-small subset $X \cup Y$ of G.

Theorem 3. *There exists a P-small subset X of the group $G = \mathbb{Q}^2$ such that $\Delta(X)$ is not near P-small.*

Proof. We use the Cartesian coordinates in G, put $X = \{(x, y) \in G :\mid x \mid \le y \le \mid x \mid +1\}$ and note that $(0, 2z) + X \cap (0, 2z') + X = \varnothing$ for all distinct $z, z' \in \mathbb{Z}$. Hence, X is P-small.

We observe that $\Delta(X)$ contains the subset $Y = \{(x, y) \in G : - \mid x \mid \le y \le \mid x \mid\}$ and $\Delta(Y) = G$, so $\Delta(X)$ is not near P-small. □

We recall [10] that a family $\mathcal{F}$ of subsets of a group G is Δ-*complete* (∇-*complete*) if $\Delta(X) \in \mathcal{F}$ for each $X \in \mathcal{F}$ ($\Delta(X) \in \mathcal{F}$ implies $X \in \mathcal{F}$). By Theorem 3, the family of all near P-small subsets of a group G need not to be Δ-complete.

Theorem 4. *For every infinite amenable group G, the family of all near P-small subsets of G is ∇-complete.*

Proof. We assume the contrary and choose a subset X of G such that $\Delta(X)$ is near P-small but X is not near P-small. Then there exists the minimal natural number n such that, for any $F \subset G$, $\mid F \mid= n$, there exist distinct $x, y \in F$ such that $xX \bigcap yX$ is infinite. By the minimality of n, there is $H \subset G$, $\mid H \mid= n-1$ such that $xX \bigcap yX$ is finite for all distinct $x, y \in H$. Given any $g \in G \setminus H$, there is $h_g \in H$ such that $gX \bigcap h_g X$ is infinite. If follows that $h_g^{-1} g \in \Delta(X)$, $G \setminus H \subseteq H\Delta(X)$ and $\Delta(X)$ is large. Hence, $\Delta(X)$ is not absolute zero and $\Delta(X)$ could not be near P-small. □

We do not know whether Theorem 4 holds for non-amenable groups.

Theorem 5. *For every countable Abelian group G, there exists a near P-small subset X which is neither weakly nor almost P-small.*

Proof. Suppose we have a sequence $(S_n)_{n\in\omega}$ of finite subsets from Sym_G such that $\mid S_n \mid> n$ and

(a) $S_k \bigcap S_i S_j = \{e\}$ for any $i, j, k \in \omega$, $k \notin \{i, j\}$.

We apply Lemma 2 to find a subset X of G such that $\Delta(X) = G \setminus \bigcup_{n\in\omega} S_n$ and $G = XX^{-1}$. We note that, for any distinct $g_1, g_2 \in G$

(b) if $g_1^{-1} g_2 \in S_n$ then $g_1 X \bigcap g_2 X$ is finite;

(c) if $g_1^{-1} g_2 \notin \bigcup_{n\in\omega} S_n$ then $g_1 X \bigcap g_2 X$ is infinite.

The condition $G = XX^{-1}$ implies that X is not weakly P-small. Since $\mid S_n \mid> n$, by (b), X is near P-small. We assume that X is almost P-small and choose an injective sequence $(x_n)_{n\in\omega}$ in G such that $x_i X \bigcap x_j X$ is finite for all distinct $i, j \in \omega$. Since $x_0 X \bigcap x_1 X$ is finite, by (c), there exists $i \in \omega$ such that $x_0^{-1} x_1 \in S_i$. Analogously, for $n > 1$, there exist $k, j \in \omega$ such that $x_0^{-1} x_n \in S_k$, $x_1^{-1} x_n \in S_j$. We note that $x_0^{-1} x_1 = (x_0^{-1} x_n)(x_n^{-1} x_1)$, $x_1^{-1} x_n = (x_1^{-1} x_0)(x_0^{-1} x_n)$, $x_0^{-1} x_n = (x_0^{-1} x_1)(x_1^{-1} x_n)$. Thus, we have got

$$x_0^{-1} x_1 \in S_i \bigcap S_k S_j, \quad x_1^{-1} x_n \in S_j \bigcap S_i S_k, \quad x_0^{-1} x_n \in S_k \bigcap S_i S_j,$$

and, in view of (a), $i = j = k$. Hence, $(x_0^{-1} x_n) \in S_i$ for any $n > 2$ that is impossible because S_i is finite, so X is not almost P-small. To conclude the proof, it remains to find $(S_n)_{n\in\omega}$ satisfying (a). Since each infinite Abelian group contains either infinite cyclic subgroup, or the Prüffer p-subgroup, or the direct product of $\aleph_0$ finite groups, the late is a routine exercise. □

It should be mentioned that initially above construction appeared to find a weakly P-small but not almost P-small subsets of G.

Theorem 6. *Every infinite group G can be partitioned into $\aleph_0$ P-small subsets.*

Proof. We take an arbitrary countable subgroup H of G, decompose G into right cosets by H and choose some set R of representatives of cosets, so $G = HR$. Then $\{hR : h \in H\}$ is a desired partition of G.

□

REFERENCES

[1] Banakh T., Lyaskovska N. *Weakly P-small not P-small subsets in groups*. Internat. J. Algebra Comput. 2008, **18** (1), 1–6. doi:10.1142/S0218196708004263

[2] Bella A., Malykhin V. *Certain subsets of a group*. Questions Answers Gen. Topology 1999, **17**, 183–187.

[3] Dikranjan D., Marconi U., Moresco R. *Groups with a small set of generators*. Appl. Gen. Topol. 2003, **4** (2), 327–350. doi:10.4995/agt.2003.2037

[4] Dikranjan D., Protasov I. *Every infinite group can be generated by P-small subset*. Appl. Gen. Topol. 2006, **7** (2), 265–268. doi:10.4995/agt.2006.1929

[5] Erde J. *A note on combinatorial derivation*. Preprint http://arxiv.org/abs/1210.7622

[6] Lutsenko Ie., Protasov I.V. *Sparse, thin and other subsets of groups*. Internat. J. Algebra Comput. 2009, **19** (4), 491–510. doi:10.1142/S0218196709005135

[7] Prodanov I. *Some minimal group topologies are precompact*. Math. Ann. 1997, **227** (2), 117–125. doi: 10.1007/BF01350188

[8] Protasov I.V. *Small systems of generators of groups*. Math. Notes 2004, **76** (3–4), 389–394. doi: 10.1023/B:MATN.0000043466.45064.97

[9] Protasov I.V. *The combinatorial derivation*. Appl. Gen. Topol. 2013, **14** (2), 171–178. doi:10.4995/agt.2013.1587

[10] Protasov I.V. *The combinatorial derivation and its inverse mapping*. Cent. Eur. J. Math. 2013, **11** (12), 2176–2181. doi:10.2478/s11533-013-0313-x

[11] Protasov I.V., Slobodianiuk S. *On the subset combinatorics of G-spaces*. Algebra Discrete Math. 2014, **17**, 98–109.

Received 18.06.2014

Протасов І.В., Протасова К.Д. *Навколо P-малих підмножин груп* // Карпатські матем. публ. — 2014. — Т.6, №2. — С. 337–341.

Підмножина X групи G називається P-малою (майже P-малою), якщо існує ін'єктивна послідовність $(g_n)_{n\in\omega}$ в G така, що підмножини $(g_nX)_{n\in\omega}$ попарно не перетинаються ($g_nX \cap g_mX$ скінченні для всіх різних n, m), і слабко P-малі, якщо для кожного $n \in \omega$, існують $g_0, \ldots, g_n \in G$ такі, що підмножини $g_0X, \ldots, g_nX$ попарно не перетинаються. Узагальнено ці поняття: підмножина X називається близько P-малою, якщо для кожного $n \in \omega$ існують $g_0, \ldots, g_n \in G$ такі, що $g_iX \cap g_jX$ скінченні для всіх різних $i, j \in \{0, \ldots, n\}$. Досліджено співвідношення між близько P-малими підмножинами і відомими типами підмножин груп, досліджено поведінку близько P-малих підмножин під дією комбінаторної похідної та її оберненого відображення.

Ключові слова і фрази: P-малі, майже P-малі, слабко P-малі, близько P-малі підмножини групи; комбінаторна похідна.

Протасов И.В., Протасова К.Д. *Вокруг P-малых подмножеств групп* // Карпатские матем. публ. — 2014. — Т.6, №2. — С. 337–341.

Подмножество X группы G называется P-малым (почти P-малым), если существует инъективная последовательность $(g_n)_{n\in\omega}$ в G такая, что подмножества $(g_nX)_{n\in\omega}$ попарно не пересекаются ($g_nX \cap g_mX$ конечны для всех различных n, m), и слабо P-малы, если для каждого $n \in \omega$, существуют $g_0, \ldots, g_n \in G$ такие, что подмножества $g_0X, \ldots, g_nX$ попарно не пересекаются. Обобщены эти понятия: подмножество X называется близко P-малым, если для каждого $n \in \omega$ существуют $g_0, \ldots, g_n \in G$ такие, что $g_iX \cap g_jX$ конечные для всех различных $i, j \in \{0, \ldots, n\}$. Изучены соотношения между близко P-малыми подмножествами и известными типами подмножеств групп, изучено поведение близко P-малых подмножеств под действием комбинаторной производной и ее обратного отображения.

Ключевые слова и фразы: P-малая, почти P-малая, слабо P-малая, близко P-малые подмножества группы; комбинаторная производная.