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GORENSTEIN TILED ORDERS

The aim of this article is to describe exponent matrices of Gorenstein tiled orders. The necessary and sufficient condition for possibility such to construct a Gorenstein tiled order which has given orders over some discrete valuation ring (the unique for the whole main diagonal) on the main diagonal and its permutation be a product of correspond cycles with given cyclic Gorenstein tiled order is considered.

Key words and phrases: Gorenstein tiled order, exponent matrix, Kirichenko's permutation.

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INTRODUCTION

The notion of tiled orders was introduced at first in [1]. Gorenstein tiled orders appeared in the first time in the article [2] and a convenient criterion for tiled order to be a Gorenstein order is also given there. It is shown in [3] that injective dimension of a Gorenstein tiled order is equal to 1. More then this tiled orders with injective dimension 1 are Gorenstein. The description of cyclic Gorenstein tiled orders has been done at [4].

The aim of this article is to describe exponent matrices of Gorenstein tiled orders. The necessary and sufficient condition for possibility such to construct a Gorenstein tiled order which has given orders over some discrete valuation ring (the unique for the whole main diagonal) on the main diagonal and its permutation be a product of correspond cycles with given cyclic Gorenstein tiled order is considered.

The necessary information about tiled orders and exponent matrices can be found in [5, 6].

1 TILED ORDERS OVER DISCRETE VALUATION RINGS

Recall [7] that a *semimaximal ring* is a semiperfect semiprime right Noetherian ring A such that for each primitive idempotent $e \in A$ the ring eAe is a discrete valuation ring (not necessarily commutative).

Denote by $M_n(B)$ the ring of all $n \times n$ matrices over a ring B.

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Theorem 1 (see [7]). *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form*

$$\Lambda = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix}, \tag{1}$$

where $n \geq 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element π, and α_{ij} are integers such that $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$, $\alpha_{ii} = 0$ for all i, j, k.

The ring $\mathcal{O}$ is embedded into its classical division ring of fractions $\mathcal{D}$, and (1) is the set of all matrices $(a_{ij}) \in M_n(\mathcal{D})$ such that

$$a_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}\Lambda e_{jj},$$

where $e_{11}, \dots, e_{nn}$ are the matrix units of $M_n(\mathcal{D})$. It is clear that $Q = M_n(\mathcal{D})$ is the classical ring of fractions of Λ. Obviously, the ring A is right and left Noetherian.

Definition 1.1. *A module M is* distributive *if its lattice of submodules is distributive, i.e.,*

$$K \cap (L + N) = K \cap L + K \cap N$$

for all submodules K, L, and N.

Clearly, any submodule and any factormodule of a distributive module are distributive modules. A *semidistributive module* is a direct sum of distributive modules. A ring A is *right* (*left*) *semidistributive* if it is semidistributive as the right (left) module over itself. A ring A is *semidistributive* if it is both left and right semidistributive (see [8]).

Theorem 2 (see [9]). *The following conditions for a semiperfect semiprime right Noetherian ring A are equivalent:*

- *A is semidistributive;*
- *A is a direct product of a semisimple artinian ring and a semimaximal ring.*

By a *tiled order* over a discrete valuation ring, we mean a Noetherian prime semiperfect semidistributive ring Λ with nonzero Jacobson radical. In this case, $\mathcal{O} = e\Lambda e$ is a discrete valuation ring with a primitive idempotent $e \in \Lambda$.

Definition 1.2. *An integer matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called*

- *an* exponent matrix *if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ and $\alpha_{ii} = 0$ for all i, j, k;*
- *a* reduced exponent matrix *if $\alpha_{ij} + \alpha_{ji} > 0$ for all i, j, $i \neq j$.*

We use the following notation $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of the ring Λ, i.e.

$$\Lambda = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O},$$

in which e_{ij} are the matrix units. If a tiled order is *reduced*, i.e., $\Lambda/R(\Lambda)$ is the direct product of division rings, then $\alpha_{ij}+\alpha_{ji}>0$ if $i\neq j$, i.e., $\mathcal{E}(\Lambda)$ is reduced.

We denote by $\mathcal{M}(\Lambda)$ the poset (ordered by inclusion) of all projective right Λ-modules that are contained in a fixed simple Q-module U. All simple Q-modules are isomorphic, so we can choice one of them. Note that the partially ordered sets $\mathcal{M}_l(\Lambda)$ and $\mathcal{M}_r(\Lambda)$ corresponding to the left and the right modules are anti-isomorphic.

The set $\mathcal{M}(\Lambda)$ is completely determined by the exponent matrix $\mathcal{E}(\Lambda)=(\alpha_{ij})$. Namely, if Λ is reduced, then

$$\mathcal{M}(\Lambda)=\{p_i^z \mid i=1,\dots n, \text{and } z\in\mathbb{Z}\},$$

where

$$p_i^z\le p_j^{z'} \iff \begin{cases} z-z'\ge\alpha_{ij} & \text{if } \mathcal{M}(\Lambda)=\mathcal{M}_l(\Lambda),\\ z-z'\ge\alpha_{ji} & \text{if } \mathcal{M}(\Lambda)=\mathcal{M}_r(\Lambda).\end{cases}$$

Obviously, $\mathcal{M}(\Lambda)$ is an infinite periodic set.

Let P be an arbitrary poset. A subset of P is called a chain if any two of its elements are related. A subset of P is called a antichain if no two distinct elements of the subset are related.

Definition 1.3. *A right (resp. left) Λ-module M (resp. N) is called a right (resp. left) Λ-lattice if M (resp. N) is a finitely generated free $\mathcal{O}$-module.*

Given a tiled order Λ we denote $Lat_r(\Lambda)$ (resp. $Lat_l(\Lambda)$) the category of right (resp. left) Λ-lattices. We denote by $S_r(\Lambda)$ (resp. $S_l(\Lambda)$) the partially ordered by inclusion set, formed by all Λ-lattices contained in a fixed simple $M_n(\mathcal{D})$-module W (resp. in a left simple $M_n(\mathcal{D})$-module V). Such Λ-lattices are called irreducible.

Let $\Lambda=\{\mathcal{O},\mathcal{E}(\Lambda)\}$ be a tiled order, W (resp. V) is a simple right (resp. left) $M_n(\mathcal{D})$-module with $\mathcal{D}$-basis $e_1,\dots,e_n$ such that $e_ie_{jk}=\delta_{ij}e_k$ ($e_{ij}e_k=\delta_{jk}e_i$).

Then any right (resp. left) irreducible Λ-lattice M (resp. N), lying in W (resp. in V) is a Λ-module with $\mathcal{O}$-basis $(\pi^{\alpha_1}e_1,\dots,\pi^{\alpha_n}e_n)$, while

$$\begin{cases} \alpha_i+\alpha_{ij}\ge\alpha_j, & \text{for the right case;}\\ \alpha_{ij}+\alpha_j\ge\alpha_i, & \text{for the left case.}\end{cases} \tag{2}$$

Thus, irreducible Λ-lattices M can be identified with integer-valued vector $(\alpha_1,\dots,\alpha_n)$ satisfying (2). We shall write $\mathcal{E}(M)=(\alpha_1,\dots,\alpha_n)$ or $M=(\alpha_1,\dots,\alpha_n)$.

The order relation on the set of such vectors and the operations on them corresponding to sum and intersection of irreducible lattices are obvious.

Remark 1.1. *Obviously, irreducible Λ-lattices $M_1=(\alpha_1,\dots,\alpha_n)$ and $M_2=(\beta_1,\dots,\beta_n)$ are isomorphic if and only if $\alpha_i=\beta_i+z$ for $i=1,\dots,n$ and $z\in\mathbb{Z}$.*

For each right (left) Λ-lattice M (N) it is defined a left (right) Λ-lattice $M^*=\mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O}_{\mathcal{O}})$ ($N^*=\mathrm{Hom}_{\mathcal{O}}(N,{}_{\mathcal{O}}\mathcal{O})$) such that $M^{**}=M$ ($N^{**}=N$) (see. [7], §3). For an arbitrary $\varphi\in M^*$ and $a\in\Lambda$ the multiplication $a\varphi$ is defined with the formula $(a\varphi)(m)=\varphi(ma)$ where $m\in M$. For every homomorphism $\psi\colon M\to N$ of right lattices it is defined a conjugated homomorphism $\psi^*\colon N^*\to M^*$ of left lattices with the rule $(\psi^*f)(m)=f(\psi m)$.

It is especially easy can be defined the duality for an irreducible Λ-lattice. Let $M=(\alpha_1,\dots,\alpha_n)$ be an irreducible Λ-lattice. Then $M^*=(-\alpha_1,\dots,-\alpha_n)^T$ is an irresucible left Λ-lattice and for $M\subset N$ we have that $N^*\subset M^*$.

Let Λ be a reduced tiled order with the exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$. Denote $P_i = (\alpha_{i1}, \dots, \alpha_{in})$ and $Q_k = (\alpha_{1k}, \dots, \alpha_{ik})^T$, where T is a transpose operation.

Definition 1.4. *A Λ-lattice M is called relatively injective if $M = P^*$ where P is a projective Λ-module. Projective Λ-lattice P is called bijective Λ-lattice if P is a projective left Λ-lattice.*

Example 1. *Let Λ be a reduced tiled order with exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$, where*

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 2 & 2 & 2 & 0 & 0 & 1 \\ 3 & 3 & 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Here $P_1^* \simeq Q_5$, $P_2^* \simeq Q_6$, $P_4^* \simeq Q_3$, $P_5^* \simeq Q_2$, $P_6^* \simeq Q_1$ but $P_3^* \not\simeq Q_4$.

2 GORENSTEIN TILED ORDERS AND THEIR EXPONENT MATRICES

Definition 2.1. *A tiled order Λ is called a Gorenstein tiled oreder if Λ is a bijective Λ-lattice i.e. if Λ^* is a projective left Λ-lattice.*

Further we will call a Gorenstein tiled order just as Gorenstein order.

Theorem 3 (see. [2, lemma 3.2]). *The following conditions are equivalent for a reduced tiled order $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda) = (\alpha_{pq})\}$:*

(a) an order Λ is Gorenstein

(b) there exists a permutation $\sigma\colon i \to \sigma(i)$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i = 1, \dots, n$; $k = 1, \dots, n$.

Denote with $M_n(\mathbb{Z})$ the ring of all square $n \times n$-matrices over integers ring $\mathbb{Z}$. Let $\mathcal{E} \in M_n(\mathbb{Z})$.

Definition 2.2. *Call a matrix $\mathcal{E} = (\alpha_{ij})$ as exponent matrix if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, \dots, n$ and $\alpha_{ii} = 0$ for $i = 1, \dots, n$. These inequalities are called ring inequalities. An exponent matrix $\mathcal{E}$ is called a reduced exponent matrix if $\alpha_{ij} + \alpha_{ji} > 0$ for all $i, j = 1, \dots, n$.*

Definition 2.3. *Two exponent matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are called* equivalent *if one can be obtained from another with a composition of transformations of the following two types:*

(1) subtracting some integer from all elements of i-th line with simultaneous adding this number to all elements of the i-th column;

(2) simultaneous permuting of two lines and two columns with the same numbers.

Definition 2.4. *The following exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called a Gorenstein matrix if there exists a permutation σ of a set $\{1, 2, \dots, n\}$ such that*

$$\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)} \qquad \text{for all } i, k.$$

The permutation σ is denoted with $\sigma(\mathcal{E})$. If a matrix $\mathcal{E}$ is a reduced Gorenstein exponent matrix then $\sigma(\mathcal{E})$ does not have fixed points.

Proposition 2.1 ([5]). *Let $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ be exponent matrices and Θ is obtained from $\mathcal{E}$ with composition of permutations of the type (1). If $\mathcal{E}$ is a reduced exponent Gorenstein matrix with correspond permutation $\sigma(\mathcal{E})$ then Θ is also reduces exponent Gorenstein matrix with a permutation $\sigma(\Theta) = \sigma(\mathcal{E})$.*

Proposition 2.2 ([5]). *If $\mathcal{E}$ is a Gorenstein matrix then any second type transformation Θ it leaves to be a Gorenstein matrix and for the new correspond permutation π we have $\pi = \tau^{-1}\sigma\tau$ i.e. $\sigma(\Theta) = \tau^{-1}\sigma(\mathcal{E})\tau$.*

Remark 2.1. *A reduced tiled order Λ is Gorenstein if and only if when its exponent matrix $\mathcal{E}(\Lambda)$ is a Gorenstein matrix.*

Definition 2.5. *A reduced Gorenstein exponent matrix $\mathcal{E}$ is called cyclic if $\sigma(\mathcal{E})$ is a cycle.*

Lemma 2.1 ([10]). *Let Λ be a reduced Gorenstein tiled order with an exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$ and Kirichenko permutation σ. Then $\alpha_{\sigma(i)\sigma(j)} + \alpha_{\sigma(j)\sigma(k)} - \alpha_{\sigma(i)\sigma(k)} = \alpha_{ij} + \alpha_{jk} - \alpha_{ik}$.*

Proof. After adding $\alpha_{\sigma(i)\sigma(j)}$ to both sides of the equality $\alpha_{i\sigma(i)} = \alpha_{ij} + \alpha_{j\sigma(i)}$ we obtain

$$\alpha_{i\sigma(i)} + \alpha_{\sigma(i)\sigma(j)} = \alpha_{ij} + \alpha_{j\sigma(i)} + \alpha_{\sigma(i)\sigma(j)} \quad \text{or} \quad \alpha_{i\sigma(i)} + \alpha_{\sigma(i)\sigma(j)} = \alpha_{ij} + \alpha_{j\sigma(j)}.$$

Represent $\alpha_{i\sigma(i)}$ and $\alpha_{j\sigma(j)}$ as sum of two summands as follows $\alpha_{ik} + \alpha_{k\sigma(i)} + \alpha_{\sigma(i)\sigma(j)} = \alpha_{ij} + \alpha_{jk} + \alpha_{k\sigma(j)}$. Whence obtain $\alpha_{k\sigma(i)} + \alpha_{\sigma(i)\sigma(j)} - \alpha_{k\sigma(j)} = \alpha_{ij} + \alpha_{jk} - \alpha_{ik}$. Further we get

$$(\alpha_{k\sigma(i)} + \alpha_{k\sigma(k)}) + \alpha_{\sigma(i)\sigma(j)} - (\alpha_{k\sigma(j)} + \alpha_{k\sigma(k)}) = \alpha_{ij} + \alpha_{jk} - \alpha_{ik}.$$

Again represent $\alpha_{k\sigma(k)}$ as two summands as follows

$$(\alpha_{k\sigma(i)} + \alpha_{k\sigma(j)} + \alpha_{\sigma(j)\sigma(k)}) + \alpha_{\sigma(i)\sigma(j)} - (\alpha_{k\sigma(j)} + \alpha_{k\sigma(i)} + \alpha_{\sigma(i)\sigma(k)}) = \alpha_{ij} + \alpha_{jk} - \alpha_{ik}.$$

We have $\alpha_{\sigma(i)\sigma(j)} + \alpha_{\sigma(j)\sigma(k)} - \alpha_{\sigma(i)\sigma(k)} = \alpha_{ij} + \alpha_{jk} - \alpha_{ik}$. □

3 CYCLIC GORENSTEIN ORDERS

Lemma 3.1 ([4]). *Let Λ be a reduced cyclic Gorenstein tiled order with an Exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$ and Kirichenko permutation $\sigma = (12 \ldots n)$. If $\alpha_{i1} = 0$ for all $i = 1, \ldots, n$ then $\alpha_{1j} = \alpha_{1,n+2-j}$ for $1 < j \leq n$.*

Proof. The lemma 2.1 gives that $\alpha_{ij} + \alpha_{ji} = \alpha_{\sigma^m(i)}\alpha_{\sigma^m(j)} + \alpha_{\sigma^m(j)}\alpha_{\sigma^m(i)}$ for an arbitrary natural m. For $i = 1$ we have

$$\alpha_{1j} + \alpha_{j1} = \alpha_{\sigma^m(1)}\alpha_{\sigma^m(j)} + \alpha_{\sigma^m(j)}\alpha_{\sigma^m(1)}.$$

As σ is a cyclic permutation then there exists an integer m such that $\sigma^m(j) = 1$. The equality $\sigma^m(j) \equiv j + m \pmod{n}$ yields $j + m = n + 1$. So $m = n + 1 - j$. This gives that $\sigma^m(1) = 1 + m = n + 2 - j$ and $\alpha_{1j} + \alpha_{j1} = \alpha_{n+2-j,1} + \alpha_{1,n+2-j}$. Now the lemma conditions yields $\alpha_{j1} = \alpha_{n+2-j,1} = 0$ whence $\alpha_{1j} = \alpha_{1,n+2-j}$. □

Proposition 3.1 ([4]). *Let $\alpha_{12}, \alpha_{13}, \ldots, \alpha_{1n}$ be an arbitrary set of real numbers. There exists the unique matrix (α_{ij}) such that these given numbers are elements of the first line of this matrix and equalities $\alpha_{kk} = \alpha_{k1} = 0$ and $\alpha_{ik} + \alpha_{ki+1} = \alpha_{ii+1}$ and $k = 1, \ldots, n; i = 1, \ldots, n-1$ hold.*

Proof. Let $\alpha_{k1} = 0$. Obtain other elements α_{km} of the matrix (α_{ij}) from the linear equations system $\alpha_{ik} + \alpha_{ki+1} = \alpha_{ii+1} \quad k = 1, \ldots, n; i = 1, \ldots, n-1$.

Really $\alpha_{kk+1} = \alpha_{k1} + \alpha_{1k+1} = \alpha_{1k+1}$ for $k < n$. The equality $\alpha_{1k} + \alpha_{k2} = \alpha_{12}$ gives that $\alpha_{k2} = \alpha_{12} - \alpha_{1k}$. As $\alpha_{k2} + \alpha_{2k+1} = \alpha_{kk+1} = \alpha_{1k+1}$ then either $\alpha_{2k+1} = \alpha_{1k+1} - \alpha_{k2} = \alpha_{1k+1} + \alpha_{1k} - \alpha_{12}$ or $\alpha_{2q} = \alpha_{1q} + \alpha_{1q-1} - \alpha_{12}$ for $q > 1$.

Further $\alpha_{2k} + \alpha_{k3} = \alpha_{23} = \alpha_{13}$ whence $\alpha_{k3} = \alpha_{13} - \alpha_{2k} = \alpha_{13} + \alpha_{12} - \alpha_{1k} - \alpha_{1k-1}$ for $k > 1$. From the equality $\alpha_{k3} + \alpha_{3k+1} = \alpha_{kk+1} = \alpha_{1k+1}$ find the α_{3k+1} as $\alpha_{3k+1} = \alpha_{1k+1} - \alpha_{k3} = \alpha_{1k+1} + \alpha_{1k} + \alpha_{1k-1} - \alpha_{13} - \alpha_{12}$ which is the same as $\alpha_{3q} = \alpha_{1q} + \alpha_{1q-1} + \alpha_{1q-2} - \alpha_{13} - \alpha_{12}$ for $q > 2$.

Assume that for $l < n$ elements of the l-th column α_{kl} for $k > l-2$ can be represented through elements of the first line as follows

$$\alpha_{kl} = \sum_{j=2}^{l} \alpha_{1j} - \sum_{j=0}^{l-2} \alpha_{1k-j}\,, k > l-2,$$

and elements of the l-th line α_{lq} for $q > l-1$ are represented through elements of the first line as follows

$$\alpha_{lq} = \sum_{j=0}^{l-1} \alpha_{1q-j} - \sum_{j=2}^{l} \alpha_{1j}\,, q > l-1.$$

Find expressions for elements $l+1$-th column and $l+1$-th line through elements of the first line.

The equality $\alpha_{lk} + \alpha_{k,l+1} = \alpha_{l,l+1} = \alpha_{1,l+1}$ yields

$$\begin{aligned}\alpha_{k,l+1} &= \alpha_{1,l+1} - \alpha_{lk} = \alpha_{1,l+1} - \left(\sum_{j=0}^{l-1} \alpha_{1k-j} - \sum_{j=2}^{l} \alpha_{1j}\right) \\ &= \sum_{j=2}^{l+1} \alpha_{1j} - \sum_{j=0}^{l-1} \alpha_{1,k-j} = \sum_{j=2}^{l+1} \alpha_{1j} - \sum_{j=0}^{(l+1)-2} \alpha_{1,k-j}, k > l-1 \quad \text{or} \quad k > (l+1)-2.\end{aligned}$$

Further have $\alpha_{k,l+1} + \alpha_{l+1,k+1} = \alpha_{k,k+1} = \alpha_{1,k+1}$. Whence,

$$\begin{aligned}\alpha_{l+1,k+1} &= \alpha_{1,k+1} - \alpha_{k,l+1} = \alpha_{1,k+1} - \left(\sum_{j=2}^{l+1} \alpha_{1j} - \sum_{j=0}^{(l+1)-2} \alpha_{1,k-j}\right) \\ &= \sum_{j=0}^{(l+1)-1} \alpha_{1,(k+1)-j} - \sum_{j=2}^{l+1} \alpha_{1j}, \quad k+1 > (l+1)-1 \quad \text{or} \\ \alpha_{l+1,q} &= \sum_{j=0}^{(l+1)-1} \alpha_{1,q-j} - \sum_{j=2}^{l+1} \alpha_{1j}, \quad q > (l+1)-1.\end{aligned}$$

Whence with the use of induction by the number of a line and column obtain find the unknown elements of the matrix (α_{ij}). They are expressed with the elements of the first line with the following formulas

$$\alpha_{km} = \sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1k-j} \text{ for } k > m-2; \tag{3}$$

$$\alpha_{km} = \sum_{j=0}^{k-1} \alpha_{1m-j} - \sum_{j=2}^{k} \alpha_{1j} \text{ for } m > k-1. \tag{4}$$

During this process we have considered all the equations $\alpha_{ik} + \alpha_{ki+1} = \alpha_{ii+1}$, $k = \overline{1,n}; i = \overline{1,n-1}$. It is obvious that the solution of the system of linear equations satisfies the equations of it. More the equation (3) yields

$$\alpha_{kk} = \sum_{j=2}^{k} \alpha_{1j} - \sum_{j=0}^{k-2} \alpha_{1,k-j} = 0\,, k = 2, \ldots, n,$$

and from the equation (4) we get

$$\alpha_{kk} = \sum_{j=0}^{k-1} \alpha_{1,k-j} - \sum_{j=2}^{k} \alpha_{1j} = 0\,, k = 2, \ldots, n.$$

□

The formula (3) yields

$$\begin{aligned} \alpha_{m-1,m} &= \sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1,m-1-j} = \sum_{t=0}^{m-2} \alpha_{1,m-t} - \sum_{t=2}^{m-1} \alpha_{1t} - \alpha_{11} \\ &= \sum_{t=0}^{(m-1)-1} \alpha_{1,m-t} - \sum_{t=2}^{m-1} \alpha_{1t}, \quad m = \overline{3,n}. \end{aligned}$$

The last equality is the expression for $\alpha_{m-1,m}$ with the use of the formula (4).

This means that the former linear equations system is consistent.

Whence we have got the matrix (α_{ij}) whose elements can be calculated with formulas

$$\alpha_{km} = \begin{cases} 0, & \text{if } m = 1, \\ \sum\limits_{j=2}^{m} \alpha_{1j} - \sum\limits_{j=0}^{m-2} \alpha_{1k-j}, & \text{if } k \geq m > 1, \\ \sum\limits_{j=0}^{k-1} \alpha_{1m-j} - \sum\limits_{j=2}^{k} \alpha_{1j}, & \text{if } 1 < k < m, \\ \alpha_{1m}, & \text{if } k = 1. \end{cases} \tag{5}$$

Corollary 3.1 ([4]). *Let (α_{ij}) be a matrix whose elements are calculated with (5) and $\alpha_{1j} = \alpha_{1,n+2-j}$ for $j = 2, \ldots, n$. Then $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$ for all $i, j = 1, \ldots, n$ where $\sigma = (12\ldots n)$.*

Proof. As $\alpha_{1j} = \alpha_{1,n+2-j}$ then

$$\alpha_{nm} = \sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1n-j} = \sum_{j=2}^{m} \alpha_{1j} - \sum_{i=2}^{m} \alpha_{1,n+2-i} = 0$$

and $\alpha_{nm} + \alpha_{m1} = \alpha_{n1} = 0$ for all m. Whence elements of the matrix (α_{ij}) satisfy the condition $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$ for all $i, j = 1, \ldots, n$ where $\sigma = (12\ldots n)$. □

Proposition 3.2 ([4]). *Let (α_{ij}) be a matrix whose elements satisfy the equality (5). Then for every triple or pairwise different numbers i, j, k there exist p, q such that $\alpha_{ij} + \alpha_{jk} - \alpha_{ik} = \alpha_{pq}$.*

Proof. Rewrite the equalities (3)–(4) as follows

$$\alpha_{km} = \sum_{j=2}^{m} \alpha_{1j} - \sum_{j=0}^{m-2} \alpha_{1k-j} = \sum_{t=2}^{m} \alpha_{1t} - \sum_{t=k-m+2}^{k} \alpha_{1t} \quad \text{for } k \geq m > 1,$$
$$\alpha_{km} = \sum_{j=0}^{k-1} \alpha_{1m-j} - \sum_{j=2}^{k} \alpha_{1j} = \sum_{t=m-k+1}^{m} \alpha_{1t} - \sum_{t=2}^{k} \alpha_{1t} \quad \text{for } m \geq k > 1.$$

Denote $S_{ijk} = \alpha_{ij} + \alpha_{jk} - \alpha_{ik}$.

It is easy to check that if the numbers i, j, k are pairwise different then the equality

$$\alpha_{pq} = \begin{cases} \alpha_{i-k+1,j-k+1}, & \text{if } \min(i,j,k) = k, \\ \alpha_{k-j,i-j+1}, & \text{if } \min(i,j,k) = j, \\ \alpha_{j-i,k-i}, & \text{if } \min(i,j,k) = i \end{cases}$$

holds. In the case when at least two indices coincide one obtain

$$S_{ijj} = \alpha_{ij} + \alpha_{jj} - \alpha_{ij} = 0,$$
$$S_{iik} = \alpha_{ii} + \alpha_{ik} - \alpha_{ik} = 0,$$
$$S_{iji} = \alpha_{ij} + \alpha_{ji} - \alpha_{ii} = \alpha_{ij} + \alpha_{ji} = \alpha_{1,|i-j|+1} > 0.$$

□

Corollary 3.2 ([4]). *A matrix (α_{ij}) with non negative elememts which satisfy the equalities* (5) *and $\alpha_{1j} = \alpha_{1,n+2-j}$ for all $2 \leq j \leq n$ is a Gorenstein reduced exponent matrix of a cyclic Gorenstein order with the Kirichenko permutation $\sigma = (1\ 2\ \cdots\ n)$.*

Proposition 3.3 ([4]). *The exponent matrix (α_{ij}) of the cyclic reduced Gorenstein tiled order Λ with the Kirichenko permutation $\sigma = (1\ 2\ \cdots\ n)$ such that $\alpha_{i1} = 0$ for $i = \overline{1,n}$ is symmetrical in the main diagonal.*

Proof. It is clear that the matrix (α_{ij}) is symmetrical in the second diagonal if $\alpha_{ij} = \alpha_{n+1-j,n+1-i}$. After the direct checking one may make sure that $\alpha_{km} - \alpha_{n+1-m,n+1-k} = 0$ for all k, m. □

Corollaries 3.1, 3.2 and the proposition 3.3 yield the following theorem which gives the whole description of the reduced cyclic Gorenstein tiled orders.

Theorem 4 ([4]). *Any cyclic reduced Gorenstein tiled order is isomorphic to a reduced order Λ with Kirichenko permutation $\sigma = (1\ 2\ \cdots\ n)$ whose exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$ has the following properties:*

1) all elements of the matrix (α_{ij}) can be expressed with the formulas (5) through $\left[\frac{n}{2}\right]$ natural parameters $\alpha_{12}, \ldots, \alpha_{1[\frac{n}{2}]+1}$;

2) $\alpha_{1j} = \alpha_{1,n+2-j}$ for all j;

3) the matrix (α_{ij}) is symmetrical in the second diagonal.

Conversely each non negative integer valued matrix (α_{ij}) which satisfies the properties 1-3 from above such that $\alpha_{ij} + \alpha_{ji} > 0$ for $(i \neq j)$ is an exponent matrix of some cyclic reduced Gorenstein tiled order with the Kirichenko permutation $\sigma = (1\ 2\ \cdots\ n)$.

The elements of the exponent matrix (α_{ij}) of the a cyclic reduced Gorenstein tiled order Λ with Kirichenko permutation $\sigma = (1\ 2\ \cdots\ n)$ satisfy the equality

$$\frac{\sum\limits_{i,j=1}^{n} \alpha_{ij}}{n^2} = \frac{\sum\limits_{i,j=1}^{n} \left(\alpha_{ij} + \alpha_{j\sigma(i)}\right)}{2n^2} = \frac{\sum\limits_{i,j=1}^{n} \alpha_{i\sigma(i)}}{2n^2} = \frac{\sum\limits_{i=1}^{n} \alpha_{i\sigma(i)}}{2n}.$$

The value $t = \frac{\sum\limits_{i=1}^{|\langle\sigma\rangle|} \alpha_{i\sigma(i)}}{|\langle\sigma\rangle|}$ is much important in the studying of cyclic Gorenstein orders.

Theorem 5 ([11])**.** *Let Λ be a reduced cyclic Gorenstein tiled order with exponent matrix $\mathcal{E}(\Lambda)$ and Kirichenko permutation σ. An order Λ is isomorphic to an order Λ' whose exponent matrix is a linear combination of powers or a matrix of the Kirichenko permutation σ if and only if $\frac{1}{|\langle\sigma\rangle|} \cdot \sum\limits_{i} \alpha_{i\sigma(i)} = t$ is a natural number.*

Proof. We will assume that $\sigma = (1\ 2\ \cdots\ n)$.

All cyclic Gorenstein tiled orders are described at the theorem 4. Elements of an exponent matrix of such order satisfy the equalities

$$\alpha_{km} = \begin{cases} 0, & \text{if } m = 1, \\ \sum\limits_{j=2}^{m} \alpha_{1j} - \sum\limits_{j=0}^{m-2} \alpha_{1k-j} & \text{if } k \geq m > 1, \\ \sum\limits_{j=0}^{k-1} \alpha_{1m-j} - \sum\limits_{j=2}^{k} \alpha_{1j} & \text{if } 1 < k < m, \\ \alpha_{1m} & \text{if } k = 1. \end{cases} \tag{6}$$

Let $\frac{1}{|\langle\sigma\rangle|} \cdot \sum\limits_{i} \alpha_{i\sigma(i)} = t$ be a natural number. Show that with the transformation of the first type the exponent matrix of the order Λ can be transformed to an exponent matrix with the necessary property. Let us do the transformations of the first type. Let us add the number x_k to all the elements of the k-th line and lets subtract if from all the elements of the k-th column. Then the equality $\alpha'_{km} = \alpha_{km} + x_k - x_m$ will appear. To find x_k we will need $\alpha'_{k\sigma(k)} = t$ for all k. Consider the linear equations system

$$\begin{aligned} &x_1 - x_2 = t - \alpha_{12}, \\ &x_2 - x_3 = t - \alpha_{23}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &x_{n-1} - x_n = t - \alpha_{n-1n}, \\ &x_n - x_1 = t - \alpha_{n1}. \end{aligned}$$

It has a solution

$$\begin{aligned} &x_2 = \alpha_{12} - t + x_1, \\ &x_3 = \alpha_{12} + \alpha_{23} - 2t + x_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &x_k = \alpha_{12} + \alpha_{23} + \cdots + \alpha_{k-1k} - (k-1)t + x_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &x_n = \alpha_{12} + \alpha_{23} + \cdots + \alpha_{n-1n} - (n-1)t + x_1. \end{aligned} \tag{7}$$

That is why the exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$ can be reduced with the transformations of the first type to the matrix$(\alpha'_{ij}) = \mathcal{E}(\Lambda')$ such that $\alpha'_{k\sigma(k)} = t$ for all k.

Now show that $\alpha'_{ij} = \alpha'_{\sigma(i)\sigma(j)}$ for alli, j. Really equalities $\alpha'_{i\sigma(i)} = t = \alpha'_{ij} + \alpha'_{j\sigma(i)}$ i $\alpha'_{j\sigma(j)} = t = \alpha'_{j\sigma(i)} + \alpha'_{\sigma(i)\sigma(j)}$ yield that $\alpha'_{ij} = \alpha'_{\sigma(i)\sigma(j)}$ for all i, j.

That is why $\mathcal{E}(\Lambda') = \sum\limits_{j=1}^{n} \alpha'_{1j} P_{\sigma}^{j-1}$ where $P_{\sigma} = \sum\limits_{j=1}^{n} e_{j\sigma(j)}$ and e_{ij} are matrix units.

The converse statement is obvious. □

4 GORENSTEIN ORDERS WITH PAIRWISE ISOMORPHIC SIMPLE CYCLES

The matrix transformations of two types change a sum of elements of a matrix. It is easy to see the following proposition.

Proposition 4.1 ([11]). *Let Λ and Λ' be two isomorphic reduced tiled order with correspond exponent matrices $\mathcal{E}(\Lambda) = (\alpha_{ij})$ and $\mathcal{E}(\Lambda') = (\alpha'_{ij})$ whose Kirichenko permutations are σ and σ' correspondingly. Then $\sum\limits_{i,j}^{n} \alpha_{ij} = \sum\limits_{i,j}^{n} \alpha'_{ij}$ and $\sum\limits_{i=1}^{n} \alpha_{i\sigma(i)} = \sum\limits_{i=1}^{n} \alpha'_{i\sigma'(i)}$.*

Lemma 4.1 ([11]). *Let Λ be a reduced Gorenstein tiled order with correspond exponent matrix $\mathcal{E}(\Lambda)$ and Kirichenko permutation σ and let $\sigma = \sigma_1 \cdot \sigma_2$ be a decomposition of σ into the product of two disjoint cycles. Then*

$$\frac{\sum\limits_{i} \alpha_{i\sigma_1(i)}}{| < \sigma_1 > |} = \frac{\sum\limits_{k} \alpha_{k\sigma_2(k)}}{| < \sigma_2 > |}.$$

Proof. We can assume that $\sigma_1 = (1\ 2\ \cdots\ n)$ and $\sigma_2 = (n+1\ n+2\ \cdots\ n+m)$ (we can reach this with the isomorphic second type transformations of lines and columns of the exponent matrix $\mathcal{E}(\Lambda)$). Then the exponent matrix $\mathcal{E}(\Lambda)$ will become of the form

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_1 & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_2 \end{pmatrix}$$

where $\mathcal{E}_1$ is an exponent matrix of a reduced cyclic tiled order with Kirichenko permutation σ_1 and $\mathcal{E}_2$ is an exponent matrix of a reduced cyclic tiled order with Kirichenko permutation σ_2. That is why $\alpha_{ij} + \alpha_{j\sigma_1(i)} = \alpha_{i\sigma_1(i)}$ for all $j = n+1, \ldots, n+m$, $i = 1, \ldots, n$ and $\alpha_{kl} + \alpha_{l\sigma_2(k)} = \alpha_{k\sigma_2(k)}$ for all $l = 1, \ldots, n$, $k = n+1, \ldots, n+m$. whence obtain

$$\begin{aligned} m \cdot \sum_{i=1}^{n} \alpha_{i\sigma_1(i)} &= \sum_{j=n+1}^{n+m} \sum_{i=1}^{n} \alpha_{i\sigma_1(i)} = \sum_{j=n+1}^{n+m} \sum_{i=1}^{n} (\alpha_{ij} + \alpha_{j\sigma_1(i)}) \\ &= \sum_{l=1}^{n} \sum_{k=n+1}^{n+m} (\alpha_{kl} + \alpha_{l\sigma_2(k)}) = \sum_{l=1}^{n} \sum_{k=n+1}^{n+m} \alpha_{k\sigma_2(k)} = n \cdot \sum_{k=n+1}^{n+m} \alpha_{k\sigma_2(k)}. \end{aligned}$$

Whence $m \cdot \sum\limits_{i=1}^{n} \alpha_{i\sigma_1(i)} = n \cdot \sum\limits_{k=n+1}^{n+m} \alpha_{k\sigma_2(k)}$ i.e. $\frac{\sum\limits_{i=1}^{n} \alpha_{i\sigma_1(i)}}{n} = \frac{\sum\limits_{k=n+1}^{n+m} \alpha_{k\sigma_2(k)}}{m}$. □

Corollary 4.1 ([11]). *If n and m are pairwise simple then*

$$\frac{\sum\limits_{i=1}^{n}\alpha_{i\sigma_1(i)}}{n}=\frac{\sum\limits_{k=n+1}^{n+m}\alpha_{k\sigma_2(k)}}{m}=t$$

where t is a natural number.

Lemma 4.1 provides the sufficient condition for the possibility to construct the Gorenstein tiled order with the correspond exponent matrix with a given Kirichenko permutation σ which decomposes into independent cycles $\sigma_1,\ldots,\sigma_s$ and two sided Piers decomposition of Λ has blocks $\Lambda_1,\ldots,\Lambda_s$ on the main diagonal. The input data for this condition is a set of the cyclic tiled orders $\Lambda_1,\ldots,\Lambda_s$ with the cyclic permutations $\sigma_1,\ldots,\sigma_s$ correspondingly. Call this condition Ω and it is following

$$(\Omega)\qquad\qquad \frac{\sum\limits_{i}\alpha_{i\sigma_1(i)}}{|\langle\sigma_1\rangle|}=\cdots=\frac{\sum\limits_{i}\alpha_{i\sigma_s(i)}}{|\langle\sigma_s\rangle|}.$$

Lemma 4.2 ([11]). *Let Λ be a reduced Gorenstein tiled order with correspond exponent matrix $\mathcal{E}=(\alpha_{ij})$ and Kirichenko permutation $\sigma=\sigma_1\cdot\sigma_2$ where $\sigma_1=\begin{pmatrix}1&2&\cdots&n\end{pmatrix}$, $\sigma_2=\begin{pmatrix}n+1&n+2&\cdots&n+m\end{pmatrix}$ such that numbers n and m are piecewise simple. Then the order Λ is isomorphic to the order Λ' with an exponent matrix*

$$\mathcal{E}'=\begin{pmatrix}\mathcal{E}_1' & x_{12}U_{12}\\ x_{21}U_{21} & \mathcal{E}_2'\end{pmatrix}$$

where $\mathcal{E}_1'$, $\mathcal{E}_2'$ are exponent matrices or reduced cyclic Gorenstein orders with permutations $\sigma_1=\begin{pmatrix}1&2&\cdots&n\end{pmatrix}$ and $\sigma_2=\begin{pmatrix}n+1&n+2&\cdots&n+m\end{pmatrix}$ which are linear combinations of powers of permutation matrices P_{σ_1}, P_{σ_2} correspondingly, U_{12}, U_{21} are such matrices that all their elements are equal to ones x_{12}, x_{21} are integers such that

$$x_{12}+x_{21}=t=\frac{\sum\limits_{i=1}^{n}\alpha_{i\sigma_1(i)}}{n}=\frac{\sum\limits_{k=n+1}^{n+m}\alpha_{k\sigma_2(k)}}{m}.$$

Proof. An exponent matrix of a Gorenstein tiled order whose Kirichenko permutation is a product of disjoint cycles has a following form

$$\mathcal{E}=\begin{pmatrix}\mathcal{E}_1 & \mathcal{E}_{12}\\ \mathcal{E}_{21} & \mathcal{E}_2\end{pmatrix}$$

where $\mathcal{E}_1=e\mathcal{E}e$, $e=e_{11}+\cdots+e_{nn}$, $f=E-e$, $\mathcal{E}_2=f\mathcal{E}f$ are exponent matrices of cyclic Gorenstein orders with Kirichenko permutations σ_1,σ_2 correspondingly. As numbers n and m are pairwise simple then the corollary 4.1 gives that

$$t=\frac{\sum\limits_{i=1}^{n}\alpha_{i\sigma_1(i)}}{n}=\frac{\sum\limits_{k=n+1}^{n+m}\alpha_{k\sigma_2(k)}}{m}$$

is a natural number. Then according to the theorem 5 matrices $\mathcal{E}_1,\mathcal{E}_2$ can be reduced to matrices $\mathcal{E}_1'$ and $\mathcal{E}_2'$ with transformations of the first type. The last matrices are linear combinations of permutational matrices P_{σ_1} and P_{σ_2} correspondingly and mire then this $\alpha_{i\sigma_1(i)} = \alpha_{k\sigma_2(k)} = t$ for all $i = 1,\dots,n$ and $k = n+1,\dots,n+m$. The conditions to be Gorenstein matrices yields $\alpha_{ij} + \alpha_{j\sigma_1(i)} = \alpha_{i\sigma_1(i)} = t$ for all $i = 1,\dots,n, j = n+1,\dots,n+m$ and $\alpha_{kl} + \alpha_{l\sigma_2(k)} = \alpha_{k\sigma_2(k)} = t$ for all $l = 1,\dots,n$ and $k = n+1,\dots,n+m$. Whence for $j = k$ and $l = \sigma_1(i)$ obtain $\alpha_{ij} + \alpha_{j\sigma_1(i)} = \alpha_{j\sigma_1(i)} + \alpha_{\sigma_1(i)\sigma_2(j)}$ i.e. $\alpha_{ij} = \alpha_{\sigma_1(i)\sigma_2(j)}$. As lengthes of permutations σ_1 and σ_2 are pairwise simple then $\alpha_{ij} = \alpha_{1n+1}$ for all $i = 1,\dots,n, j = n+1,\dots,n+m$. Denote $\alpha_{1n+1} = x_{12}$ a rational number. Then $\alpha_{kl} = \alpha_{n+11} = t - x_{12}$. Whence the exponent matrix $\mathcal{E}'$ will be of the following form

$$\mathcal{E}' = \begin{pmatrix} \mathcal{E}'_1 & x_{12}U_{12} \\ x_{21}U_{21} & \mathcal{E}'_2 \end{pmatrix}$$

where $\mathcal{E}'_1$ and $\mathcal{E}'_2$ are reduced exponent matrices of cyclic Gorenstein orders with permutations $\sigma_1 = \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} n+1 & n+2 & \cdots & n+m \end{pmatrix}$ correspondingly which are linear combinations of powers of permutational matrices P_{σ_1}, P_{σ_2} correspondingly U_{12} and U_{21} are matrices whose all elements are equal to one. So,

$$x_{12} + x_{21} = t = \frac{\sum\limits_{i=1}^{n} \alpha_{i\sigma_1(i)}}{n} = \frac{\sum\limits_{k=n+1}^{n+m} \alpha_{k\sigma_2(k)}}{m}.$$

□

Theorem 6 ([11])**.** *Let Λ be a reduced Gorenstein tiled oreder with an exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$ whose Kirichenko permutation σ is a product of cycles which do not intersect $\sigma = \sigma_1 \cdots \sigma_m$ and whose lengthes are pairwise simple. Then the order Λ is isomorphic to an order Λ' with Kirichenko permutation $\sigma' = \sigma'_1 \cdots \sigma'_m$ and exponent matrix $\mathcal{E}(\Lambda') = (\alpha'_{ij})$ of the form*

$$\begin{pmatrix} \mathcal{E}_1 & x_{12}U_{12} & \cdots & x_{1m}U_{1m} \\ x_{12}U_{21} & \mathcal{E}_2 & \cdots & x_{2m}U_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1}U_{m1} & x_{m2}U_{m2} & \cdots & \mathcal{E}_m \end{pmatrix},$$

where

$$x_{ij} + x_{ji} = t = \frac{\sum\limits_{k=1}^{|<\sigma_i>|} \alpha_{k\sigma_i(k)}}{|<\sigma_i>|}$$

for all $i, j = 1,\dots m$, $i \neq j$ and $x_{ij} + x_{js} \geq x_{is}$ for all pairwise different $i, j, s = 1,\dots m$, $\mathcal{E}_k$ are exponent matrices of cyclic Gorenstein orders with Kirichenko permutations which are conjugate to those Kirichenko one σ'_k whici are linear combinations of permutational matrices $P_{\sigma'_k}$ ($k = 1,\dots,m$).

Proof. A Gorenstein tiled order Λ is isomorphic to an order Λ'' with Kirichenko permutation $\sigma' = \sigma'_1 \cdots \sigma'_m$ where each permutation σ'_k acts on a set of natural numbers. Then the exponent matrix of the order Λ'' will be of the form

$$\mathcal{E}(\Lambda'') = \begin{pmatrix} \mathcal{E}''_1 & \mathcal{E}''_{12} & \cdots & \mathcal{E}''_{1m} \\ \mathcal{E}''_{21} & \mathcal{E}''_2 & \cdots & \mathcal{E}''_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}''_{m1} & \mathcal{E}''_{m2} & \cdots & \mathcal{E}''_m \end{pmatrix}.$$

As orders of permutations σ'_k are pairwise simple then

$$t=\frac{\sum\limits_{k=1}^{|<\sigma'_i>|}\alpha''_{k\sigma'_i(k)}}{|<\sigma'_i>|}=\frac{\sum\limits_{k=1}^{|<\sigma_i>|}\alpha_{k\sigma_i(k)}}{|<\sigma_i>|}$$

for all $i=1,\dots m$ and t is a natural number. Whence exponent matrices $\mathcal{E}''_k$ with Kirichenko permutations σ'_k can be reduced to the linear combination of powers of permutational matrices $P_{\sigma'_k}$ with transformations of the first type.

For arbitrary i and j ($i\neq j$) consider a Gorenstein tiled order with a Kirichenko permutation $\sigma'_i\cdot\sigma'_j$ and an exponent matrix

$$\begin{pmatrix}\mathcal{E}''_i & \mathcal{E}''_{ij}\\ \mathcal{E}''_{ji} & \mathcal{E}''_j\end{pmatrix}.$$

According to the lemma 4.2 its exponent matrix can be reduced to the form

$$\begin{pmatrix}\mathcal{E}'_i & x_{ij}U_{ij}\\ x_{ji}U_{ji} & \mathcal{E}'_j\end{pmatrix},$$

were matrices $\mathcal{E}'_i$ and $\mathcal{E}'_j$ are linear combinations of powers of their permutational matrices. Whence the exponent matrix is of the form

$$\mathcal{E}(\Lambda')=\begin{pmatrix}\mathcal{E}_1 & x_{12}U_{12} & \cdots & x_{1m}U_{1m}\\ x_{12}U_{21} & \mathcal{E}_2 & \cdots & x_{2m}U_{2m}\\ \cdots & \cdots & \cdots & \cdots\\ x_{m1}U_{m1} & x_{m2}U_{m2} & \cdots & \mathcal{E}_m\end{pmatrix}.$$

Inequalities $x_{ij}+x_{js}\geq x_{is}$ come from ring inequalities $\alpha_{ij}+\alpha_{jk}\geq\alpha_{ik}$. □

5 GORENSTEIN TILED ORDERS WITH MUTUALLY PRIME LENGTHES OF CYCLES

Let $\Lambda=\{\mathcal{O},\mathcal{E}(\Lambda)=(\alpha_{ij})\}$ be a reduced Gorenstein tiled order with Kirichenko permutation σ where $\sigma=\sigma_1\cdots\sigma_s$ is the decomposition of σ into the product of cycles, m_k is a length of the σ_k. We can assume that $\sigma_k=\begin{pmatrix}g_k+1 & g_k+2 & \cdots & g_k+m_k\end{pmatrix}$ (we can reach this with the transformations of the second type) where $g_k=\sum\limits_{j=1}^{k-1}m_j$ for $k>1$, $g_1=0$. Let $f_k=e_{g_k+1g_k+1}+e_{g_k+2g_k+2}+\cdots+e_{g_k+m_kg_k+m_k}$ and $1=f_1+\cdots+f_s$ be the ring unit of Λ into the sum of pair wise orthogonal idempotents. Two sided Piers decomposition of Λ has the following form

$$\Lambda=\begin{pmatrix}\Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1s}\\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2s}\\ \cdots & \cdots & \cdots & \cdots\\ \Lambda_{s1} & \Lambda_{s2} & \cdots & \Lambda_{ss}\end{pmatrix},$$

where Λ_{kk} is reduced cyclic tiled order with Kirichenko permutation $\sigma'=\begin{pmatrix}1 & 2 & \cdots & m_k\end{pmatrix}$. So the exponent matrix is of the form

$$\mathcal{E}=\begin{pmatrix}\mathcal{E}_{11} & \mathcal{E}_{12} & \cdots & \mathcal{E}_{1s}\\ \mathcal{E}_{21} & \mathcal{E}_{22} & \cdots & \mathcal{E}_{2s}\\ \cdots & \cdots & \cdots & \cdots\\ \mathcal{E} & \mathcal{E}_{s2} & \cdots & \mathcal{E}_{ss}\end{pmatrix},$$

where $\mathcal{E}_{kk}$ is an exponent matrix of cyclic Gorenstein tiled order Λ_{kk}. Write out the condition to be Gorenstein order for Λ in the matrix form as follows $\mathcal{E} + P_\sigma \mathcal{E}^T = \mathrm{diag}\{\alpha_{k\sigma(k)}\}U$ or

$$\begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \cdots & \mathcal{E}_{1s} \\ \mathcal{E}_{21} & \mathcal{E}_{22} & \cdots & \mathcal{E}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}_{s1} & \mathcal{E}_{s2} & \cdots & \mathcal{E}_{ss} \end{pmatrix} + \begin{pmatrix} P_{\sigma'_1} & O & \cdots & O \\ O & P_{\sigma'_2} & \cdots & O \\ \cdots & \cdots & \cdots & \cdots \\ O & O & \cdots & P_{\sigma'_s} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{E}_{11}^T & \mathcal{E}_{21}^T & \cdots & \mathcal{E}_{s1}^T \\ \mathcal{E}_{12}^T & \mathcal{E}_{22}^T & \cdots & \mathcal{E}_{s2}^T \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}_{1s}^T & \mathcal{E}_{2s}^T & \cdots & \mathcal{E}_{ss}^T \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{diag}\{\alpha'_{k\sigma_1(k)}\} & O & \cdots & O \\ O & \mathrm{diag}\{\alpha'_{k\sigma_2(k)}\} & \cdots & O \\ \cdots & \cdots & \cdots & \cdots \\ O & O & \cdots & \mathrm{diag}\{\alpha'_{k\sigma_s(k)}\} \end{pmatrix} \cdot \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ U_{s1} & U_{s2} & \cdots & U_{ss} \end{pmatrix}.$$

Whence we get the following system of matrix equations

$$\begin{cases} \mathcal{E}_{jj} + P_{\sigma'_j}\mathcal{E}_{jj}^T = \mathrm{diag}\{\alpha_{k\sigma'_j(k)}\}U_{jj}, \\ \mathcal{E}_{ij} + P_{\sigma'_i}\mathcal{E}_{ji}^T = \mathrm{diag}\{\alpha_{k\sigma'_i(k)}\}U_{ij}, \\ \mathcal{E}_{ji} + P_{\sigma'_j}\mathcal{E}_{ij}^T = \mathrm{diag}\{\alpha_{k\sigma'_j(k)}\}U_{ji}. \end{cases}$$

Reduced cyclic Gorenstein orders are described at theorem 4. Elements of exponent matrix $\mathcal{E}$ of the cyclic reduced Gorenstein tiled order Λ with correspond Kirichenko permutation $\sigma = (12\cdots n)$ whose the first line is zero are expressed with the following formulas

$$\alpha_{km} = \begin{cases} 0 & \text{if} \quad m = 1, \\ \sum\limits_{j=2}^{m} \alpha_{1j} - \sum\limits_{j=0}^{m-2} \alpha_{1k-j} & \text{if} \quad k \geq m > 1, \\ \sum\limits_{j=0}^{k-1} \alpha_{1m-j} - \sum\limits_{j=2}^{k} \alpha_{1j} & \text{if} \quad 1 < k < m, \\ \alpha_{1m} & \text{if} \quad k = 1, \end{cases}$$

and more then this $\alpha_{1j} = \alpha_{1n+2-j}$ for all j. The exponent matrix for a reduced tiled order does not contain two zero lines and does not contain two zero columns. Elements of the exponent matrix $\mathcal{E}$ of a cyclic reduced Gorenstein tiled order Λ with Kirichenko permutation $\sigma = (12\cdots n)$ and the arbitrary the first column can be expressed with the following formulas

$$\alpha'_{km} = \alpha_{km} + x_k - x_m.$$

More then this, elements of the block matrices $\mathcal{E}_{11}, \mathcal{E}_{22}, \ldots, \mathcal{E}_{ss}$ satisfy the condition (Ω) i.e. the following equality

$$\frac{\sum\limits_{k=1}^{m_1} \alpha_{k\sigma'_1(k)}}{m_1} = \frac{\sum\limits_{k=1}^{m_2} \alpha_{k\sigma'_2(k)}}{m_2} = \cdots = \frac{\sum\limits_{k=1}^{m_s} \alpha_{k\sigma'_s(k)}}{m_s}$$

holds.

We have the following system of simultaneous equations for finding $\mathcal{E}_{ij}$

$$\begin{cases} \mathcal{E}_{ij} + P_{\sigma'_i}\mathcal{E}_{ji}^T = \mathrm{diag}\{\alpha_{k\sigma'_i(k)}\}U_{ij}, \\ \mathcal{E}_{ji} + P_{\sigma'_j}\mathcal{E}_{ij}^T = \mathrm{diag}\{\alpha_{k\sigma'_j(k)}\}U_{ji}. \end{cases} \tag{8}$$

Whence, either

$$\mathcal{E}_{ij} + P_{\sigma'_i}\left(\mathrm{diag}\{\alpha_{k\sigma'_j(k)}\}U_{ji} - P_{\sigma'_j}\mathcal{E}_{ij}^T\right)^T = \mathrm{diag}\{\alpha_{k\sigma'_i(k)}\}U_{ij}$$

or

$$\mathcal{E}_{ij} - P_{\sigma'_i}\mathcal{E}_{ij}P_{\sigma'_j}^T = \mathrm{diag}\{\alpha_{k\sigma'_i(k)}\}U_{ij} - P_{\sigma'_i}^T U_{ij}\mathrm{diag}\{\alpha_{k\sigma'_j(k)}\}.$$

As permutational matrix just permutes either lines or columns when we multiply by it then the following is true

$$\mathcal{E}_{ij} - P_{\sigma'_i}\mathcal{E}_{ij}P_{\sigma'_j}^T = \mathrm{diag}\{\alpha_{k\sigma'_i(k)}\}U_{ij} - U_{ij}\mathrm{diag}\{\alpha_{k\sigma'_j(k)}\}. \tag{9}$$

The general solution non homogeneous equation equals to sum of the general solution of the homogeneous equation $\mathcal{E}_{ij} - P_{\sigma'_i}\mathcal{E}_{ij}P_{\sigma'_j}^T = 0$ and some partial solution of non homogeneous equation. Consider the solution of the homogeneous equation.

Lemma 5.1 ([12]). *Let $\sigma_1 = (1\ 2\ \cdots\ n)$, $\sigma_2 = (1\ 2\ \cdots\ m)$, $(n,m) = d$, $n = du$, $m = dv$. The solution of an equation $X - P_{\sigma_1}XP_{\sigma_2}^T = 0$ has the following block form $X = \begin{pmatrix} F & \cdots & F \\ \cdots & \cdots & \cdots \\ F & \cdots & F \end{pmatrix} \in M_{u\times v}(F)$ where $F \in M_d(\mathbb{Z})$ and F is a linear combination of permutational matrix $P_\tau = \sum\limits_{k=1}^{d} e_{k\tau(k)}$, $\tau = (12\cdots d)$ powers with arbitrary coefficients.*

Proof. The equation $X = P_{\sigma_1}XP_{\sigma_2}^T$ can be represented in the form $x_{ij} = x_{\sigma_1(i)\sigma_2(j)}$. Whence $x_{ij} = x_{\sigma_1^k(i)\sigma_2^k(j)}$ for arbitrary integer k. Show that for arbitrary integers p and q such that $0 < i + pd \leqslant n$, $0 < j + qd \leqslant m$ the equality $x_{ij} = x_{i+pd,j+qd}$ holds.

For arbitrary integer k the following equalities hold $\sigma_1^k(i) \equiv i + k (\mathrm{mod}\ n)$, $\sigma_2^k(j) \equiv j + k(\mathrm{mod}\ m)$. As numbers u are v mutually prime then there exist natural numbers a and b such that $p - q = bv - au$. Put $k = pd + an + cnm = qd + bm + cnm$. Then $\sigma_1^k(i) \equiv i + k \equiv i + pd(\mathrm{mod}\ n)$, $\sigma_2^k(j) \equiv j + k \equiv j + qd(\mathrm{mod}\ m)$. This implies $x_{ij} = x_{i+pd,j+qd}$. The las means that the matrix X is decomposed to uv equal blocks F or the dimension d (u blocks are in the block line of X and v blocks in its block column). So, $X = \begin{pmatrix} F & \cdots & F \\ \cdots & \cdots & \cdots \\ F & \cdots & F \end{pmatrix} \in M_{u\times v}(F)$. Now show that $x_{ij} = x_{\tau(i)\tau(j)}$ for $i, j \leqslant d$ where $\tau = (12\cdots d)$.

As $i, j < d$ then for $k = 1$ we have $x_{ij} = x_{\sigma_1^k(i)\sigma_2^k(d)} = x_{i+1,j+1} = x_{\tau(i)\tau(j)}$.

Let $i < d$, $j = d$. As numbers u and v are pairwise simple then $1 = \delta v - \gamma u$ for some integers δ and γ. Take $k = 1 + \gamma n + cnm = 1 - d + \delta m + cnm$. Then $\sigma_1^k(i) \equiv i + k \equiv i + 1 + \gamma n + cnm \equiv i + 1(\mathrm{mod}\ n)$, $\sigma_2^k(d) \equiv d + k \equiv d + 1 - d + \delta m + cnm \equiv 1(\mathrm{mod}\ m)$. Whence $x_{\tau(i)\tau(d)} = x_{i+1,1} = x_{\sigma_1^k(i)\sigma_2^k(d)} = x_{id}$.

In the same way for $i = d$ and $j < d$ take $k = 1 - d - \gamma n + cnm = 1 - \delta m + cnm$. Then $\sigma_1^k(d) \equiv d + k \equiv 1 - \gamma n + cnm \equiv 1(\mathrm{mod}\ n)$, $\sigma_2^k(j) \equiv j + k \equiv j + 1 - \delta m + cnm \equiv j + 1(\mathrm{mod}\ m)$. Whence $x_{\tau(d)\tau(j)} = x_{1,j+1} = x_{\sigma_1^k(d)\sigma_2^k(j)} = x_{dj}$.

For $i = d$ $j = d$ state $k = 1 - d + cnm$. Then $\sigma_1^k(d) \equiv d + k \equiv d + 1 - d + cnm \equiv 1(\mathrm{mod}\ n)$, $\sigma_2^k(d) \equiv d + k \equiv d + 1 - d + cnm \equiv 1(\mathrm{mod}\ m)$. Whence $x_{\tau(d)\tau(d)} = x_{1,1} = x_{\sigma_1^k(d)\sigma_2^k(d)} = x_{dd}$.

Whence $x_{ij} = x_{\tau(i)\tau(j)}$ for $i, j \leqslant d$. That is why $F = x_{11}E + x_{12}P_\tau + x_{13}P_\tau^2 + \cdots + x_{1d}P_\tau^{d-1}$. $\square$

Proposition 5.1 ([12]). *Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda) = (\alpha_{ij})\}$ be a reduced Gorenstein tiled order with Kirichenko permutation σ, $\sigma = \sigma_1 \cdots \sigma_s$ be a decomposition of σ into a product of cycles which do not intersect and their lengthes σ_k are mutually prime i.e. $GCF(|\langle\sigma_1\rangle|, \cdots, |\langle\sigma_s\rangle|) = 1$. Then $\frac{1}{|\langle\sigma_1\rangle|}\sum\limits_{i}^{\alpha_{i\sigma_1(i)}} = \cdots = \frac{1}{|\langle\sigma_s\rangle|}\sum\limits_{j}^{\alpha_{j\sigma_2(j)}} = t$ when t is a natural number.*

Proof. Let m_k be a length of a cycle σ_k, $Y_k = \sum\limits_i \alpha_{i\sigma_k(i)}$. As $(m_1, \ldots, m_s) = 1$ then there exist integers $a_1, \ldots, a_s$ such that $a_1 \quad m_1 + \ldots + a_s \quad m_s = 1$. According to the lemma 4.1 $\frac{Y_p}{m_p} = \frac{Y_q}{m_q}$ which is the same as $m_q Y_p = m_p Y_q$. Multiply this equality with a_q and obtain $a_q m_q Y_p = a_q m_p Y_q$. Whence $\sum\limits_{q\neq p} a_q m_q Y_p = \sum\limits_{q\neq p} a_q m_p Y_q$ which is $Y_p \cdot \sum\limits_{q\neq p} a_q m_q = m_p \cdot \sum\limits_{q\neq p} a_q Y_q$. Taking into attention the equality $(m_1, \ldots, m_s) = 1$ get $Y_p \cdot (1 - a_p m_p) = m_p \cdot \sum\limits_{q\neq p} a_q Y_q$. Numbers $1 - a_p m_p$ and m_p are mutually prime and that is why Y_p is divisible by m_p for all p. □

Theorem 7 ([12]). *Let $\Lambda = \{\mathcal{O},\ \mathcal{E}(\Lambda) = (\alpha_{ij})\}$ be a reduced Gorenstein tiled order with Kirichenko permutation σ and let $\sigma = \sigma_1 \cdots \sigma_s$ be a decomposition of σ to a product of cycles which do not intersect. Let m_k be a length of the cycle $\sigma_k = (g_k + 1\, g_k + 2\, \ldots\, g_k + m_k)$ where $g_k = \sum\limits_{j=1}^{k-1} m_j$ for $k > 1$ and $g_1 = 0$. Let $d_{ij} = GCF(m_i, m_j)$ be a maximal common factor of numbers m_i, m_j and $GCF(m_1, \ldots, m_s) = 1$. Then the order Λ is isomorphic to an order Λ' with Kirichenko permutation $\sigma' = \sigma'_1 \cdots \sigma'_s$ and the exponent matrix $\mathcal{E} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \cdots & \mathcal{E}_{1s} \\ \mathcal{E}_{21} & \mathcal{E}_{22} & \cdots & \mathcal{E}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}_{s1} & \mathcal{E}_{s2} & \cdots & \mathcal{E}_{ss} \end{pmatrix}$ where $\frac{1}{m_i}\sum\limits_{k=1}^{m_i} \alpha'_{k\sigma'(k)} = t \in \mathbb{N}$ for all $i = 1, \ldots s$. The matrix $\mathcal{E}_{kk}$ is a exponent matrix of a cyclic Gorenstein order with Kirichenko permutation $\sigma'_k = (1\,2\,\ldots\,m_k)$ and it is a linear combination of powers of an permutational matrix $P_{\sigma'_k}$ $(k = 1, \ldots, s)$. The matrix $\mathcal{E}_{kl} = \begin{pmatrix} F_{kl} & \cdots & F_{kl} \\ \cdots & \cdots & \cdots \\ F_{kl} & \cdots & F_{kl} \end{pmatrix}$, $k \neq l$, is a $\frac{m_k}{d_{kl}} \times \frac{m_l}{d_{kl}}$ block matrix where F_{kl} is a square $d_{kl} \times d_{kl}$ matrix and it is a linear combination of powers of permutational matrix $P_{\tau_{kl}}$ with $\tau_{kl} = (1\,2\,\ldots\,d_{kl})$.*

Proof. According to the proposition 5.1 $\frac{1}{|\langle\sigma_1\rangle|}\sum\limits_{i}^{\alpha_{i\sigma_1(i)}} = \cdots = \frac{1}{|\langle\sigma_s\rangle|}\sum\limits_{j}^{\alpha_{j\sigma_2(j)}} = t$ where t is a natural number. Then each diagonal matrix $\mathcal{E}_{kk}$ can be reduced with the isomorphic transformations to a matrix $\mathcal{E}'_{kk}$ which is a linear combination of permutational matrix $P_{\sigma'_k}$ such that $\alpha_{i\sigma_k(i)} = t$ for i and k. Then the matrix equation $\mathcal{E}_{ij} - P_{\sigma'_i}\mathcal{E}_{ij}P^T_{\sigma'_j} = \mathrm{diag}\{\alpha_{k\sigma'_i(k)}\}U_{ij} - U_{ij}\mathrm{diag}\{\alpha_{k\sigma'_j(k)}\}$ for $\mathcal{E}_{ij}$ will get the form $\mathcal{E}_{ij} - P_{\sigma'_i}\mathcal{E}_{ij}P^T_{\sigma'_j} = 0$. With using the lemma 5.1 obtain the proposition of the theorem. □

Remark 5.1. *The theorem 7 describes the reduced Gorenstein tiled orders $\Lambda = \{\mathcal{O},\ \mathcal{E}(\Lambda) = (\alpha_{ij})\}$ with the Kirichenko permutation σ where $\sigma = \sigma_1 \cdots \sigma_s$ is the decomposition of σ into a product of cycles which do not intersect whose lengthes is mutually prime. Really this theorem describes also a part of those reduced Gorenstein tiled orders for which there is no any restrictions on cycles but the following "more strict" (Ω) condition holds:*

$$\frac{\sum\limits_i \alpha_{i\sigma_1(i)}}{|\langle\sigma_1\rangle|} = \cdots = \frac{\sum\limits_i \alpha_{i\sigma_s(i)}}{|\langle\sigma_s\rangle|} = t, \text{ where } t \text{ is a natural number.}$$

6 THE NUMBER OF INDEPENDENT PARAMETERS TO EXPRESS ALL ELEMENTS OF THE GORENSTEIN MATRIX

The general solution of non homogeneous equation (9) is a sum of a general solution of homogeneous equation $\mathcal{E}_{ij} - P_{\sigma_i'}\mathcal{E}_{ij}P^T_{\sigma_j'} = 0$ and the partial solution of non homogeneous solution. Consider the solution of the homogeneous equation.

Note that a partial solution of the non homogeneous equation depends on permutations σ_i', σ_j' and elements $\alpha_{k\sigma_i'}(k)$ $\alpha_{k\sigma_j'}(k)$ which belong to diagonal blocks $\mathcal{E}_{ii}$ $\mathcal{E}_{jj}$. The general solution of homogenous equation is independent on elements of diagonal blocks. That is why we can represent the matrix $\mathcal{E}$ as a sum $\mathcal{E} = A + B$ where $A = (A_{ij})$, $B = (B_{ij})$ are block $(s \times s)$ matrices and $A_{ii} = 0$ for all i, A_{ij} is a general solution of the homogeneous equation $B_{ii} = \mathcal{E}_{ii}$ for all i, B_{ij} is a partial solution of non homogenous solution.

According to the theorem 4 matrix $\mathcal{E}_{kk}$ depends on $\left[\frac{m_k}{2}\right]$ parameters. As they satisfy the (Ω)-condition then there are only

$$b = \left[\frac{m_1}{2}\right] + \left[\frac{m_2}{2}\right] + \cdots + \left[\frac{m_s}{2}\right] - (s-1)$$

independent between them. So, elements od B can be expressed on b independent parameters.

Blocks A_{ij} and A_{ji} are solutions of the system $A_{ij} + P_{\sigma_i'}A^T_{ji} = 0$, $A_{ji} + P_{\sigma_j'}A^T_{ij} = 0$. That is why $A_{ji} = -P_{\sigma_j'}A^T_{ij}$ and elements A_{ij} and A_{ji} can be expressed with the same set of parameters.

According to the lemma 5.1 the solution of the equation is a block $(u \times v)$ matrix

$$A_{ij} = \begin{pmatrix} F_{ij} & \cdots & F_{ij} \\ \cdots & \cdots & \cdots \\ F_{ij} & \cdots & F_{ij} \end{pmatrix},$$

where $F_{ij} \in M_{d_{ij}}(\mathbb{Z})$, $d_{ij} = (m_i, m_j)$, $m_i = d_{ij}u$, $m_j = d_{ij}v$.

In this case the matrix F_{ij} is a linear combination of powers of permutational matrix $P_{\tau_{ij}}$ where $\tau_{ij} = (1\ 2\ \ldots\ d_{ij})$, i.e.

$$F_{ij} = \sum_{k=1}^{d_{ij}} a_k^{(ij)} (P_{\tau_{ij}})^k.$$

Whence matrices $\mathcal{E}_{ij}$ and $\mathcal{E}_{ji}$ depend on d_{ij} parameters. Then matrix A depends on $a = \sum\limits_{i<j} d_{ij}$ parameters. Whence we have got that $\mathcal{E}$ depends on $a + b = \sum\limits_{k=1}^{s} \left[\frac{m_k}{2}\right] - (s-1) + \sum\limits_{1\leq i<j\leq s} (m_i, m_j)$ parameters.

Equivalent transformation of the second type do not change values of elements of a matrix (they change only position of elements) and that is why the general quantity of parameters $a + b$ does not change.

Equivalent the first type transformations of the matrix $\mathcal{E}$ can be used either over the matrix A or over the matrix B. Diagonal blocks $\mathcal{E}_{ii}$ of the matrix B already are of the special form (the first column is zero) with the help of which they depend on the minimal quantity of parameters. Parameters $a_1^{(ij)},\dots,a_{d_{ij}}^{(ij)}$ are in each line of the matrix A_{ij} i.e. in each line of the i-th block band of the matrix A. Use the following the first type transformation. Subtract the integer t from all numbers of the i-th horizontal block band and add this t to all numbers of the i-th vertical block line

Under such transformation the form of diagonal blocks of $\mathcal{E}_{ii}$ will not be changed and so as diagonal blocks of B. The i-th block line and i-th block column will be consisted of new matrices $\bar{A}_{ij}=A_{ij}-tU_{ij}$, $\bar{A}_{ji}=A_{ji}+tU_{ji}$.

As $A_{ji}=-P_{\sigma_j'}A_{ij}^T$ then $\bar{A}_{ij}=-P_{\sigma_j'}A_{ij}^T+tU_{ji}=-P_{\sigma_j'}A_{ij}^T+tP_{\sigma_j'}U_{ji}=-P_{\sigma_j'}(A_{ij}-tU_{ij})^T$. In this case $\bar{F}_{ij}=F_{ij}-tU_{ij}=\sum\limits_{k=1}^{d_{ij}}(a_k^{(ij)}-t)(P_{\tau_{ij}})^k$.

Denote $\bar{a}_k^{(ij)}=a_k^{(ij)}-t$, $k=1,2,\dots,d_{ij}$. The new matrices $\bar{A}_{ij}$ and $\bar{A}_{ji}$ depends also on d_{ij} parameters $\bar{a}_1^{(ij)},\dots,\bar{a}_{d_{ij}}^{(ij)}$. If consider $t=a_k^{(ij)}$ for some k then $\bar{a}_k^{(ij)}=0$ and the number of parameters for express all the elements of the matrix $\bar{A}_{ij}$ will decrease by one.

The matrix A contains s horizontal and vertical block lines and columns. Subtract $t_i=a_1^{(i1)}$ from all elements of i-th horizontal band ($i=2,\dots,s$) and add it to all elements of the i-th vertical block band. Then the number of parameters of A will decrease by $s-1$.

Whence the number of parameters to express all the elements of a reduced Gorenstein exponent matrix with equals to

$$\sum_{k=1}^{s}\left[\frac{m_k}{2}\right]-2(s-1)+\sum_{1\le i<j\le s}(m_i,m_j)$$

parameters.

This result coincides with one which is got at [13]. Nevertheless we state the elements of matrix which can be considered as independent.

This result is obtained at [13] in a following way. Consider an exponent matrix $A=(a_{ij})$ with correspond Kirichenko permutation σ which can be decomposed to independent cycles of lengthes $l_1,\dots,l_q$ and q is the number of cycles.

Denote $x_k=a_{k,1}$ for every k, $2\le k\le n$. For arbitrary r, $0<r<q$ and for every k, $n_r+2\le k\le n$ denote also $z_{k,r}=a_{n_r+1,k}$. Consider the variables x_k and $z_{k,r}$ as parameters.

After this one can get formulas which are analogous for (5). Nevertheless parameters are not independent and there are some linear equalities which contain parameters. Finding the defect of the linear equations system which is consisted of these equations gives the quantity of independent parameters.

This system of equations and correspond quantity is found in [13].

7 The sufficiency of the condition (Ω) for constructing the exponent matrix with given cyclic Gorenstein matrices on the main block diagonal

Consider an exponent matrix $\mathcal{E}_1$ with the permutation $\sigma_1=(1\,2\,\dots\,m)$ and an exponent matrix $\mathcal{E}_2$ with the permutation $\sigma_2=(1\,2\,\dots\,n)$. Let the condition (Ω)

$$\frac{\sum\limits_i\alpha_{i\sigma_1(i)}}{|\langle\sigma_1\rangle|}=\frac{\sum\limits_i\alpha_{i\sigma_s(i)}}{|\langle\sigma_s\rangle|}$$

holds. We will show that there exists a reduced Gorenstein exponent matrix

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_1 & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_2 \end{pmatrix}$$

with correspond permutation $\sigma = \sigma_1\sigma_2'$, where $\sigma_2' = (m+1\; m+2\; \ldots\; m+n)$.

From the above the matrix $\mathcal{E}_{12}$ is f dimension $m \times n$ and the matrix $\mathcal{E}_{21}$ is of dimension $n \times m$.

We have an equations system using which may find the matrices $\mathcal{E}_{12}$ and $\mathcal{E}_{21}$

$$\begin{cases} \mathcal{E}_{12} + P_{\sigma_1'}\mathcal{E}_{21}^T = \mathrm{diag}\{\alpha_{k\sigma_1'(k)}\}U_{12}, \\ \mathcal{E}_{21} + P_{\sigma_2'}\mathcal{E}_{12}^T = \mathrm{diag}\{\alpha_{k\sigma_2'(k)}\}U_{21}. \end{cases}$$

This system can be rewritten as

$$\left(\begin{array}{cc|c} E & A & \bar{a} \\ B & E & \bar{b} \end{array}\right), \tag{10}$$

where $E, A, B \in M_{mn}(\mathbb{Z})$, $A = (A_{ij})$ is a block $m \times n$ matrix, $B = (B_{ij})$ is a block $n \times m$ matrix A_{ij} are matrices of the dimension $n \times m$, B_{ij} are matrices of the dimension $m \times n$ such that $A_{ij} = e_{j\sigma_1(i)}$, $B_{ij} = e_{j\sigma_2(i)}$, $\bar{a} = \begin{pmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_m \end{pmatrix}$, $\bar{b} = \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_n \end{pmatrix}$, $\bar{a}_i = \alpha_{i\sigma_1(i)}\bar{u}_n$ and $\bar{a}_i \in \mathbb{Z}^n$, $\bar{b}_i = \alpha_{i\sigma_2(i)}\bar{u}_m \in \mathbb{Z}^m$.

Multiply the first block line of the system with $(-B)$ and add it to the second line. After this obtain

$$\left(\begin{array}{cc|c} E & A & \bar{a} \\ 0 & E - BA & \bar{b} - B\bar{a} \end{array}\right). \tag{11}$$

Here $BA = C$ is a block $n \times n$ matrix where $C_{ij} = \sum\limits_{k=1}^{m} B_{ik}A_{kj} = \sum\limits_{k=1}^{m} e_{k\sigma_2(i)}e_{j\sigma_1(k)}$. More exactly $C_{ij} = 0$ for $j \neq \sigma_2(i)$ and $C_{i\sigma_2(i)} = \sum\limits_{k=1}^{m} e_{k\sigma_2(i)}e_{\sigma_2(i)\sigma_1(k)} = \sum\limits_{k=1}^{m} e_{k\sigma_1(k)} = P_{\sigma_1}$.

The linear equations (10) is consistent if and only if the linear equations system (11) is also consistent.

Consider the linear equation system $\left(E - BA \;\middle|\; \bar{b} - B\bar{a} \right)$, where $\bar{b} - B\bar{a} = \bar{c} = \begin{pmatrix} \bar{c}_1 \\ \vdots \\ \bar{c}_n \end{pmatrix}$, $\bar{c}_i = \bar{b}_i - \sum\limits_{k=1}^{m} B_{ik}\bar{a}_k = \alpha_{i\sigma_2(i)}\bar{u}_m - \sum\limits_{k=1}^{m} e_{k\sigma_2(i)}\alpha_{k\sigma_1(k)}\bar{u}_n$.

As $e_{k\sigma_2(i)}\bar{u}_n = \bar{e}_k = \begin{pmatrix} \underbrace{0\; 0\; \ldots\; 0}_{k-1} & 1 & 0 & \ldots & 0 \end{pmatrix}^T$, then

$$\bar{c}_i = \alpha_{i\sigma_2(i)}\bar{u}_m - \sum_{k=1}^{m} \alpha_{k\sigma_1(k)}\bar{e}_k = \begin{pmatrix} \alpha_{i\sigma_2(i)} - \alpha_{1\sigma_1(1)} \\ \alpha_{i\sigma_2(i)} - \alpha_{2\sigma_1(2)} \\ \vdots \\ \alpha_{i\sigma_2(i)} - \alpha_{m\sigma_1(m)} \end{pmatrix}.$$

.

The system $\left(\; E - BA \;\middle|\; \bar{b} - B\bar{a} \;\right)$ appears to be of the form

$$\left(\begin{array}{cccccc|c} E & -P_{\sigma_1} & & & & & \bar{c}_1 \\ & E & -P_{\sigma_1} & & & & \bar{c}_2 \\ & & \ddots & \ddots & & & \vdots \\ & & & E & -P_{\sigma_1} & & \bar{c}_{m-1} \\ -P_{\sigma_1} & & & & E & & \bar{c}_m \end{array} \right). \tag{12}$$

Multiply the k-th block of the system by $P_{\sigma_1}^k, \quad k = \overline{1, m-1}$ and add it to the last one. After this obtain

$$\left(\begin{array}{ccccc|c} E & -P_{\sigma_1} & & & & \bar{c}_1 \\ & E & -P_{\sigma_1} & & & \bar{c}_2 \\ & & \ddots & \ddots & & \vdots \\ & & & E & -P_{\sigma_1} & \bar{c}_{n-1} \\ 0 & 0 & \dots & 0 & E - P_{\sigma_1}^n & P_{\sigma_1}\bar{c}_1 + P_{\sigma_1}^2\bar{c}_2 + \ldots + P_{\sigma_1}^{n-1}\bar{c}_{n-1} + \bar{c}_n \end{array} \right).$$

The linear equations system (12) is consistent if and only if so as

$$\left(\; E - P_{\sigma_1}^n \;\middle|\; P_{\sigma_1}\bar{c}_1 + P_{\sigma_1}^2\bar{c}_2 + \ldots + P_{\sigma_1}^{m-1}\bar{c}_{m-1} + \bar{c}_m \;\right). \tag{13}$$

Consider the square matrix $E - P_{\sigma_1}^n \in M_m(\mathbb{Z})$. Let $m = du$, $n = dv$ where $d = (n, m)$, $(u, v) = 1$. We can consider that $m > n$.

Then

$$E - P_{\sigma_1}^n = \begin{array}{cl} \overbrace{\hphantom{E \quad E \quad \ddots \quad E}}^{v} & \\ \left[\begin{array}{cccc|cccc} E & & & & -E & & & \\ & E & & & & -E & & \\ & & \ddots & & & & \ddots & \\ & & & E & & & & -E \\ \hline -E & & & & E & & & \\ & -E & & & & E & & \\ & & \ddots & & & & \ddots & \\ & & & -E & & & & E \end{array} \right] & \begin{array}{l} \left.\vphantom{\begin{array}{c} E \\ E \\ \ddots \\ E \end{array}}\right\} u - v \\ \vphantom{\begin{array}{c} E \\ E \\ \ddots \\ E \end{array}} \end{array} \end{array}$$

where $E \in M_d(\mathbb{Z})$.

Add all the block lines of the matrix $E - P_{\sigma_1}^n$ to the last one and obtain the zero block line. Whence $rank(E - P_{\sigma_1}^n) \leq n - d$. From another hand the lemma 5.1 yields that the solution X of the equation $X - P_{\sigma_1} X P_{\sigma_2} = 0$ belongs on d parameters. This lets to find the matrix defect of X as $def\ X = d$.

Whence $def\,(E - P_{\sigma_1}^n) = d$ and $rank\,(E - P_{\sigma_1}^n) = m - d$.

Divide the system (13) into u bands of the width d and add all the lines to the last one. Then the last band will become zero.

The system (13) is consistent if and only if the last band of the expanded linear equations system is equal to zero.

So we get $P_{\sigma_1^k} = \sum\limits_{k=1}^{m} e_{i\sigma_1^k(i)}$. Then

$$P_{\sigma_1^k} \cdot \bar{c}_k = \sum_{k=1}^{m} e_{i\sigma_1^k(i)} \cdot \begin{pmatrix} \alpha_{i\sigma_2(i)} - \alpha_{1\sigma_1(1)} \\ \alpha_{i\sigma_2(i)} - \alpha_{2\sigma_1(2)} \\ \vdots \\ \alpha_{i\sigma_2(i)} - \alpha_{m\sigma_1(m)} \end{pmatrix} = \alpha_{i\sigma_2(i)}\bar{u}_m - \begin{pmatrix} \alpha_{k+1\,\sigma_1(k+1)} \\ \vdots \\ \alpha_{n\,\sigma_1(n)} \\ \alpha_{1\,\sigma_1(1)} \\ \vdots \\ \alpha_{k\,\sigma_1(k)} \end{pmatrix}.$$

$$\bar{c}_n + P_{\sigma_1}\bar{c}_1 + P_{\sigma_1}^2\bar{c}_2 + \ldots + P_{\sigma_1}^{n-1}\bar{c}_{n-1} = (\alpha_{1\sigma_2(1)} + \ldots + \alpha_{n\sigma_2(n)})\bar{u}_n$$
$$- \left(\begin{pmatrix} \alpha_{1\sigma_1(1)} \\ \alpha_{2\sigma_1(2)} \\ \vdots \\ \alpha_{m\sigma_1(m)} \end{pmatrix} + \begin{pmatrix} \alpha_{2\sigma_1(2)} \\ \vdots \\ \alpha_{m\sigma_1(m)} \\ \alpha_{1\sigma_1(1)} \end{pmatrix} + \begin{pmatrix} \alpha_{3\sigma_1(3)} \\ \vdots \\ \alpha_{1\sigma_1(1)} \\ \alpha_{2\sigma_1(2)} \end{pmatrix} + \ldots + \begin{pmatrix} \alpha_{n\sigma_1(n)} \\ \vdots \\ \alpha_{m\sigma_1(m)} \\ \alpha_{1\sigma_1(1)} \\ \vdots \\ \alpha_{n-1\sigma_1(n-1)} \end{pmatrix} \right).$$

Add all the bands of the width d to the last one and obtain

$$u\sum_{k=1}^{n} \alpha_{k\sigma_2(k)}\bar{u}_d - v\sum_{k=1}^{m} \alpha_{k\sigma_2(k)}\bar{u}_d = \bar{0}_d.$$

Whence the system (13) is consistent. Then the former equations system is also consistent. This means that the condition (Ω) is sufficient for constructing an exponent matrix with given cyclic Gorenstein matrices on the main block diagonal.

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Розглядаються горенштейнові черепичні порядки. Доводиться, що необхідна умова для побудови горенштейнового черепичного порядку, у якого на головній блочній діагоналі стоять задані циклічні горенштейнові черепичні порядки, є і достатньою.

Ключові слова і фрази: горенштейновий черепичний порядок, матриця показників, підстановка Кириченка.

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Рассматриваются горенштейновы черепичные порядки. Доказывается, что необходимое условие для построения горенштейнова черепичного порядка, у которого на главной блочной диагонали стоят заданные циклические горенштейновы порядки, является и достаточным.

Ключевые слова и фразы: горенштейнов черепичный порядок, матрица показателей, подстановка Кириченка.